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Negative Powers of Laguerre Operators

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Abstract. We study negative powers of Laguerre differential operators in $\mathbb{R}^d$, $d \geq 1$. For these operators we prove two-weight $L^p - L^q$ estimates with ranges of q depending on p. The case of the harmonic oscillator (Hermite operator) has recently been treated by Bongioanni and Torrea by using a straightforward approach of kernel estimates. Here these results are applied in certain Laguerre settings. The procedure is fairly direct for Laguerre function expansions of Hermite type, due to some monotonicity properties of the kernels involved. The case of Laguerre function expansions of convolution type is less straightforward. For half-integer type indices α we transfer the desired results from the Hermite setting and then apply an interpolation argument based on a device we call the *convexity principle* to cover the continuous range of $\alpha \in [-1/2, \infty)^d$. Finally, we investigate negative powers of the Dunkl harmonic oscillator in the context of a finite reflection group acting on $\mathbb{R}^d$ and isomorphic to $\mathbb{Z}_2^d$. The two weight $L^p - L^q$ estimates we obtain in this setting are essentially consequences of those for Laguerre function expansions of convolution type.

1 Introduction

Consider the fractional integral operator (also referred to as the Riesz potential)

$$I^\sigma f(x) = \int_{\mathbb{R}^d} \frac{1}{\|x-y\|^{d-\sigma}} f(y)\,dy, \qquad x \in \mathbb{R}^d,$$

$0 < \sigma < d$, defined for any function f for which the above integral is convergent x-a.e.; for instance, $f \in L^p(\mathbb{R}^d)$ with $1 \leq p < d/\sigma$ is good enough.

Then with an appropriate constant c_σ,

$$(-\Delta)^{-\sigma/2} f = c_\sigma I^\sigma f, \qquad f \in \mathcal{S}(\mathbb{R}^d),$$

where $\Delta = \sum_{j=1}^d \partial_j^2$ is the standard Laplacian in $\mathbb{R}^d$, $d \geq 1$, and the negative power $(-\Delta)^{-\sigma/2}$ is defined in $L^2(\mathbb{R}^d)$ by means of the Fourier transform.

The following theorem is a classical result concerning I^σ; see *e.g.,* [7, 22].

Theorem 1.1 (Hardy–Littlewood–Sobolev) *Let* $0 < \sigma < d$, $1 \leq p < \frac{d}{\sigma}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\sigma}{d}$. *Then for* $p > 1$ *we have the strong type* (p, q) *estimate*

$$\|I^\sigma f\|_q \lesssim \|f\|_p, \qquad f \in L^p(\mathbb{R}^d),$$

while for $p = 1$ *the weak type* $(1, q)$ *estimate holds,*

$$\big|\big\{x \in \mathbb{R}^d : |I^\sigma f(x)| > \lambda\big\}\big| \lesssim \Big(\frac{\|f\|_1}{\lambda}\Big)^q, \qquad \lambda > 0, \quad f \in L^1(\mathbb{R}^d).$$

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The Hardy–Littlewood–Sobolev theorem was extended to a two-weight setting in [24].

Theorem 1.2 (E. M. Stein and G. Weiss) *Let* $0 < \sigma < d$, $1 < p \le q < \infty$, $a < d/p'$, $b < d/q$, $a + b \ge 0$, *and* $\frac{1}{q} = \frac{1}{p} - \frac{\sigma - a - b}{d}$. *Then*

$$\left\| I^\sigma f \right\|_{L^q(\|x\|^{-bq})} \lesssim \left\| f \right\|_{L^p(\|x\|^{ap})}, \qquad f \in L^p(\mathbb{R}^d, \|x\|^{ap}).$$

Note that the conditions $\frac{1}{q} = \frac{1}{p} - \frac{\sigma}{d}$ or $\frac{1}{q} = \frac{1}{p} - \frac{\sigma - a - b}{d}$ appearing in the above theorems are in fact necessary and forced by a homogeneity type argument.

Numerous analogues of the Euclidean fractional integral operator were investigated in various settings, including spaces of homogeneous type, orthogonal expansions, *etc.* For instance, in the seminal article of Muckenhoupt and Stein [17] the case of ultraspherical expansions was treated. Gasper and Trebels (and one of the authors of this article) studied fractional integration for one dimensional Hermite and Laguerre function expansions [9, 10]; the Laguerre case was also considered by Kanjin and Sato [14]. Recently, Bongioanni and Torrea [2] obtained $L^p - L^q$ estimates for negative powers of the harmonic oscillator. In a more general context Bongioanni, Harboure, and Salinas [4] investigated weighted inequalities for negative powers of Schrödinger operators with weights satisfying the reverse Hölder inequality. Our present results generalize significantly those of [9, 10, 14].

