REGULARIZATION OF THE EQUATIONS OF MOTION IN A CENTRAL FORCE-FIELD. APPLICATION TO THE ZONAL EARTH SATELLITE.

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Abstract. Within the framework of linear and regular celestial mechanics, we revise a recent method of Belen'kii (1981). We generalize some of his results, giving a new regularizing function.

We make an application to the zonal earth satellite, considering the hamiltonian function through the harmonic J_4. After the angular variable u has been removed, we introduce a new time and we reduce the problem to a linear equation.

1. INTRODUCTION

In this paper, the method of regularization given by Belen'kii (1981) is revised. We propose a function g(r) that generalizes the one studied by him. Then an application to the zonal earth satellite, considering harmonics through J_4, is made.

We use the canonical set of variables (P_r, P_u, P_h, r, u, h) of Hill (1913) and, in order to apply that regularization, the angular variable u is eliminated (Caballero, 1975) using von Zeipel's method. As a consequence the new hamiltonian is

$$\bar{H}(\bar{P}_r, \bar{P}_u, \bar{P}_h, \bar{r}, -, -) = \frac{1}{2}(\bar{P}_r^2 + \frac{\bar{P}_u^2}{\bar{r}^2}) - (\frac{a_1}{\bar{r}} + \frac{a_3}{\bar{r}^3} + \frac{a_4}{\bar{r}^4} + \frac{a_5}{\bar{r}^5})$$

where $\bar{P}_u$, a_i are constant.

We make a transformation of time $d\tau = g_5^{-1}(\bar{r})\, dt$ that reduces the problem to a linear equation.

Other analytical theories have been proposed based on canonical elements associated with a suitable time regularization (Kustaanheimo-Stiefel, 1965; Scheifele-Graf, 1974; Deprit, 1981). In particular, regularizations linearizing the equations, that have also applications in other dynamics problems, have been considered by Stiefel-Scheifele (1971), Belen´kii (1981), Szebehely (1976).

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2. BELEN'KII REGULARIZATION

In certain problems of Celestial Mechanics, the hamiltonian of the relative motion of a particle in a central force-field has the form

$$H = \frac{1}{2}\left(P_r^2 + \frac{1}{r^2} P_\phi^2\right) + V_0(r) \tag{1}$$

where $P_r = \dot{r}$ denotes the radial velocity, $P_\phi = r^2\dot{\phi} = c$ is the angular momentum and

$$V_0 = -\sum_1^n \frac{a_i}{r^i} \tag{2}$$

is the potential function.

The energy integral $H = h$, may be written as

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = h - \left\{ V_0(r) + \frac{c^2}{2r^2}\right\} = h - V(r) \tag{3}$$

and Belen'kii introduces a new independent variable τ, by means of the relation

$$d\tau = g^{-1}(r)\ dt \tag{4}$$

with $g(r) > 0$ and $g(r) \in C^{(1)}$. Then, (3) can be written in the form

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = g^2(r)\{h - V(r)\} \tag{5}$$

Differenciating (5) with respect to τ, and after dividing by the nonzero factor $dr/d\tau$, Belen'kii equals the result to a linear expression, obtaining

$$\frac{d^2r}{d\tau^2} = \frac{d}{dr}\left(g^2(r)\{h - V(r)\}\right) = 2c_1 r + c_2 \tag{6}$$

where we have written $2c_1$ for subsequent simplifications.

A full study of the linear equation (6) for the three cases $c_2/c_1 \gtrless 0$, has been given by Belen'kii (1981.a; Section 2).

Integrating (6), the regularizing function must satisfy the relation

$$g^2(r)\{h - V(r)\} = c_1 r^2 + c_2 r + c_3 \tag{7}$$

Belen'kii has applied (7) to the potentials

$$V_0 = V_2 = -\frac{a_1}{r} - \frac{a_2}{r^2} \quad ; \quad V_0 = V_3 = -\frac{a_1}{r} - \frac{a_2}{r^2} - \frac{a_3}{r^3}$$

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where the corresponding regularizing functions are

$$g_2(r) = r \quad ; \qquad g_3(r) = r^{3/2} (1 + \beta r)^{-1/2}$$

respectively. The parameter β depends on a_1, a_2, a_3 and h.

