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# On Frankel's Theorem

Peter Petersen and Frederick Wilhelm

*Abstract.* In this paper we show that two minimal hypersurfaces in a manifold with positive Ricci curvature must intersect. This is then generalized to show that in manifolds with positive Ricci curvature in the integral sense two minimal hypersurfaces must be close to each other. We also show what happens if a manifold with nonnegative Ricci curvature admits two nonintersecting minimal hypersurfaces.

## 1 Introduction

Recall that one of the basic properties of planar elliptic geometries is that two distinct "lines" must intersect. Examples of this behavior can be found among the projective spaces $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$ and $CaP^2$ where "lines" are interpreted as being totally geodesic spheres of dimension 1, 2, 4 and 8 respectively. Note that these submanifolds lie in the middle dimension. Thus Frankel's Theorem (see [6]) gives a far reaching generalization of this phenomenon:

***Theorem 1*** **(Frankel)** *In a complete connected Riemannian n-manifold of positive sectional curvature two closed totally geodesic submanifolds of dimension $n_1$ and $n_2$ must intersect provided $n_1 + n_2 \geq n$.*

The proof is a simple consequence of Synge's second variation formula.

***Theorem 2*** **(Synge)** *Let $c\colon [0,l] \to M$ be a unit speed geodesic and $V(s,t)$ a variation of $c$ with the properties that $V(0,t) = c(t)$ and the variational field $\frac{\partial V}{\partial s}(0,t) = E(t)$ is a unit, normal, parallel field along $c$, then for the arclength functional*

$$L(s) = \int_0^l \left| \frac{\partial V}{\partial t} \right|$$

*we have*

$$\frac{dL}{ds}(0) = 0,$$

$$\frac{d^2L}{ds^2}(0) = -\int_0^l \sec(E,\dot{c})\,dt + \left\langle \dot{c}, \left(\nabla_{\frac{\partial V}{P\partial s}} \frac{\partial V}{\partial s}\right)(0,t) \right\rangle \Big|_0^l.$$

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To see how this formula yields Frankel's theorem simply proceed by contradiction. First choose the variation so that $c$ minimizes the distance between the two submanifolds and then $E$ so that it is tangent to these submanifolds at either end. The curvature condition then ensures us that

$$-\int_0^l \sec(E, \dot{c})\, dt < 0,$$

while the fact that the submanifolds are totally geodesic means that

$$\left\langle \dot{c}, \left(\nabla_{\frac{\partial V}{\partial s}} \frac{\partial V}{\partial s}\right)(0,t) \right\rangle \Big|_0^l = 0.$$

Thus the total contribution is negative and we have arrived at a contradiction.

The purpose of this note is to extend this result to the case of positive Ricci curvature, nonnegative Ricci curvature and positive Ricci curvature in the integral sense.

***Theorem 3*** *In a complete connected Riemannian manifold of positive Ricci curvature any two minimal hypersurfaces must intersect.*

This results appears in [7, Exercise 5.8.d], but the hypersurfaces there are assumed to be totally geodesic.

In nonnegative Ricci curvature the following rigidity phenomenon occurs in case the minimal hypersurfaces do not intersect. The proof is based on the ideas used in the Cheeger-Gromoll splitting theorem (see [2]). The reader should also compare this theorem with [5, Thereom 3] and the result for manifolds with boundary presented in [4].

***Theorem 4*** *Let $M$ be a complete connected Riemannian manifold of nonnegative Ricci curvature and $N_1$, $N_2$ two closed, connected minimal hypersurfaces. If the two hypersurfaces do not intersect, then they are both totally geodesic and one of the following cases will occur*

