Canad. Math. Bull. Vol. **54** (4), 2011 pp. 663–675
doi:10.4153/CMB-2011-075-1
© Canadian Mathematical Society 2011

Admissible Sequences for Twisted Involutions in Weyl Groups

Ruth Haas and Aloysius G. Helminck

Abstract. Let W be a Weyl group, Σ a set of simple reflections in W related to a basis Δ for the root system Φ associated with W and θ an involution such that $\theta(\Delta) = \Delta$. We show that the set of θ-twisted involutions in W, $\mathfrak{I}_\theta = \{w \in W \mid \theta(w) = w^{-1}\}$ is in one to one correspondence with the set of regular involutions $\mathfrak{I}_{\mathrm{Id}}$. The elements of $\mathfrak{I}_\theta$ are characterized by sequences in Σ which induce an ordering called the Richardson–Springer Poset. In particular, for Φ irreducible, the ascending Richardson–Springer Poset of $\mathfrak{I}_\theta$, for nontrivial θ is identical to the descending Richardson–Springer Poset of $\mathfrak{I}_{\mathrm{Id}}$.

1 Introduction

Let W be a Weyl group generated by a set of reflections Σ. We will assume that Σ comes from a basis Δ for the root system associated with W, *i.e.*, (W, Σ) is a finite Coxeter System corresponding to a root system. If θ is an involution of W then the set $\mathfrak{I}_\theta = \{w \in W \mid \theta(w) = w^{-1}\}$ is called the set of θ-twisted involutions in W. This set is important in the study of orbits of minimal parabolic k-subgroups acting on symmetric k-varieties. The geometry of these orbits and their closures induce a poset structure on the set $\mathfrak{I}_\theta$. Understanding this poset structure is key to understanding the structure of the orbits. This connection will be described in Section 2. There we will also show that it is sufficient to consider only those involutions θ such that $\theta(\Delta) = \Delta$, and hence $\theta(\Sigma) = \Sigma$. Thus we restrict our attention to involutions θ such that $\theta(\Delta) = \Delta$. The results of this paper imply a simple algorithm for computing the poset and elements of $\mathfrak{I}_\theta$. In particular, for any θ, the poset of $\mathfrak{I}_\theta$, can be quickly derived from the poset of $\mathfrak{I}_{\mathrm{Id}}$, where Id is the identity automorphism of W. Notice that $\mathfrak{I}_{\mathrm{Id}} = \{w \in W \mid w^2 = e\}$.

The Weyl Group W acts on the set $\mathfrak{I}_\theta$ by θ-*twisted conjugation*, which is defined as $w * a = wa\theta(w)^{-1}$ where $w \in W$ and $a \in \mathfrak{I}_\theta$. If $s \in \Sigma$ and $a \in \mathfrak{I}_\theta$ then define $s \circ a = sa$ (group multiplication) if $s * a = a$ and $s \circ a = s * a$ otherwise. In both cases $s \circ a$ is a twisted involution in $\mathfrak{I}_\theta$. A sequence $\mathbf{s} = (s_1, \dots, s_k)$ in Σ induces a sequence in $\mathfrak{I}_\theta$ defined by induction as follows: $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$, where $a_0 = e$ and $a_i = s_i \circ a_{i-1} = s_i \circ \cdots \circ s_1 \circ e$ for $i \in [1, k]$. It will be important to keep track of for which elements in $\mathbf{s}$ the $*$ operation is used. Thus for $s \in \Sigma$, $a \in \mathfrak{I}_\theta$ we will use the notation $\overline{s_i}$ if $s_i \circ a_{i-1} = s_i * a_{i-1}$. Define $\overline{\Sigma} := \{\bar{s} \mid s \in \Sigma\}$ and let $\mathbf{r_s} = (r_1, r_2, \dots, r_k)$ be the sequence in $\Sigma \cup \overline{\Sigma}$ defined by $r_i = \overline{s_i}$ if $s_i \circ a_{i-1} = s_i * a_{i-1}$ in $\mathbf{a}(\mathbf{s})$ and $r_i = s_i$ otherwise. The sequence $\mathbf{r_s}$ will be called an admissible sequence for $\mathfrak{I}_\theta$ and the sequence $\mathbf{s}$ will be called an underlying admissible sequences for $\mathfrak{I}_\theta$.

Received by the editors November 7, 2008.
Published electronically April 25, 2011.
The second author was partially supported by N.S.F. Grant DMS-0532140.
AMS subject classification: **20G15**, 20G20, 22E15, 22E46, 43A85.

https://doi.org/10.4153/CMB-2011-075-1 Published online by Cambridge University Press

Recall $l(w)$, the *length* of w with respect to Σ, is the number of elements in a minimal expression of w as a product of basis elements. Note that this is not generally equal to the number of elements in $\Sigma \cup \overline{\Sigma}$ in a sequence determining w. The sequence $\mathbf{r_s}$ is called an *admissible ascending sequence* for $\mathcal{I}_\theta$ if $0 = l(a_0) < l(a_1) < \cdots < l(a_k)$. We will often refer to these as simply admissible sequences when the rest is clear from context. Richardson and Springer [RS90] showed that every element in $\mathcal{I}_\theta$ can be represented by admissible sequences. An element may be represented by several admissible sequences, and determining all of these is crucial to determine the orbit closures described in Section 2.

Remark 1.1 From the classification of involutions of simple algebraic groups in [Hel88] it follows that if the involution θ comes from the root system of a maximal torus as in [Hel88], then for Φ of type D_{2n} the condition $\theta(\Delta) = \Delta$ implies that $\theta = id$. So the case of a non-trivial diagram automorphism does not occur.

In this paper we show the following.

Theorem 1.2 *Let W be a Weyl group not of type D_{2n}, with Σ, Δ and Φ as above, and θ an involution such that $\theta(\Delta) = \Delta$. The sequence $\mathbf{r} = (r_1, r_2, \dots, r_n)$ in $\Sigma \cup \overline{\Sigma}$ is a maximal admissible ascending sequence for $\mathcal{I}_\theta$ if and only if $\hat{\mathbf{r}} = (r_n, \dots, r_1)$ is a maximal admissible ascending sequence for $\mathcal{I}_{\mathrm{Id}}$. Where in the sequence $\mathbf{r}$, $\bar{s}a$ denotes the operation $sa\theta(s)$, while in the sequence $\hat{\mathbf{r}}$, $\bar{s}a$ denotes the operation sas.*

The full power of this theorem comes from the fact that every partial sequence of a maximal admissible ascending sequence will also be an admissible ascending sequence, and indeed, all admissible sequences are partial sequences of maximal ones. Note that the $*$ and $\circ$ action on $\mathcal{I}_\theta$ as used in the sequences depend on the involution, and are therefore different for the two sequences in the theorem. Computing the sequences for $\mathcal{I}_{\mathrm{Id}}$ will be significantly easier than doing it directly in $\mathcal{I}_\theta$. To show that there is a one-to-one correspondence between sequences we will need some additional definitions.