Likewise, Ferrer and Elipe (1982), studying these potentials have considered the following regularizing function

$$g_3(r) = r^{3/2} (\beta + r)^{-1/2}$$

which allows the treatment of these cases in a more uniform manner.

3. A NEW REGULARIZING FUNCTION

In a more general problem, with the potential

$$- V_0 = - V_n = \frac{a_1}{r} + \frac{a_3}{r^3} + \dots + \frac{a_n}{r^n}$$

we have

$$V = V_0 + \frac{c^2}{2r^2} = - \sum_1^n \frac{a_i}{r^i} \tag{8}$$

where $a_2 = - c^2/2$. In this case we propose the following regularizing function

$$g_n(r) = r^{n/2} \{ r^{n-2} + \alpha_1 r^{n-3} + \alpha_2 r^{n-4} + \dots + \alpha_{n-2} \}^{-1/2} \tag{9}$$

where α_1, α_2, ... α_{n-2}, are parameters which depend on a_i, h, and must be suitably chosen to have $g_n(r) > 0$.

Inserting (8), (9) in (7), we arrive at the equation

$$hr^n + \sum_1^n a_i \, r^{n-i} = \{ r^{n-2} + \sum_1^{n-2} \alpha_k r^{n-k-2} \} \{ c_1 r^2 + c_2 r + c_3 \}$$

Equating the coefficients of the same powers of r in both sides, we have the system

$$h = c_1 \quad ; \quad a_1 = c_1\alpha_1 + c_2 \quad ; \quad a_2 = c_1\alpha_2 + c_2\alpha_1 + c_3$$

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$$a_i = c_1\alpha_i + c_2\alpha_{i-1} + c_3\alpha_{i-2} \qquad (i = 3, \dots , n-2) \tag{10}$$

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$$a_{n-1} = c_2\alpha_{n-2} + c_3\alpha_{n-3} \quad ; \quad a_n = c_3\alpha_{n-2}$$

Solving (10) with respect to the coefficients c_1, c_2, c_3, α_1, α_2, ... α_{n-2}, we get the expression of the coefficients in terms of a_1, a_2, ... a_n, h.

The study of this last system is difficult and it seems that the more practical way of solving it is by a numerical method.

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In particular, we have studied the system (10) for n = 5, taking

$$g_5(r) = r^{5/2} (r^3 + \alpha r^2 + \beta r + \gamma)^{-1/2} \qquad (11)$$

In this case, the equations of that system are given by

$$\begin{aligned} h &= c_1 && (12_1) & a_3 &= c_1\gamma + c_2\beta + c_3\alpha && (12_4) \\ a_1 &= c_1\alpha + c_2 && (12_2) & a_4 &= c_2\gamma + c_3\beta && (12_5) \\ a_2 &= c_1\beta + c_2\alpha + c_3 && (12_3) & a_5 &= c_3\gamma && (12_6) \end{aligned} \qquad (12)$$

From (12_1), (12_2), (12_3) we obtain

$$\begin{aligned} c_1 &= h \quad ; \quad c_2 = a_1 - h\alpha = c_2(\alpha) \\ c_3 &= a_2 - a_1\alpha + h\alpha^2 - h\beta = c_3(\alpha, \beta) \end{aligned} \qquad (13)$$

From (12_6), if $a_5 \neq 0$, we get: $c_3 \neq 0$, $\gamma \neq 0$. Then $\gamma = a_5/c_3(\alpha, \beta)$.

Finally, substituting the above expressions in (12_4), (12_5), we have the system

$$\begin{aligned} C\beta^2 + B\beta + A &= 0 \\ D'\beta^3 + C'\beta^2 + B'\beta + A' &= 0 \end{aligned} \qquad (14)$$

where

$$C = h^2\alpha^5 - 2ha_1\alpha^4 + (a_1^2 + 2ha_2)\alpha^3 - (2a_1a_2 + a_3h)\alpha^2 + (a_2^2 + a_1a_3)\alpha + ha_5 - a_2a_3$$

$$B = - h(2h + 1)\alpha^3 + 4ha_1\alpha^2 - (3ha_2 + a_1^2)\alpha + a_1a_1 + ha_3$$