1) *Both hypersurfaces are 2-sided and divide $M$ into 3 connected components; the region between the two hypersurfaces splits as a product $N_1 \times [a, b]$ with boundary at $N_1 \simeq N_1 \times \{a\}$ and $N_2 \simeq N_1 \times \{b\}$.*
2) *Both hypersurfaces are 2-sided and divide $M$ into 2 connected components; both hypersurfaces are isometric to each other and $M$ is isometric to a mapping torus*

$$\frac{N_1 \times [a, b]}{(x, b) \sim (y, b) \text{ iff } \phi(x) = y},$$

*where $\phi \colon N_1 \to N_1$ is an isometry.*

3) *One hypersurface, say, $N_1$ is 1-sided and the other 2-sided. The region between the two hypersurfaces splits in the following way: There is a Riemannian 2-fold covering map $\pi \colon N_2 \to N_1$ and the region between the hypersurfaces is isometric to the mapping cylinder*

$$\frac{N_2 \times [a, b]}{(x, b) \sim (y, b) \text{ iff } \pi(x) = \pi(y)}.$$

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4) *Both hypersurfaces are* 1*-sided. There are Riemannian* 2*-fold covering maps* $\pi_i$: $N \to N_i$ *from a totally geodesic hypersurface in* $M$ *and* $M$ *is isometric to a double mapping cylinder*

$$\frac{N \times [a_1, a_2]}{(x, a_i) \sim (y, a_i) \text{ iff } \pi_i(x) = \pi_i(y)}.$$

We show how this theorem ties in with the topology of the hypersurfaces in a later section.

The integral curvature case requires some definitions. First consider the function

$$\rho = \max\{(n-1)\kappa - \mathrm{Ric}_-, 0\},$$

where $\mathrm{Ric}_-$ is the function that records the lowest eigenvalue for the Ricci endomorphism $TM \to TM$. Clearly $\rho$ measures the extent to which the Ricci curvature is greater than $(n-1)\kappa$. Next define

$$\bar{k}(p, \kappa, R) = \sup_{x \in M} \frac{1}{\mathrm{vol}\, B(x, R)} \int_{B(x,R)} \rho^p.$$

This quantity measures in the $L^p$ sense how much Ricci curvature lies below $(n-1)\kappa$ on the scale of $R$. In particular, $\bar{k}(p, \kappa, R) = 0$ iff $\mathrm{Ric} \geq (n-1)\kappa$. We are now ready to state a generalization of the above result.

***Theorem 5*** *Suppose* $p > n/2$. *For every* $\varepsilon, \kappa > 0$ *there is an explicit* $\delta(n, p, R, \varepsilon, \kappa) > 0$ *such that any two minimal hypersurfaces in a complete connected Riemannian manifold with* $\bar{k}(p, \kappa, R) \leq \delta$ *must be less than* $\varepsilon$ *apart from each other.*

We also have a discussion on what happens if the intermediate curvature are positive and a discussion on a tricky conjecture with some justifications included.

## 2 Examples

It is easy to see that Theorem 5 is optimal and that all four cases in Theorem 4 occur.

First consider a sphere which is flattened near the equator so that it has a family of equidistant totally geodesic equators. More precisely one considers the unit sphere $S^n$ as a warped product $dr^2 + \sin^2(r)\, ds^2_{n-1}$, where $ds^2_{n-1}$ is the metric of the unit sphere $S^{n-1}$. Now change $\sin(r)$ to a smooth concave function $\phi_\varepsilon(r)$ on $[0, \pi]$ which is symmetric around $\pi/2$, constant on $[\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$ and $\sin(r)$ on $[0, \frac{\pi}{2} - 2\varepsilon] \cup [\frac{\pi}{2} + 2\varepsilon, \pi]$.

For $\varepsilon \to 0$ this example has $\bar{k}(p, 1, R) \to 0$ thus showing that Theorem 5 is optimal.

The Ricci curvature of this metric is nonnegative; so fixing $\varepsilon$ yields an example where two totally geodesic hypersurfaces divide $M$ into 3 components and the region between the hypersurfaces splits. Further, note that the antipodal map on $S^n$ is still an isometry. The quotient will then yield an example of a manifold with a 1-sided and a 2-sided hypersurface with the region in between them splitting as a mapping cylinder.

Given any manifold $N$ with nonnegative Ricci curvature and an isometry $\phi\colon N \to N$ we have a mapping torus

$$\frac{N \times [a_1, a_2]}{(x, a_i) \sim (y, a_i) \ \text{iff} \ \phi(x) = y}$$

Examples of the case where there are two 1-sided totally geodesic hypersurfaces are equally easy to come by. Simply pick a manifold $N$ and two Riemannian 2-fold covering maps $\pi_i\colon N \to N_i$, then the double mapping cylinder

$$\frac{N \times [a_1, a_2]}{(x, a_i) \sim (y, a_i) \ \text{iff} \ \pi_i(x) = \pi_i(y)}$$

yields a Riemannian manifold with the desired properties.

