THE DIFFERENCE BETWEEN CONSECUTIVE PRIME NUMBERS V

by R. A. RANKIN

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Let p_n denote the nth prime and let ε be any positive number. In 1938 (**3**) I showed that, for an infinity of values of n,

$$p_{n+1}-p_n > (\tfrac{1}{3}-\varepsilon)\log p_n \frac{\log_2 p_n \log_4 p_n}{(\log_3 p_n)^2},$$

where, for $k \geqq 1$, $\log_{k+1} x = \log(\log_k x)$ and $\log_1 x = \log x$. In a recent paper (**4**) Schönhage has shown that the constant $\frac{1}{3}$ may be replaced by the larger number $\frac{1}{2}e^{\gamma}$, where γ is Euler's constant; this is achieved by means of a more efficient selection of the prime moduli used. Schönhage uses an estimate of mine for the number B_1 of positive integers $n \leqq u$ that consist entirely of prime factors $p \leqq y$, where

$$u = \alpha x \log x \log_3 x/(\log_2 x)^2, \quad y = \exp(\delta \log x \log_3 x/\log_2 x).$$

Here x is large and α and δ are positive constants to be chosen suitably.

However, an improved estimate for B_1 can be obtained from the work of de Bruijn (**1**, **2**). It follows from formulæ (1.3) and (1.4) of (**1**), that $B_1 \sim u\rho(v)$, where $v = (\log u)/\log y$; by formula (1.8) of (**2**), $\log \rho(v) \sim -v \log v$. It follows that $B_1 < u/\log x$, as required by Schönhage, if $\delta < 1$ and x is sufficiently large; this improves the previous estimate of $\delta < \frac{1}{2}$. The succeeding steps in the argument are unaltered, so that α may be chosen, as before, to be any number less than δe^{γ}. This means that, for an infinity of values of n,

$$p_{n+1}-p_n > (e^{\gamma}-\varepsilon)\log p_n \frac{\log_2 p_n \log_4 p_n}{(\log_3 p_n)^2}.$$

Similar improvements can be made in Theorems II and III of my earlier paper (**3**).

REFERENCES

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(4) A. Schönhage, Eine Bemerkung zur Konstruktion grosser Primzahllücken, *Arch. Math.*, **14** (1963), 29-30.

The University
Glasgow