$$A = h(2h - a_1)$$

$$D' = h^2$$

$$C' = - 2h(h\alpha^2 - a_1\alpha + a_2)$$

$$B' = h^2\alpha^4 - 2ha_1\alpha^3 + (a_1^2 + 2ha_2)\alpha^2 - 2a_1a_2\alpha + a_2^2 + ha_4$$

$$A' = - a_4h\alpha^2 + (a_1a_4 - ha_5)\alpha + a_1a_5 - a_2a_4$$

Eliminating β in (14) we get an equation of the form $P(\alpha) = 0$ where $P(\alpha)$ is a polynomial in α of eighteenth degree. Thus it seems convenient to solve (14) by numerical methods.

4. SOME PARTICULAR CASES FOR THE NEW REGULARIZING FUNCTION

i) $a_5 = 0$

In this case, from (12_6) it follows that we can take $\gamma = 0$. Then, the last three equations of (12) reduce to

$$a_3 = c_2\beta + c_3\alpha \qquad a_4 = c_3\beta \tag{15}$$

Then, substituting c_3, given by (13), in (15_1), we get

$$\beta = \frac{a_3 - a_2\alpha + a_1\alpha^2 - h\alpha^3}{a_1 - 2h\alpha} \tag{16}$$

and substituting β in (15_2), we have a sixth degree equation

$$\sum_{0}^{6} A_n\alpha^n = 0$$

where

$$A_6 = 2h^3$$

$$A_5 = -5a_1h^2$$

$$A_4 = 4(a_1^2 + ha_2)h$$

$$A_3 = -\{a_1^3 + 6a_1a_2h + (2a_3 + 1)h^2\}$$

$$A_2 = 2a_1^2a_2 + (3a_1a_3 - 2a_2^2 - a_1)h - 4a_4h^2$$

$$A_1 = -\{(a_1a_3 - a_2^2)a_1 + (4a_1a_4 + 2a_2a_3 + a_2)h\}$$

$$A_0 = a_1a_2a_3 - a_1^2a_4 - a_3h$$

Then, the regularizing function is

$$g_4(r) = r^2(r^2 + \alpha r + \beta)^{-1/2}$$

and we must take the values α,β in such a way that $r^2 + \alpha r + \beta > 0$ or else, we must find the range of r for which the regularization is well defined.

ii) $a_4 = a_5 = 0$

Again, it is sufficient to take $\gamma = \beta = 0$ in (12). Then, the parameter α verifies the cubic equation

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$$h\alpha^3 - a_1\alpha 2 + a_2\alpha - a_3 = 0 \tag{17}$$

The regularizing function is now

$$g_3(r) = r^{3/2}(r + \alpha)^{-1/2} \tag{18}$$

A study and application of (17) and (18) has been made by Ferrer-Elipe (1982).

iii) $a_3 = a_4 = a_5 = 0$

In this case it is sufficient to take $\gamma = \beta = \alpha = 0$. Then, the system (12) reduces to

$$c_1 = h \quad ; \quad c_2 = a_1 \quad ; \quad c_3 = a_2$$

where

$$g_2(r) = r$$

is the regularizing function of Sundman.

5. AN APPLICATION TO THE ZONAL EARTH SATELLITE

I.- It is well known that the kinetic energy T and the potential V of an artificial zonal satellite of the Earth, in the canonical set of variables (P_r, P_u, P_h, r, u, h) of Hill (1913), are given by the equations

$$T = \frac{1}{2}\left(P_r^2 + \frac{P_u^2}{r^2}\right)$$

$$V = -\frac{\mu}{r}\left\{1 - \sum_{n>2} J_n \left(\frac{1}{r}\right)^n P_n(\sin\phi)\right\}$$