Recall that W has a unique element of maximum length with respect to Σ, denoted here by w_0. We will show that $w_0 \in \mathcal{I}_\theta$, for all θ and if $\mathbf{s} = (s_1, \dots, s_n)$ is a maximal admissible sequence then $w_0 = s_n \circ \cdots \circ s_1 \circ e$. It will be convenient to define partial sequences descending from w_0 as follows. If $\mathbf{t} = (t_1, \dots, t_k)$ is a sequence in Σ, then define a sequence $(b_0, b_1, \dots, b_k)$ in $\mathcal{I}_\theta$ by induction as follows. Let $b_0 = w_0$ and for $i \in [1,\ k]$ let $b_i = t_i \circ b_{i-1}$. As before, let $\mathbf{r_t} = (r_1, r_2, \dots, r_k)$ be the sequence in $\Sigma \cup \overline{\Sigma}$ defined by $r_i = \overline{t_i}$ if $t_i \circ b_{i-1} = t_i * a_{i-1}$ and $r_i = t_i$ otherwise. The sequence $\mathbf{r_t}$ will be called a *descending sequence* for $\mathcal{I}_\theta$. Admissible descending sequences in $\mathcal{I}_\theta$ are those in which the lengths strictly decrease, *i.e.*, if $l(w_0) = l(b_0) > l(b_1) > \cdots > l(b_k)$.

We use xy to denote regular group multiplication of x and y, as well, it will be convenient to denote the action $s * x = sxs$ by $\bar{s}x$. For a sequence $\mathbf{r} = (r_1, r_2, \dots, r_k)$ and $w \in W$, define $\mathbf{r} \cdot w := r_k \cdots r_2 r_1 w$.

Theorem 1.3 *Let W be a Weyl group not of type D_{2n}, with Σ, Δ and Φ as above, and θ an involution such that $\theta(\Delta) = \Delta$. Denote by $\cdot$ the action of a sequence $\mathbf{r} \in \Sigma \cup \overline{\Sigma}$ on $\mathcal{I}_\theta$, and $\cdot'$ the action of $\mathbf{r}$ on $\mathcal{I}_{Id}$.*

https://doi.org/10.4153/CMB-2011-075-1 Published online by Cambridge University Press

1. *A sequence* $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ *in* $\Sigma \cup \overline{\Sigma}$ *is an admissible ascending sequence for* $\mathfrak{I}_\theta$ *if and only if it is an admissible descending sequence for* $\mathfrak{I}_{\mathrm{Id}}$.
2. *A sequence* $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ *in* $\Sigma \cup \overline{\Sigma}$ *is an admissible descending sequence for* $\mathfrak{I}_\theta$ *if and only if it is also an admissible ascending sequence for* $\mathfrak{I}_{\mathrm{Id}}$.
3. *Let* $\mathbf{q}$ *and* $\mathbf{r}$ *be two sequences in* $\Sigma \cup \overline{\Sigma}$. *They are two admissible ascending sequences for* $\mathfrak{I}_\theta$ *and* $\mathbf{q} \cdot e = \mathbf{r} \cdot e$ *in* $\mathfrak{I}_\theta$ *if and only if they are two admissible descending sequences for* $\mathfrak{I}_{\mathrm{Id}}$ *and* $\mathbf{q} \cdot' w_0 = \mathbf{r} \cdot' w_0$ *in* $\mathfrak{I}_{\mathrm{Id}}$.

The sequences induce a partial order on the elements of $\mathfrak{I}_\theta$. If $a, b \in \mathfrak{I}_\theta$, then define $a \succ b$ if there exists an admissible ascending sequence $(r_1, \ldots, r_n)$ and $j < n$ such that $a = r_n \cdots r_2 r_1 e$ and $b = r_j \cdots r_2 r_1 e$. This is a combinatorial description of the Richardson–Springer order on $\mathfrak{I}_\theta$, see Section 2. Following the proofs of the main results we give some additional results about the structure of the the Richardson–Springer Poset and its sequences in Section 5.

The usual Bruhat order in the Weyl group also induces a poset structure on the set of involutions. This poset structure was studied by Incitti, Hultman and others (see, *e.g.*, [Hul05], [Inc04], [Inc06]). However this order is not the same order as the one that was used by Richardson and Springer [RS90] to describe the orbits (and their closures) of parabolic subgroups acting on symmetric varieties. In this paper we consider the poset structure induced by the Richardson–Springer order which we refer to as the Richardson–Springer poset. An example of the difference in the posets is given in Section 3.

The results in this paper will lead in a variety of directions. For example, the duality for the twisted involution posets implies a duality for representations of the associated symmetric spaces. The structure developed here leads to a significant simplification in designing algorithms for computations related to symmetric spaces. Many interesting combinatorial questions can be asked about these posets including rank formulas, path formulas, *etc.* Forthcoming papers of the authors and their students address these ([CHHW10], [CHH+10], [HH010]).

2 Context

In this section we describe how twisted involutions and their admissible sequences help to characterize orbits of a minimal parabolic k-subgroup acting on a symmetric k-variety. We start by discussing these orbits and their applications.