Note that the above examples can be found on $S^2$, $\mathbb{R}P^2$, $T^2$, and the Klein bottle. In the last two cases with flat metrics. Note, *e.g.*, that all flat metrics on $T^2$ are mapping tori of the form

$$\frac{S^1 \times [0, b]}{(x, 0) \sim (y, b) \ \text{iff} \ \phi(x) = y},$$

where $\phi$ is a rotation.

## 3 Proofs

We start by giving a proof of the positive Ricci curvature case using the variational method.

**Proof of Theorem 3** Suppose that $N_1, N_2 \subset M$ are minimal hypersurfaces and that $p_i \in N_i$ are points in these hypersurfaces closest to each other. If $p_1 \neq p_2$ choose, as in Frankel's theorem, a unit speed geodesic $c\colon [0, l] \to M$ from $p_1$ to $p_2$. Next select an orthonormal frame of parallel fields $E_1, \dots, E_n$ along $c$ with $E_n = \dot{c}$. At the end points $E_1, \dots, E_{n-1}$ are therefore tangent to the hypersurfaces. Now pick variations $V_1, \dots, V_{n-1}$ with the property that $V_j(s, 0) \in N_1$, $V_j(s, l) \in N_2$ for small $s$ and $\frac{\partial V_j}{\partial s}(0, t) = E_j$. Adding up the contributions for the $n - 1$ resulting variations of arclength yields

$$\sum_{j=1}^{n-1} \frac{d^2 L_j(0)}{ds^2} = \sum_{j=1}^{n-1} - \int_0^l \sec(E_j, \dot{c})\, dt + \sum_{j=1}^{n-1} \Big\langle \dot{c}, \Big( \nabla_{\frac{\partial V_j}{\partial s}} \frac{\partial V_j}{\partial s} \Big) (0, t) \Big\rangle \Big|_0^l.$$

Now observe that

$$\sum_{j=1}^{n-1} \Big\langle \dot{c}, \Big( \nabla_{\frac{\partial V_j}{\partial s}} \frac{\partial V_j}{\partial s} \Big) (0, 0) \Big\rangle,$$

$$\sum_{j=1}^{n-1} \Big\langle \dot{c}, \Big( \nabla_{\frac{\partial V_j}{\partial s}} \frac{\partial V_j}{\partial s} \Big) (0, l) \Big\rangle$$

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are the mean curvatures of $N_1$ at $p_1$ and $N_2$ at $p_2$ respectively. Thus these contributions are zero and we obtain the desired contradiction as follows:

$$\sum_{j=1}^{n-1} \frac{d^2 L_j(0)}{ds^2} = \sum_{j=1}^{n-1} -\int_0^l \sec(E_j, \dot{c})\, dt$$
$$= -\int_0^l \operatorname{Ric}(E_j, \dot{c})\, dt$$
$$< 0.$$

A different proof of this comes to mind if one uses Calabi's idea of finding generalized upper bounds for the Laplacian of the distance functions (see [1], [9]). This is the proof that will be used to establish the two more general results.

**Alternate Proof of Theorem 3** If $d_i$ is the distance to $N_i$ and the Ricci curvature of $M$ is positive then $\Delta d_i < 0$ on $M - N_i$ in the barrier sense since $N_i$ is minimal. In particular, $d_1 + d_2$ has negative Laplacian in the barrier sense. If the two hypersurfaces don't intersect the function $d_1 + d_2$ has a global minimum on $M - (N_1 \cup N_2)$. This however contradicts the Laplacian estimate as $\Delta(d_1 + d_2) \geq 0$ at all minima.

**Proof of Theorem 4** In case $M$ only has nonnegative Ricci curvature we have that $\Delta d_i \leq 0$ on $M - N_i$. So if $d_1 + d_2$ as an interior minimum on a component $O$ of $M - (N_1 \cup N_2)$, then $d_1 + d_2$ is constant on that component. But then it follows that $\Delta d_i = 0$ on $O$ and hence both $d_i$ are smooth. Since they both satisfy the Riccati equation

$$\nabla_{\nabla d_i} \Delta d_i + |\operatorname{Hess} d_i|^2 = -\operatorname{Ric}(\nabla d_i, \nabla d_i),$$

we have the further property that Hess $d_i = 0$ on $O$. This immediately implies that $O$ is isometric to $N \times (0, d)$, where $d = d_1 + d_2$ and $N$ is any of the isometric level sets of $d_i$. This level set is totally geodesic as Hess $d_i = 0$.