The corresponding hamiltonian with the harmonics J_2, J_3, J_4 is given by the expression

$$H = H(P_r, P_u, P_h, r, u, -) = H_0 + H_1 + H_2$$

where

$$H_0 = \frac{1}{2}\left(P_r^2 + \frac{P_u^2}{r^2}\right) - \frac{\mu}{r}$$

$$H_1 = -\frac{\mu}{r^3} J_2 (B_{20} + B_{22}\cos 2u)$$

$$H_2 = \frac{\mu}{8r^4} J_3\sqrt{1 - \theta^2}\{3(1 - 5\theta^2)\sin u - 5(1 - \theta^2)\sin 3u\} +$$

$$\frac{3\mu}{8r^5} J_4 \left\{ \left(\frac{3}{8} - \frac{15}{4}\theta^2 + \frac{35}{8}\theta^4\right) + \left(-\frac{5}{6} + \frac{20}{3}\theta^2 - \frac{35}{6}\theta^4\right)\cos 2u + \right.$$

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$$\frac{35}{24}\,(1 - \theta^2)^2 \cos 4u \,\}$$

and where we use the notation

$$B_{20} = -\frac{1}{4}\,(1 - 3\theta^2) \quad ; \quad B_{22} = \frac{3}{4}\,(1 - \theta^2) \quad ; \quad \theta = \frac{P_h}{P_u} = \cos I$$

The equatorial radius of the Earth, has been taken as unity.

The elimination of the variable u has been done by Caballero (1975) using the method of von Zeipel. The new hamiltonian

$$\bar{H} = \bar{H}\,(\,\bar{P}_r,\ \bar{P}_u,\ \bar{P}_h,\ \bar{r}\ ,\ -,\ -\,)$$

takes the form

$$\bar{H} = \frac{1}{2}\,(\,\bar{P}_r^2 + \frac{\bar{P}_u^2}{\bar{r}^2}\,) - (\,\frac{a_1}{\bar{r}} + \frac{a_3}{\bar{r}^3} + \frac{a_4}{\bar{r}^4} + \frac{a_5}{\bar{r}^5}\,)$$

or

$$\bar{H} = \frac{1}{2}\,(\frac{d\bar{r}}{dt})^2 - \sum_1^5 \frac{a_i}{\bar{r}^i} \qquad (19)$$

where

$$a_1 = \mu \ ; \quad a_2 = \frac{\bar{P}_u^2}{2} \ ; \quad a_3 = J_2\mu B_{20} + \frac{J_2^2\mu^3\, B_{22}}{16\bar{P}_u^2}\,(3 - 7\theta^2\,)$$

$$a_4 = \frac{J_2^2\,{}^2B_{22}}{48\bar{P}_u^2}\,(-\,21 + 69\,\theta^2) \quad ; \quad a_5 = -\,\frac{9J_4\mu}{64}\,(1 - 10\theta^2 + \frac{35}{3}\,\theta^4)$$

Since $\bar{u}$ and $\bar{h}$ are cyclic, $\bar{P}_u$, $\bar{P}_h$ are constant. Hence the coefficients a_i are constant too. Then we can apply to (19) the study made in section 3. (Cid et al., 1982).

II.- As we have said, the solution of $P(\alpha) = 0$ as well as the effective calculation of the values of α, β, γ which determines a regularizing function $g_5(r)$ with $g_5(r) > 0$, seems to need numerical methods.

Now we give two numerical examples, obtained through the system (12), which show the feasibility of the method proposed. The existence of $g_5(r)$ for a wide set of values for the orbital parameters a, e, I, remains to be analyzed.

Data:

$\mu = 0.00553$ $\qquad J_2 = 1.082631\ 10^{-3}$ $\qquad J_4 = -\ 1.65\ 10^{-6}$

example 1:	$a = 2$	$e = 0.1$	$I = 80°$
example 2:	$a = 4$	$e = 0.1$	$I = 80°$

Results:

	Case 1	Case 2
γ	-0.295732	-0.1425387
β	$-0.400184\ 10^1$	$-0.8003057\ 10^1$
α	$0.245787\ 10^3$	$0.4726564\ 10^2$
c_1	$-0.2765\ 10^{-2}$	$-0.1382\ 10^{-2}$
c_2	$-0.2553\ 10^{-5}$	$-0.2113\ 10^{-5}$
c_3	$-0.3076\ 10^{-8}$	$-0.6383\ 10^{-8}$

We have also checked that the variation of the eccentricity e in the range $0.01 \leqslant e \leqslant 0.3$ has small influence on the values of the last table. It is easy to see that in the two cases considered we have $g_5(r) > 0$, because $r \geqslant 1$.

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