Throughout this section let G be a connected reductive algebraic group defined over a field k of characteristic not 2, θ an involution of G defined over k (*i.e.*, of order two) and $H = G_\theta = \{g \in G \mid \theta(g) = g\}$ the set of fixed points of θ. Let G_k, H_k denote the sets of k-rational points of G, H. The orbits of a minimal parabolic k-subgroup P acting on the symmetric k-variety G_k/H_k play a fundamental role in the study of representations associated with these symmetric k-varieties. These orbits were studied for many fields and can be characterized in several equivalent ways. They can be characterized as the P_k-orbits acting on the symmetric k-variety G_k/H_k by θ-twisted conjugation (*i.e.*, if $x, g \in G_k$ then define $g * x := gx\theta(g)^{-1}$), or as the number of H_k-orbits acting on the flag variety G_k/P_k by conjugation, or also as the set $P_k \backslash G_k / H_k$ of (P_k, H_k)-double cosets in G_k. The last is the same as the set of $P_k \times H_k$-

https://doi.org/10.4153/CMB-2011-075-1 Published online by Cambridge University Press

orbits on G_k. For k algebraically closed these orbits were characterized by Springer [Spr85], for $k = \mathbb{R}$ characterizations were given by Matsuki [Mat79] and Rossmann [Ros79] and for general fields these orbits were characterized by Helminck and Wang [HW93]. For general fields one can first consider the set of (P, H)-double cosets in G. Then the (P_k, H_k)-double cosets in G_k can be characterized by the (P, H)-double cosets in G defined over k plus an additional invariant describing the decomposition of a (P, H)-double coset into (P_k, H_k)-double cosets.

The following result gives a characterization of the (P, H)-double cosets in G. Let $A \subset P$ be a θ-stable maximal k-split torus of G (which exists by [HW93]), N the normalizer of A in G_k, Z the centralizer of A in G_k, let $W = W(A) = N/Z$ be the Weyl group of A in G_k, and let $\mathcal{I}_\theta = \{w \in W \mid \theta(w) = w^{-1}\}$ be the set of twisted involutions in W.

Theorem 2.1 *The (P, H)-double cosets in G defined over k can be characterized by the pairs $(\mathcal{O}, w)$, where $\mathcal{O}$ is a closed orbit and $w \in \mathcal{I}_\theta \subset W$.*

Proof For k algebraically closed and P a Borel subgroup of G this result follows by combining [Spr85] and [RS90]. For k not algebraically closed and P a minimal parabolic k-subgroup of G the result follows by combining the characterization of the orbits in [HW93], [Hel00] and [Hel04]. ∎

Remark 2.2 It follows from this result that to classify the $P \times H$ orbits on G one needs to determine both the closed orbits and the twisted involutions that occur. In many cases there is a unique closed orbit and then the $P \times H$ orbits on G are completely characterized by the twisted involutions in W. In this paper we prove a number of results about these twisted involutions in the Weyl group.

2.3 Orbit Closures

These twisted involutions also play an essential role in the study of the orbit closures. The closure of a $P \times H$ orbit on G decomposes as the union of other $P \times H$ orbits on G and these orbit closures are of fundamental importance in the study of Harish-Chandra modules, see [Vog83]. There is a natural order $\succ$ on the set of $P \times H$ orbits on G, called the *Bruhat order*, which is defined as follows. If $\mathcal{O}_1, \mathcal{O}_2$ are orbits, then $\mathcal{O}_1 \succ \mathcal{O}_2$ if and only if $\mathcal{O}_2$ is contained in the closure of the orbit $\mathcal{O}_1$. This order on the orbits induces an order on the related set of twisted involutions, which we call the *Richardson–Springer order* on $\mathcal{I}_\theta$. In [RS90] Richardson and Springer gave a combinatorial description of these Bruhat orders in terms of sequences of reflections in simple roots, which is exactly the sequence order as we defined above.

2.4 Reduction to Diagram Automorphisms

In this subsection we show that it suffices to consider involutions θ such that $\theta(\Delta) = \Delta$. For k an algebraically closed field, Steinberg [Ste68] proved that there exists a θ-stable maximal torus A and a basis Δ of $\Phi(T)$ such that $\theta(\Delta) = \Delta$. In this subsection we show that this result can be extended to involutions of groups defined over non algebraically closed fields and maximal k-split tori as above. First we need some more

https://doi.org/10.4153/CMB-2011-075-1 Published online by Cambridge University Press

notation. In the remainder of this subsection let G be a reductive algebraic group defined over a field k, θ a k-involution of G, A a θ-stable maximal k-split torus of G, $X = X^*(A)$ the group of characters of A, $\Phi = \Phi(A)$ the set of roots of A with respect to G and Φ^+ the set of positive roots with respect to a basis Δ of Φ. Let $w_\theta \in W$ such that

$$\theta(\Phi^+) = w_\theta(\Phi^+),$$

and let $\theta' = \theta w_\theta$. Then w_θ and θ' satisfy the following conditions:

Proposition 2.5 *Let Φ, Φ^+, θ, w_θ and θ' be as above.*

1. *$w_\theta \in \mathfrak{I}_\theta$.*
2. *θ' is an involution of Φ and $\theta'(\Delta) = \Delta$.*
3. *θ' is $W(A)$-conjugate to θ if and only if $w_\theta = \theta(x)x^{-1}$ for some $x \in W(A)$.*

Proof Since $\theta(\Phi^+) = w_\theta(\Phi^+)$ it follows that $w_\theta^{-1}\theta(\Phi^+) = \Phi^+$ and consequently $\theta(w_\theta^{-1})(\Phi^+) = \theta w_\theta^{-1}\theta(\Phi^+) = \theta(\Phi^+) = w_\theta(\Phi^+)$. So $\theta(w_\theta^{-1}) = w_\theta$, which proves (1) and (2). As for (3) note that $\theta' = x\theta x^{-1}$ for some $x \in W(A)$ if and only if $\theta' = \theta \cdot \theta(x)x^{-1}$. Since $\theta' = \theta w_\theta$ the result follows. ■

Although $w_\theta \in \mathfrak{I}_\theta$ the involutions θ' and θ are usually not conjugate since not all elements of $\mathfrak{I}_\theta$ can be written as $\theta(x)x^{-1}$ for some $x \in W(A)$. To remove the element w_θ we need to pass to a θ-stable conjugate of the torus A and choose a suitable basis Δ of Φ. We first describe this basis, which will enable us to get a detailed description of w_θ. For this we need some more notation.