We now have to check how $O$ behaves near the boundary where it meets the two hypersurfaces $N_1$ and $N_2$. Clearly $N$ is diffeomorphic to a component of the unit normal bundle to either of the hypersurfaces. Therefore, if $N_i$ is 2-sided it follows that $N$ is diffeomorphic to $N_i$ while if $N_i$ is 1-sided there is a 2-fold covering map $N \to N_i$. Given that $O$ splits we have the added rigidity that in the first case $N$ and $N_i$ are isometric, while in the second case the 2-fold covering map is Riemannian, *i.e.*, a local isometry.

It now remains to check how $M$ can be reconstructed from this information. If both hypersurfaces are 1-sided then $M - (N_1 \cup N_2)$ is connected so we are finished in that case. If, say, $N_1$ is 1-sided and $N_2$ is 2-sided, then $O \neq M - (N_1 \cup N_2)$. Otherwise the splitting of $O$ would imply that the unit normal bundles of the $N_i$s are diffeomorphic. Thus $O$ is the region which borders both hypersurfaces. The other component of $M - (N_1 \cup N_2)$ borders only $N_2$ and we can't say anything more about this region. Finally we have the situation where both hypersurfaces are 2-sided. In this case $M - (N_1 \cup N_2)$ always has a component $O$ isometric to $N \times (0, d)$ with the property that its closure is isometric to $N \times [0, d]$. This will identify one component of each of the unit normal bundles with $N$. This means that if $M - (N_1 \cup N_2)$ is connected then $M$ has boundary, a contradiction. Thus $M - (N_1 \cup N_2)$ has either 3 or 2 components. In the former case each of the two other components border only one of the hypersurfaces, thus there isn't anything else we can say. In the latter situation, there must be a minimizing geodesic between the two hypersurfaces in the remaining component $O'$, thus $d_1 + d_2$ will also have an interior minimum on $O'$. The above analysis then shows that $O'$ is also isometric to $N_1 \times (0, d')$. Thus $M - N_1$ is isometric to $N_1 \times (0, d + d')$. To reconstruct $M$ we have to glue $N_1$ back in. Following geodesics normal to $N_1$ around from one side to the other gives the desired $\phi \colon N_1 \to N_1$ which exhibits $M$ as a mapping torus. Equivalently there is a Riemannian submersion $M \to S^1$ onto a circle of length $d + d'$ such that the fibers are totally geodesic. Note that any closed connected minimal hypersurface $H$ which is 2-sided, but doesn't divide $M$ into two components yields such a Riemannian submersion. For such a hypersurface there are two naturally defined distance functions $d_\pm$. One measures the distance to one side of $H$ and the other the distance to the opposite side. Clearly $\Delta d_\pm \leq 0$ as before. So the proof is completed in the same way.

We now proceed to the situation where $\bar{k}(p, \kappa, R)$ is small. The required Laplacian estimates for the distance functions are obtained in [10, Section 2] and the method of proofs can also be gleaned from [10, Section 3].

**Proof of Theorem 5** First suppose for simplicity that $\kappa = 1$. We need to use the diameter estimate in [10] to see that the two hypersurfaces are no more than $\pi + O(\delta)$ apart from each other. Then we must choose a new $\kappa \ll 1$ as the comparison curvature so that one can obtain Laplacian estimates as in [10] on all of $M$. Given this, one has that $d_i$ satisfies:

$$\Delta d_i \leq -(n-1)\frac{\sin(\sqrt{\kappa} d_i)}{\sqrt{\kappa}\cos(\sqrt{\kappa} d_i)} + \psi_i$$

on $M - N_i$, where

$$\psi_i = \max\left\{ \Delta d_i + (n-1)\frac{\sin(\sqrt{\kappa} d_i)}{\sqrt{\kappa}\cos(\sqrt{\kappa} d_i)}, 0 \right\}$$

satisfies

$$\frac{1}{\operatorname{vol} B(x,r)} \int_{B(x,r)} \psi_i^{2p} \leq C_1(n,p,\kappa,R,r) \cdot \delta.$$