Let $X_0(-\theta) = \{\chi \in X \mid \theta(\chi) = -\chi\}$, $\Phi_0(-\theta) = \Phi \cap X_0(-\theta)$ and π the natural projection from X to $X/X_0(-\theta)$. A linear order $\succ$ on X is called a θ^--order if it has the following property:

$$\text{if } \chi \in X, \chi \succ 0, \quad \text{and} \quad \chi \notin X_0(\theta), \text{then } \theta(\chi) \succ 0.$$

A basis Δ of Φ with respect to a θ^--order on X will be called a θ^--basis of Φ. A θ^--order on X induces orders of $\Phi_0(-\theta)$ and $\bar{\Phi}_\theta := \pi\big(\Phi - \Phi_0(-\theta)\big)$. Conversely, linear orders on $\Phi_0(-\theta)$ and $\bar{\Phi}_\theta$ induce a unique θ^--order on X. Let $W\big(\Phi_0(-\theta)\big)$ denote the finite subgroup of $W(A)$ generated by the s_α for $\alpha \in \Phi_0(-\theta)$. We have the following:

Lemma 2.6 *Let Δ be a θ^--basis of Φ, $w_0(-\theta)$ the longest element of $W\big(\Phi_0(-\theta)\big)$ with respect to $\Delta_0(-\theta) := \Delta \cap \Phi_0(-\theta)$ and $\theta^* := \theta \cdot w_0(-\theta)$.*

1. *$w_\theta = w_0(-\theta) \in \mathfrak{I}_\theta$.*
2. *$\theta' = \theta^*$, $(\theta^*)^2 = \mathrm{id}$.*

Proof The result is immediate from the observation that $\theta(\Phi^+) = w_0(-\theta)(\Phi^+)$. ■

This result gives us a nice description of the element w_θ. To remove w_θ we find a suitable conjugate of the torus A. Let $A^- = \{a \in A \mid \theta(a) = a^{-1}\}^0$ and for $\alpha \in \Phi(A)$ let $A_\alpha = \{a \in A \mid s_\alpha(a) = a\}^0$ and $G_\alpha = Z_G(A_\alpha)$ the centralizer of A_α.

https://doi.org/10.4153/CMB-2011-075-1 Published online by Cambridge University Press

Theorem 2.7 *Let A, θ,* etc., *be as above.*

1. *There exists a θ-stable conjugate A_1 of A such that $\Phi_0(\theta, A_1) := \{\alpha \in \Phi(A_1) \mid \theta(\alpha) = -\alpha\} = \varnothing$.*
2. *Let Δ_1 be a θ^--basis of $\Phi(A_1)$. Then $\theta(\Delta_1) = \Delta_1$.*

Proof Let Δ be a θ^--basis of Φ, $w_0(-\theta)$ the longest element of $W\big(\Phi_0(-\theta)\big)$ with respect to $\Delta_0(-\theta) := \Delta \cap \Phi_0(-\theta)$ and $\theta^* := \theta \cdot w_0(-\theta)$. Since $w_0(-\theta)$ is an involution it follows from [Hel91, 2.7] that there exist $\alpha_1, \dots, \alpha_n \in \Phi_0(-\theta)$ strongly orthogonal roots, such that $w_0(-\theta) = s_{\alpha_1} \cdots s_{\alpha_n}$. From [Hel97, 3.8] it follows that there exists $x \in G_{\alpha_1}$ such that $A_0 = xAx^{-1}$ is a θ-stable torus with $\dim A^- = \dim A_0^- + 1$. Let $A^-_{w_0(-\theta)} := \{a \in A \mid w_0(-\theta)(a) = a^{-1}\}^0$. Using induction it follows that there exists $x \in Z_G(A^-_{w_0(-\theta)})$ such that $A_1 = xAx^{-1}$ is θ-stable, $\dim A_1^- = \dim A^- - n$ and $\Phi_0(\theta, A_1) := \{\alpha \in \Phi(A_1) \mid \theta(\alpha) = -\alpha\} = \varnothing$. By choosing a θ^--basis of Φ_1 the result follows from Lemma 2.6. ∎

In the remainder of this paper we assume that Δ is θ-stable.

3 Example

We consider as an example S_4 whose corresponding Dynkin diagram is of type A_3. Throughout this example, $*$, $\circ$ and $\cdot$ will refer to actions of W on $\mathcal{I}_{\mathrm{Id}}$. Let the generators of S_4 be $\Sigma = \{s_1 = (12), s_2 = (23), s_3 = (34)\}$. We compute the maximal ascending sequences for $\mathcal{I}_{\mathrm{Id}}$ directly. We make frequent use of the facts that $s_1 s_3 = s_3 s_1$, $s_1 s_2 s_1 = s_2 s_1 s_2$ and $s_3 s_2 s_3 = s_2 s_3 s_2$. As well, note that the same element will not occur twice in a row in an underlying admissible sequence. Let the *size* of a sequence denote the number of elements in it.

Since $s_1 * e = s_2 * e = s_3 * e = e$, the admissible ascending sequences of size 1 are s_1, s_2 and s_3. There will be 6 admissible ascending sequences of size 2 representing 3 different elements. Since $s_2 * s_1 = s_2 s_1 s_2 = s_1 s_2 s_1 = s_1 * s_2$; $s_3 * s_2 = s_2 * s_3$ while $s_3 * s_1 = s_3 s_1 s_3 = s_1$ so $s_3 \circ s_1 = s_1 s_3 = s_1 \circ s_3$ and the admissible ascending sequences are $\bar{s}_2 s_1 = \bar{s}_1 s_2$; $s_3 s_1 = s_1 s_3$; and $\bar{s}_2 s_3 = \bar{s}_3 s_2$.

Since $s_3 * \bar{s}_2 s_1 = s_3 s_2 s_1 s_2 s_3$ we get the admissible ascending sequence $\bar{s}_3 \bar{s}_2 s_1$. Since $s_2 * s_3 s_1 = s_2 s_3 s_1 s_2$, we get the admissible ascending sequence $\bar{s}_2 s_3 s_1$. The only sequence of size 3 we must still consider is $s_1 * \bar{s}_2 s_3$. Noticing that $s_3 s_2 s_1 s_2 s_3 = s_3 s_1 s_2 s_1 s_3 = s_1 s_3 s_2 s_3 s_1$ shows that the admissible ascending sequence $\bar{s}_1 \bar{s}_2 s_3 = \bar{s}_3 \bar{s}_2 s_1$. There are 3 cases to consider for possible sequences of size 4.

(i) $s_2 * \bar{s}_3 \bar{s}_2 s_1 = s_2 s_3 s_2 s_1 s_2 s_3 s_2 = s_3 s_2 s_3 s_1 s_3 s_2 s_3 = s_3 s_2 s_1 s_2 s_3$, so the admissible ascending sequence is $s_2 \bar{s}_3 \bar{s}_2 s_1$.

(ii) $s_1 * \bar{s}_2 s_3 s_1 = s_1 s_2 s_3 s_1 s_2 s_1$, giving the admissible ascending sequence $\bar{s}_1 \bar{s}_2 s_3 s_1$.

(iii) $s_3 * \bar{s}_2 s_3 s_1 = s_3 s_2 s_3 s_1 s_2 s_3$, giving the admissible ascending sequence $\bar{s}_3 \bar{s}_2 s_3 s_1$.

All three are actually equal to the same element, as can be seen:

$$\begin{aligned} s_3 s_2 s_3 s_1 s_2 s_3 &= s_2 s_3 s_2 s_1 s_2 s_3 = s_2 s_3 s_1 s_2 s_1 s_3 = s_2 s_1 s_3 s_2 s_3 s_1 = s_2 s_1 s_2 s_3 s_2 s_1 \\ &= s_1 s_2 s_1 s_3 s_2 s_1 = s_1 s_2 s_3 s_1 s_2 s_1. \end{aligned}$$

https://doi.org/10.4153/CMB-2011-075-1 Published online by Cambridge University Press

There can be no bigger sequences in $\mathfrak{I}_{\mathrm{Id}}$ since these three start with the three possible elements of Σ. Further this must be w_0 the longest element in $\mathfrak{I}_{\mathrm{Id}}$ and the unique longest in W. From the computation, and more easily, from the poset drawn in Figure 1 we see that there are actually 8 maximal sequences.

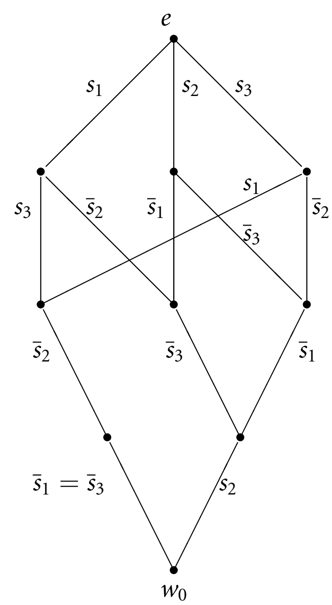

Figure 1: Poset of $\mathfrak{I}_{\mathrm{Id}}$ for S_4.

By Theorems 1.2 and 1.3, we can read all admissible sequences for $\mathfrak{I}_\theta$ directly from our poset for $\mathfrak{I}_{\mathrm{Id}}$. Some samples follow. There will be 8 maximal admissible descending sequences for $\mathfrak{I}_\theta$. Following the left-most path down from the top we get $\bar{s}_1\bar{s}_2 s_3 s_1 w_0$ where the $\bar{s}$ is an action in θ, is one maximal admissible descending sequence for $\mathfrak{I}_\theta$. Two admissible descending sequences in $\mathfrak{I}_\theta$ represent the same element in $\mathfrak{I}_\theta$ if and only if the sequences represent the same element as admissible ascending sequence in $\mathfrak{I}_{\mathrm{Id}}$. Hence, $\bar{s}_2 s_3 w_0 = \bar{s}_3 s_2 w_0$.

Instead of the above order on the set of twisted involutions one can also consider the order induced from the usual Bruhat order in Weyl group. This leads to a different poset, which was studied by Incitti, Hultman and others (see, *e.g.*, [Hul05], [Inc04], [Inc06]). In Figure 2 we give the poset on the set $\mathfrak{I}_{\mathrm{Id}}$ for S_4 induced by the strong Bruhat order. We indicate each additional edge in the above Richardson–Spinger poset with a dashed line.

4 Proofs

Throughout this section we assume that the root system, Weyl Group, *etc.*, come from a maximal k-split torus of G as described in Section 2. We also assume that the

https://doi.org/10.4153/CMB-2011-075-1 Published online by Cambridge University Press

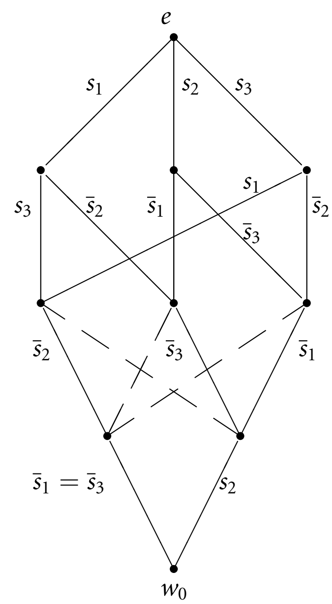

Figure 2: Poset of $\mathcal{I}_{\mathrm{Id}}$ for S_4 induced by the strong Bruhat order.

involution θ is the restriction of an involution of the group G.

Let Φ denote a root system in the Euclidean Vector Space, E, Δ a basis of Φ, Φ^+ and Φ^- the positive and negative roots, respectively, W the Weyl group of Φ and $\Sigma = \{s_\alpha \mid \alpha \in \Delta\}$, where s_α denotes the reflection through α. If $\tau \in \mathrm{Aut}(\Phi)$ is an involution then τ induces an involution of W as follows, $\tilde{\tau}(w) := \tau w \tau$. Following the standard abuse of notation we will write τ for $\tilde{\tau}$.

As in Section 2, let θ be an involution such that $\theta(\Delta) = \Delta$, *i.e.*, θ is either the identity or a diagram automorphism. By the following remark, it suffices to prove our theorems for the case that Φ is irreducible, which we assume from here on.