Now assume that the two hypersurfaces are $r_0$ apart from each other and choose $r < \min\{r_0/3, R\}$. Then consider the function $f = d_1 + d_2$ and let $x_0$ be a minimum point for $f$ which is distance $r_0/2$ from both hypersurfaces. On the ball $B(x_0, r)$ the function $f$ then satisfies

$$\Delta f \leq -C_2(n,\kappa) \cdot r_0 + \psi,$$

$$\frac{1}{\operatorname{vol} B(x,r)} \int_{B(x,r)} \psi^{2p} \leq C_1(n,p,\kappa,R,r) \cdot \delta.$$

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Now pick $u$ so that

$$\Delta u = C_2(n,\kappa)\cdot r_0,$$
$$u = 0 \text{ on } \partial B(x_0,r).$$

Since $\Delta u > 0$ it must follow that $u < 0$ on $B(x_0,r)$. In fact as explained in [10, Section 3] $u$ can be compared with the rotationally symmetric solution to the same problem in constant curvature $\kappa$. This yields the following estimate

$$\begin{aligned} u(x_0) &\le -c_1(n,k,r,r_0) + C_3(n,p,\kappa,R,r)(\delta)^{\frac{1}{2p}} \\ &\le -c(n,k,r,r_0), \end{aligned}$$

provided $\delta$ is sufficiently small compared with $c_1(n,k,r,r_0)$. Then we have that

$$\Delta(f+u) \le \psi,$$
$$f(x)+u(x) = f(x) \ge r_0 \text{ on } \partial B(x_0,r),$$
$$f(x_0)+u(x_0) \le r_0 - c(n,k,r,r_0).$$

As in [10, Section 3] the generalized maximum principle implies an estimate in the opposite direction

$$\begin{aligned} \inf_{B(x_0,r)}(f+u) &\ge \inf_{\partial B(x_0,r)}(f+u) - C_4(n,p,\kappa)\left(\frac{1}{\operatorname{vol} B(x,r)}\int_{B(x,r)}\psi^{2p}\right)^{\frac{1}{2p}} \\ &\ge r_0 - C_5(\delta)^{\frac{1}{2p}}. \end{aligned}$$

Thus we have

$$r_0 - C_5(\delta)^{\frac{1}{2p}} \le r_0 - c(n,k,r,r_0),$$

which forces $r_0 \to 0$ as $\delta \to 0$.

## 4 Topology of Hypersurfaces

In this section we expound a little on Theorem 4 and see how the hypotheses are equivalent to topological conditions on how the hypersurfaces sit in $M$.

First note that each closed hypersurface $H \subset M$ has a $\mathbb{Z}_2$ fundamental class, *i.e.*, $H_{n-1}(H,\mathbb{Z}_2) = \mathbb{Z}_2$. By abuse of notation we'll refer to the image of this fundamental class in $H_{n-1}(M,\mathbb{Z}_2)$ as $[H]$. The long exact sequence

$$0 \to H_n(M,\mathbb{Z}_2) \to H_n(M,H,\mathbb{Z}_2) \to H_{n-1}(H,\mathbb{Z}_2) \to H_{n-1}(M,\mathbb{Z}_2) \to \cdots$$

tells us that $H_n(M,H,\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ iff $[H] = 0$ in $H_{n-1}(M,\mathbb{Z}_2)$. Since $H_n(M,H,\mathbb{Z}_2)$ is isomorphic to $\bar{H}^0(M-H,\mathbb{Z}_2)$ via Poincare duality we see that $H$ divides $M$ into two components iff $[H] = 0$ in $H_{n-1}(M,\mathbb{Z}_2)$.

The four cases in Theorem 4 can now be characterized as follows:

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1) happens iff both $[H_1] = [H_2] = 0$ in $H_{n-1}(M, \mathbb{Z}_2)$,
2) happens iff $[H_1] = [H_2] \neq 0$ in $H_{n-1}(M, \mathbb{Z}_2)$,
3) happens iff $[H_1] \neq 0$, $[H_2] = 0$ or $[H_2] \neq 0$, $[H_1] = 0$ in $H_{n-1}(M, \mathbb{Z}_2)$,
4) happens iff $[H_1] \neq [H_2]$ and both are nonzero in $H_{n-1}(M, \mathbb{Z}_2)$.