Remark 4.1 All possible Dynkin diagrams consist of a set of connected components, each of which corresponds to an irreducible root system. From [Hel88] it follows that an involution either fixes a connected component of the Dynkin diagram or exchanges two identical copies. In the case where the involution exchanges two connected components, the Weyl group for these two components is $W = W_1 \times W_1$, where W_1 is the Weyl group of the irreducible component. Also, $\Sigma = \Sigma_1 \times \Sigma_1$, where Σ_1 is the set of generators for W_1. Further, the set $\mathcal{I}_\theta = \{(w, w^{-1}) \mid w \in W_1\}$ and the Richardson–Springer order on $\mathcal{I}_\theta$ is the usual Bruhat order on W_1. In particular, the admissible sequence for an element $(w, w^{-1}) \in \mathcal{I}_\theta$ will be a sequence of the form $(r_1, \dots, r_n) \in \Sigma \cup \overline{\Sigma}$, such that $r_i \in \overline{\Sigma}$ for all i. Moreover, each r_i is of the form $r_i = \overline{(s_j, e)}$, for some $s_j \in \Sigma_1$. Thus, the presentations of $(w, w^{-1}) \in \mathcal{I}_\theta$ as admissible sequences are in one-to-one correspondence with the presentations of w as reduced expressions in Σ_1.

https://doi.org/10.4153/CMB-2011-075-1 Published online by Cambridge University Press

Remark 4.2 If Φ is of type $A_1, B_n, C_n, E_7, E_8, F_4$, or G_2, then there are no non-trivial diagram automorphisms. In these cases looking at twisted involutions is the same as looking at regular involutions. If Φ is of type D_n, $n \geq 5$, A_n, $n \geq 2$ or E_6 there is a unique non-trivial diagram automorphism of order 2 which we shall denote by θ. For D_4 there are 3 non-trivial diagram automorphisms of order 2. We shall only consider involutions of Φ coming from involutions of the group G. By the classification theorem of involutions of reductive algebraic groups (see [Hel88]) for n even, no involution of the group G induces a diagram automorphism of order 2 of the Dynkin diagram of type D_n.

Lemma 4.3 *Let Φ be a root system with Weyl Group W. Then, $-\operatorname{Id} \notin W$ if and only if Φ is of type A_n for $n \geq 2$, D_n for n odd, or E_6.*

Proof This result can be found in [Hel91]. One can also check it case by case using the tables in [Bou81]. ∎

Recall $l(w)$ is the length of w with respect to Σ. It is well known (see for example [Bou81]) that the $l(w) = |w(\Phi^+) \cap \Phi^-|$. Recall that w_0 is defined to be the longest element in W, with respect to Σ. This is precisely the unique element such that $w_0(\Phi^+) = \Phi^-$. Notice that if $-\operatorname{Id} \in W$ then $-\operatorname{Id} = w_0$.

Lemma 4.4 *Let $\theta \neq \operatorname{Id}$ be a non-trivial diagram automorphism of order 2 as above, then $\theta w_0 = w_0\theta = -\operatorname{Id}$. Further, $w_0 \in \mathfrak{I}_\theta$.*

Proof If $w_0 = -\operatorname{Id}$ then by Lemma 4.3, Φ is of type $A_1, B_n, C_n, E_7, E_8, F_4$ or G_2 or D_{2n}. By Remark 4.2 none of these cases has a non-trivial diagram automorphism. Hence, $w_0 \neq -\operatorname{Id}$. Since $\theta(\Delta) = \Delta$ we get that $\theta(\Phi^+) = \Phi^+$. Then $\theta w_0 \theta(\Phi^+) = \Phi^-$, hence, $\tilde{\theta}(w_0) := \theta w_0 \theta = w_0$. For the second equality we have that $-\operatorname{Id} w_0(\Phi^+) = \Phi^+$, so since this is not the identity map it must be the unique non-trivial diagram automorphism θ.

Now, $\theta w_0 \theta w_0 = (-\operatorname{Id})^2 = \operatorname{Id}$ and by definition $\theta(w_0) = \theta w_0 \theta$. Hence $w_0^{-1} = \theta(w_0)$, so $w_0 \in \mathfrak{I}_\theta$. ∎

Lemma 4.5 *If $(r_1, \dots, r_k) \in \Sigma \cup \overline{\Sigma}$ is an admissible ascending sequence and $(r_1, \dots, r_k) \cdot e \neq w_0$ then there exists an element $r_{k+1} \in \Sigma \cup \overline{\Sigma}$ such that $(r_1, \dots, r_k, r_{k+1})$ is also an admissible ascending sequence.*

To keep clear when we are computing under the two different involutions θ and Id we use some additional notation. Recall the an element $w \in W$ acts on $a \in \mathfrak{I}_\tau$ by *twisted conjugation* which is defined as $w * a = wa\tau(w)^{-1}$. As before denote by $*$ the twisted action of W on $\mathfrak{I}_\theta$, and $*'$ the twisted action of W on $\mathfrak{I}_{Id}$ (which is just the usual conjugation). For $s \in \Sigma$, $a \in \mathfrak{I}_\theta$ and $b \in \mathfrak{I}_{\mathrm{Id}}$ we will use the notation $\bar{s}a := s * a$, and $\bar{s}'b := s *' b$. Define $\overline{\Sigma} := \{\bar{s} \mid s \in \Sigma\}$ and $\overline{\Sigma}' := \{\bar{s}' \mid s \in \Sigma\}$. If $\mathbf{r} = (r_1, r_2, \dots, r_k)$ be a sequence in $\Sigma \cup \overline{\Sigma}$ then define the action of $\mathbf{r}$ on $a \in \mathfrak{I}_\theta$ by $\mathbf{r} \cdot a := r_k \cdots r_2 r_1 a$. If $\mathbf{r}'$ is a sequence in $\Sigma \cup \overline{\Sigma}'$ then define the action of $\mathbf{r}'$ on $b \in \mathfrak{I}_{\mathrm{Id}}$ similarly.

Lemma 4.6 *Let $\mathbf{r} = (r_1, r_2, \dots, r_k)$ be a sequence in $\Sigma \cup \overline{\Sigma}$, and define $\mathbf{r}' := (r'_1, r'_2, \dots, r'_k)$ to be the sequence in $\Sigma \cup \overline{\Sigma}'$, where $r'_i = s$ if $r_i = s \in \Sigma$ and $r'_i = \bar{s}'$ if $r_i = \bar{s} \in \overline{\Sigma}$. Then $\mathbf{r} \cdot e = (\mathbf{r}' \cdot w_0)w_0$.*

https://doi.org/10.4153/CMB-2011-075-1 Published online by Cambridge University Press

Proof It is useful to write that for all i, $r_i = s_i$ or $r_i = \bar{s}_i$. Let $(r_{i_1}, r_{i_2}, \dots, r_{i_l})$ be the subsequence of $\mathbf{r}$ consisting of precisely the elements in $\overline{\Sigma}$, and so $(r'_{i_1}, r'_{i_2}, \dots, r'_{i_l})$ will be the subsequence of precisely the elements in $\overline{\Sigma}'$. The left hand side above is then:

$$\begin{aligned}\mathbf{r}\cdot e &= r_k\cdots r_2r_1e = s_k\cdots s_2s_1\theta(s_{i_1})\theta(s_{i_2})\cdots\theta(s_{i_l})\\ &= s_k\cdots s_2s_1\theta s_{i_1}s_{i_2}\cdots s_{i_l}\theta.\end{aligned}$$

The right hand side above is

$$\begin{aligned}(\mathbf{r}'\cdot w_0)w_0 &= (r'_k\cdots r'_2r'_1w_0)w_0\\ &= s_k\cdots s_2s_1w_0s_{i_1}s_{i_2}\cdots s_{i_l}w_0\\ &= s_k\cdots s_2s_1(-\mathrm{Id})\theta s_{i_1}s_{i_2}\cdots s_{i_l}(-\mathrm{Id})\theta\\ &= s_k\cdots s_2s_1\theta s_{i_1}s_{i_2}\cdots s_{i_l}\theta.\end{aligned}$$ ■

Proof of Theorem 1.2 If $(r_1, r_2, \dots, r_n)$ is a maximal admissible ascending sequence in $\mathcal{I}_{\mathrm{Id}}$ then $(r_1, r_2, \dots, r_n)\cdot e = w_0$. Hence, by Lemma 4.6,

$$\big((r'_1, r'_2, \dots, r'_n)\cdot w_0\big)\,w_0 = w_0$$

and so $(r'_1, r'_2, \dots, r'_n)\cdot w_0 = e$. Thus, $(r'_1, r'_2, \dots, r'_n)$ is a maximal admissible descending sequence for $\mathcal{I}_\theta$. It is easy to verify that the sequence $(r'_1, r'_2, \dots, r'_n)$ is a maximal admissible descending sequence in $\mathcal{I}_\theta$ if and only if $(r'_n, r'_{n-1}, \dots, r'_1)$ is a maximal admissible ascending sequence in $\mathcal{I}_\theta$. ■

Proof of Theorem 1.3 By Lemma 4.6, for all $j \le n$,

$$(r_1, r_2, \dots, r_j)\cdot e = \big((r'_1, r'_2, \dots, r'_j)\cdot w_0\big)\,w_o.$$

Hence,

$$l\big((r_1, r_2, \dots, r_j)\cdot e\big) = l(w_0) - l\big((r'_1, r'_2, \dots, r'_j)\cdot w_0\big).$$

Consequently,

$$l\big((r_1, r_2, \dots, r_j)\cdot e\big) > l\big((r_1, r_2, \dots, r_{j-1})\cdot e\big)$$

if and only if

$$l\big((r'_1, r'_2, \dots, r'_j)\cdot w_0\big) < l\big((r'_1, r'_2, \dots, r'_{j-1})\cdot w_0\big).$$

This proves (1), and the argument for (2) is similar. Statement (3) follows again from Lemma 4.6. ■

https://doi.org/10.4153/CMB-2011-075-1 Published online by Cambridge University Press

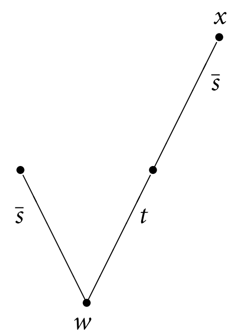

Figure 3: Proof of Theorem 5.2.

5 Properties of the Richardson–Springer Order

We collect some properties about admissible ascending sequences.

Corollary 5.1 *If Φ is of type $A_1, B_n, C_n, E_7, E_8, F_4$, or G_2, then every admissible ascending sequence is also an admissible descending sequence and vice versa. That is, the poset is symmetric.*

Proof By Lemma 4.3, $w_0 = -\operatorname{Id}$ in W if Φ is of type A_1, B_n, C_n, E_7, E_8, F_4, or G_2. By an argument similar to that of Lemma 4.6, the result follows. ∎

We next show that all elements of $\mathfrak{I}_{\mathrm{Id}}$ can be obtained by admissible ascending sequences in which all elements of Σ occur before all elements in $\overline{\Sigma}$.

Theorem 5.2 *Assume Φ is simply-laced. Then for every $w \in \mathfrak{I}_{\mathrm{Id}}$ there is an admissible ascending sequence $\mathbf{r} = (r_1, r_2, \dots, r_k)$ with $w = \mathbf{r} \cdot e$, such that the subsequence of $\mathbf{r}$ consisting of precisely the elements in Σ is $(r_1, r_2, \dots, r_\nu)$ for some ν.*

Proof The proof is by induction on the size of an admissible ascending sequence for w. Note that each admissible sequences of size 1 contains one element from Σ. Suppose $w = t\bar{s}\mathbf{r}\cdot e$, where $(r_1, \dots, r_{k-2}, \bar{s}, t)$ is an admissible ascending sequence such that $\bar{s} \in \overline{\Sigma}$ and $t \in \Sigma$. It will be convenient to write $x := \mathbf{r} \cdot e$. This is schematical shown in Figure 3. We show that $\bar{s}w$ is also in $\mathfrak{I}_{\mathrm{Id}}$ and of smaller length than w. First, if $st = ts$ then $w = \bar{s}t\mathbf{r} \cdot e$ and we're done. So assume $st \neq ts$.

Notice that $\bar{s}w = stsx$, so $l(\bar{s}w) \leq l(w)$. Assume these lengths are equal, to get a contradiction. If they are equal then $sw = st\bar{s}x \in \mathfrak{I}_{\mathrm{Id}}$. By assumption $tsxst = sxs \neq x$ since w came from an ascending sequence, hence $sw = st\bar{s}x = stsxs = s(sxs)t = xst$, and $tsx \in \mathfrak{I}_{\mathrm{Id}}$.