Only one thing needs to be justified here and that is the fact that $[H_1] \neq [H_2]$ in case 4). To see this use the structure of $M$ to decompose it as follows $M = U_1 \cup U_2$ where $H_i \subset U_i$ and $U_1 \cap U_2 = H$ is a hypersurface with 2-fold covering maps onto $H_i$. The Meyer-Vietories sequence for $U_1$ and $U_2$ is

$$0 \to H_n(M, \mathbb{Z}_2) \to H_{n-1}(H, \mathbb{Z}_2) \to H_{n-1}(U_1, \mathbb{Z}_2) \oplus H_{n-1}(U_2, \mathbb{Z}_2) \\ \to H_{n-1}(M, \mathbb{Z}_2) \to \cdots .$$

Since both $H_n(M, \mathbb{Z}_2)$ and $H_{n-1}(H, \mathbb{Z}_2)$ are equal to $\mathbb{Z}_2$ this sequence reduces to

$$0 \to H_{n-1}(U_1, \mathbb{Z}_2) \oplus H_{n-1}(U_2, \mathbb{Z}_2) \to H_{n-1}(M, \mathbb{Z}_2) \to \cdots .$$

Since $U_i$ deformation retracts to $H_i$ it follows that we have an injection

$$0 \to H_{n-1}(H_1, \mathbb{Z}_2) \oplus H_{n-1}(H_2, \mathbb{Z}_2) \to H_{n-1}(M, \mathbb{Z}_2)$$

and hence that $[H_1] \neq [H_2]$ in $H_{n-1}(M, \mathbb{Z}_2)$.

Note that in case $M$ is orientable we know that $H_{n-1}(M, \mathbb{Z}) = H^1(M, \mathbb{Z})$ and further that $H^1(M, \mathbb{R}) = H^1(M, \mathbb{Z}) \otimes \mathbb{R}$. Thus the rank of $H_{n-1}(M, \mathbb{Z})$ is simply the first Betti number $b_1$. It follows from the Bochner technique that there is a Riemannian submersion

$$M \to T^{b_1}$$

onto a flat torus of dimension $b_1$ with the property that the fibers are totally geodesic. This result is clearly related to case 2) of Theorem 4, as any nontrivial class in $H_{n-1}(M, \mathbb{Z})$ that is represented by a connected minimal hypersurface will yield a Riemannian submersion $M \to S^1$.

The next thing to note is that in cases 2) and 4) of Theorem 4 there are some interesting short exact sequences for the fundamental group of $M$. In case 2) we are simply talking about the short exact sequence for the fibration $M \to S^1$ with fiber $H$, *i.e.*,

$$1 \to \pi_1(H) \to \pi_1(M) \to \mathbb{Z} \to 1.$$

In case 4) we use Van Kampen's theorem together with the fact that $U_i$ deformation retracts to $H_i$ to get a diagram

$$\begin{array}{ccc} \pi_1(H) & \longrightarrow & \pi_1(H_1) \\ \downarrow & & \downarrow \\ \pi_1(H_2) & \longrightarrow & \pi_1(M) \end{array}$$

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Here the two maps $\pi_1(H) \to \pi_1(H_i)$ are induced from a 2-fold covering and are therefore injections where the image has index 2. This means that we get a short exact sequence

$$1 \to \pi_1(H) \to \pi_1(M) \to \mathbb{Z}_2 * \mathbb{Z}_2 \to 1,$$

where $\mathbb{Z}_2 * \mathbb{Z}_2$ is the free product of two groups of order 2. If $a_i$, $i = 1, 2$ denote the generators for $\mathbb{Z}_2 = \pi_1(H_i)/\pi_1(H)$. Then $a_1a_2$ is an element of infinite order with the property that it generates a cyclic subgroup of index 2. The preimage $G < \pi_1(M)$ also has index 2. Note that $\pi_1(H) < G$ and $\pi_1(H_i) \cap G = \pi_1(H)$. Now pass to the 2-fold cover $\bar{M}$ of $M$ which has $G$ as fundamental group. The preimage of $H$ in $\bar{M}$ consists of two copies of $H$ and $\bar{M}$ splits along $H$ according to case 2).