Now it cannot be that both $tsx \in \mathfrak{I}_{\mathrm{Id}}$ and $t\bar{s}x \in \mathfrak{I}_{\mathrm{Id}}$. The former implies $tsxtsx = e$, while the latter implies $tsxstsxs = tsxtstxs = e$. Together these force $x = txs$. By the exchange property, there exists a minimal representation of x that either begins with t or ends in s. If there is a minimal representation of x that begins with t, then $l(tx) < l(x)$, but since $xs = tx$ then $l(xs) < l(x)$ as well. Hence there must be a minimal representation of x ending with s as well. Thus $x = \sigma_1 \cdots \sigma_h s$, $\sigma_i \in \Sigma$ but then $\bar{s}x = s\sigma_1 \cdots \sigma_h ss = s\sigma_1 \cdots \sigma_h$ which has the same length as x, a contradiction. ∎

https://doi.org/10.4153/CMB-2011-075-1 Published online by Cambridge University Press

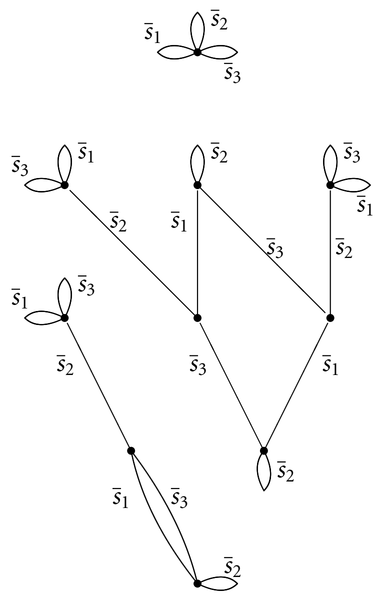

Figure 4: Orbit-Stabilizer Graph of $\mathcal{I}_\theta$ for S_4.

5.3 Orbits and Stabilizers

The Richardson–Springer Poset of $\mathcal{I}_\theta$ provides a method to determine the orbits and stabilizers under θ twisted action. It is useful to consider an edge-labeled graph G defined as follows. Vertices of G are precisely the elements of $\mathcal{I}_\theta$. There is an edge (v, w) labeled $\bar{s}_i$ in G precisely when $\bar{s}_i v = w$. This may result in multiple edges. Further, if $s_i v = w$ then put loops labeled $\bar{s}_i$ at v and w. An example of this orbit stabilizer graph for S_4 is given in Figure 4. Note by Theorem 1.3, the graph and edge labels do not depend on θ. Recall that a *walk* in a graph is a sequence $(v_1, \dots, v_k)$ of vertices in G such that v_i, v_{i+1} is an edge (or loop) in G. The walk is *closed* if further $v_1 = v_k$.

Proposition 5.4 *For θ, G,* etc., *as defined above, two elements of $\mathcal{I}_\theta$ are in the same orbit under twisted action by elements of W if and only if they are in the same connected component of G. An element $s_{i_1} s_{i_2} \cdots s_{i_k}$ in W is in the stabilizer of x under conjugation if and only if there is a closed walk at x whose edge labels are precisely $\bar{s}_{i_1}, \bar{s}_{i_2}, \dots, \bar{s}_{i_k}$.*

References

[Bou81] N. Bourbaki, *Groupes et algébres de Lie.* Éléments de Mathématique, ch. Chapitres 4, 5 et 6, Masson, Paris, 1981.

[CHH+10] P. Cahn, R. Haas, A. G. Helminck, J. Li and J. Schwartz, *A data structure for the exceptional Weyl groups of type F_4 and G_2*. In preparation.

https://doi.org/10.4153/CMB-2011-075-1 Published online by Cambridge University Press

[CHHW10] C. Cooley, R. Haas, A. G. Helminck and N. Williams, *Combinatorial properties of the Richardson-Springer involution poset*. In preparation.

[HH010] R. Haas and A. G. Helminck, *Algorithms for twisted involutions in Weyl groups.* Proceedings of the International Conference on Algebraic Groups and Related Topics, Algebraic Colloquium, Peking University, 2010, to appear.

[Hel88] A. G. Helminck, *Algebraic groups with a commuting pair of involutions and semisimple symmetric spaces*. Adv. in Math. **71**(1988), 21–91. doi:10.1016/0001-8708(88)90066-7

[Hel91] ______, *Tori invariant under an involutorial automorphism I*. Adv. in Math. **85**(1991), 1–38. doi:10.1016/0001-8708(91)90048-C

[Hel97] ______, *Tori invariant under an involutorial automorphism II*. Adv. in Math. **131**(1997), 1–92. doi:10.1006/aima.1997.1633

[Hel00] ______, *Computing orbits of minimal parabolic k-subgroups acting on symmetric k-varieties*. J. Symbolic Comput. **30**(2000), 521–553. doi:10.1006/jsco.2000.0395

[Hel04] ______, *Combinatorics related to orbit closures of symmetric subgroups in flag varieties*. CRM Proc. Lecture Notes **35**(2004), 71–90.

[Hul05] A. Hultman, *Fixed points of involutive automorphisms of the Bruhat order*. Adv. Math. **195**(2005), 283–296. doi:10.1016/j.aim.2004.08.011

[HW93] A. G. Helminck and S. P. Wang, *On rationality properties of involutions of reductive groups*. Adv. in Math. **99**(1993), 26–96. doi:10.1006/aima.1993.1019

[Inc04] F. Incitti, *The Bruhat order on the involutions of the symmetric group*. J. Algebraic Combin. **20**(2004), 243–261. doi:10.1023/B:JACO.0000048514.62391.f4

[Inc06] ______, *Bruhat order on the involutions of classical Weyl groups*. Adv. in Appl. Math. **37**(2006), 68–111. doi:10.1016/j.aam.2005.11.002

[Mat79] T. Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*. J. Math. Soc. Japan **31**(1979), 331–357. doi:10.2969/jmsj/03120331

[Ros79] W. Rossmann, *The structure of semisimple symmetric spaces*. Canad. J. Math. **31**(1979), 157–180.

[RS90] R. W. Richardson and T. A. Springer, *The Bruhat order on symmetric varieties*. Geom. Dedicata **35**(1990), 389–436.

[Spr85] T. A. Springer, *Some results on algebraic groups with involutions*. In: Algebraic groups and related topics (Kyoto/Nagoya, 1983), North-Holland, Amsterdam, 1985, 525–543.

[Ste68] R. Steinberg, *Endomorphisms of linear algebraic groups*. Mem. Amer. Math. Soc. **80**, Amer. Math. Soc., Providence, RI, 1968.

[Vog83] D. A. Vogan, *Irreducible characters of semi-simple Lie groups III. Proof of the Kazhdan–Lusztig conjectures in the integral case*. Invent. Math. **71**(1983), 381–417. doi:10.1007/BF01389104

Department of Mathematics, Smith College, Northampton, MA 01063, USA
e-mail: rhaas@math.smith.edu

Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA
e-mail: loek@unity.ncsu.edu

https://doi.org/10.4153/CMB-2011-075-1 Published online by Cambridge University Press