## 5 Intermediate Curvatures

Frankel's theorem and our extension to manifolds with positive Ricci curvature also allow for a family of intermediate results. The proofs are fairly obvious given the concepts involved and the proofs of the two extreme cases.

The $k$-th intermediate curvatures are said to be positive provided that for any choice of $k+1$ orthonormal vectors $(e_0, e_1, \ldots, e_k)$ the sum $\sum_{i=1}^k \sec(e_0, e_i)$ is positive (see also [12], [11]). When $k = 1$ this is simply saying that the sectional curvature is positive, while when $k = n - 1$ this means that the Ricci curvature is positive. Note that if $k < n - 1$ and the $k$-th intermediate curvatures are constant then in fact the sectional curvatures are constant.

There is also a concept of intermediate mean curvatures. We say that the submanifold $N \subset M$ is $k$-minimal if for all choices of orthonormal frames $\{E_0, E_1, \ldots, E_k\}$ with $E_0$ perpendicular to $N$ and $\{E_1, \ldots, E_k\}$ tangent to $N$ we have that $\sum_{j=1}^k \langle e_0, \nabla_{e_j} e_j \rangle = 0$. Note that if $k = 1$ this condition is the same as saying that $N$ is totally geodesic, while if $k = \dim N$ it says that $N$ is minimal. As with constant intermediate curvature we see that unless $k = \dim N$ the condition of $k$-minimality simply implies that the submanifold is totally geodesic.

The desired intermediate theorem is

***Theorem 6*** *In a complete connected Riemannian n-manifold with positive k-th intermediate curvatures two closed k-minimal submanifolds of dimension $n_1$ and $n_2$ must intersect provided $n_1 + n_2 \geq n + k - 1$*

## 6 Further Discussion

It is worthwhile mentioning some further problems related to the results obtained here.

One issue that comes to mind is what one can say about a manifold $M$ with nonnegative sectional curvature that contains two non intersecting totally geodesic submanifolds whose dimensions add up to or exceed the dimension of $M$.

The first thing that should be mentioned in this context is what happens if a nonnegatively curved manifold $M$ contains a totally geodesic hypersurface $N$. Note that this is a slightly more general situation than the one studied in [12, Theorem 4]. The

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distance function to $N$ is concave just as in the proof of the Cheeger-Gromoll Soul Theorem (see [3]). One should therefore be able to say a good deal about the topology of $M$. As in the soul theorem $M - N$ will have a soul. In fact there might be two souls in case $M - N$ is disconnected. Thus $M - N$ is diffeomorphic to the normal bundle over this soul. The topology at $N$ depends on the sidedness of $N$ and looks topologically like the situations discussed before. Namely, either there is a 2-1 identification onto $N$, or $N$ is 2-sided in which case $M - N$ is either connected or disconnected. In case $N$ is 2-sided and $M - N$ is connected it follows that $M$ is a mapping torus. In case $N$ is 2-sided and divides $M$ into two components we get two souls. Each component of $M - N$ has totally geodesic boundary $N$ so it is not hard to see that Perel'man's rigidity theorem (see [8]) gives us a $C^1$ Riemannian submersion onto the soul. As we have seen, the presence of two minimal hypersurfaces yields a totally geodesic hypersurface and consequently a detailed picture of both the topology and geometry on the manifold $M$.

Returning to the more general situation, where the dimensions of the two totally geodesic submanifolds add up to more than the dimension of $M$, it is tempting to conjecture that $M$ still has a very rigid geometry and topology. It is possible that $M$ contains two totally geodesic submanifolds (not necessarily the given ones) with the property that the complement of either submerses onto the other.

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*Department of Mathematics*
*University of California-Los Angeles*
*Los Angeles, California 90095-1555*
*U.S.A.*
*e-mail: petersen@math.ucla.edu*

*Department of Mathematics*
*University of California-Riverside*
*Riverside, California 92521-0135*
*U.S.A.*
*e-mail: fred@math.ucr.edu*