In this paper we focus on negative powers of "Laplacians" associated with multi-dimensional Laguerre function expansions. For these operators we prove two-weight $L^p - L^q$ estimates in the spirit of Theorem 1.2. Such estimates are of interest, for instance, in the study of higher order Riesz transforms or Sobolev spaces related to Laguerre expansions. In all the cases we discuss, spectra of self-adjoint extensions of the considered operators are subsets of $(0, \infty)$, which are discrete and separated from zero. Hence their negative powers are well defined in appropriate L^2 spaces just by means of the spectral theorem. The relevant extensions to weighted L^p spaces of the negative powers are given by suitable integral representations. The emerging integral operators are called the *potential operators* (sometimes also referred to as the *fractional integral operators*). Also, we take an opportunity to slightly enhance the result obtained by Bongioanni and Torrea for the harmonic oscillator by stating and proving a weighted counterpart (with power weights) to their result.

In the Laguerre case we consider two different systems of Laguerre functions, $\{\varphi_k^\alpha\}$ and $\{\ell_k^\alpha\}$. The first one leads to so-called Laguerre function expansions of a Hermite type. It occurs that to some extent in this setting the problem of $L^p - L^q$ estimates for the potential operator almost reduces to the Hermite case. This is due to the fact that the heat kernels corresponding to different multi-indices of type $\alpha \in [-1/2, \infty)^d$ possess a certain monotonicity property with respect to α. Thus it suffices to consider only the specific multi-index $\alpha_o = (-1/2, \dots, -1/2)$, which corresponds to Hermite function expansions.

The second system of Laguerre functions is related to so-called Laguerre expansions of a convolution type. In this case our approach is quite different, and in fact a more involved analysis is necessary. We first deal with half-integer multi-indices α

and transfer the desired results from the Hermite setting. Then to cover the continuous range of $\alpha \in [-1/2, \infty)^d$, we derive a certain interpolation argument, which we call the *convexity principle*. This method is of independent interest and can be applied in other situations.

The paper is organized as follows. In Section 2 we gather known facts concerning the potential kernel and the potential operator related to the harmonic oscillator. Then we state and prove a two-weight $L^p - L^q$ estimate for the Hermite potential operator in the spirit of Theorem 1.2. In Section 3 we discuss potential operators associated with Laguerre function expansions of Hermite type. Section 4 is devoted to Laguerre function expansions of convolution type. Section 5 establishes the convexity principle that allows us to give proofs of the main results of Section 4. In Section 6 we take an opportunity to study negative powers of the Dunkl harmonic oscillator in the context of a finite reflection group acting on $\mathbb{R}^d$ and isomorphic to $\mathbb{Z}_2^d$. The results of this section contain as special cases those of Sections 2 and 4, and are strongly connected with the estimates of Section 4. Finally, in Section 7 we gather various additional observations and remarks. Comments explaining how our present results generalize those of [9, 10, 14] are located throughout the paper.

We use a standard notation with essentially all symbols referring to either $\mathbb{R}^d$ or $\mathbb{R}^d_+ = (0, \infty)^d$, $d \geq 1$, depending on the context. Thus Δ denotes either the Laplacian in $\mathbb{R}^d$ or its restriction to $\mathbb{R}^d_+$, and $\|\cdot\|$ stands for the Euclidean norm. By $\langle f, g\rangle$ we denote $\int_{\mathbb{R}^d} f(x)\overline{g(x)}\, dx$ (or the same, but with the integration restricted to $\mathbb{R}^d_+$) whenever the integral makes sense. For a nonnegative weight function w on either $\mathbb{R}^d$ or $\mathbb{R}^d_+$, by $L^p(\mathbb{R}^d, w)$ or $L^p(\mathbb{R}^d_+, w)$, $1 \leq p \leq \infty$, or simply by $L^p(w)$, we denote the usual Lebesgue spaces related to the measure $dw(x) = w(x)dx$ (we will often abuse the notation slightly and use the same symbol w to denote the measure induced by a density w). If $w \equiv 1$, we simply write $L^p(\mathbb{R}^d)$ or $L^p(\mathbb{R}^d_+)$. Beginning in Section 4, Lebesgue measure dx on $\mathbb{R}^d_+$ is replaced by $\mu_\alpha(dx)$, where $\alpha \in (-1, \infty)^d$ is a multi-index, hence some symbols previously related to dx are then related to $\mu_\alpha(dx)$. A similar situation occurs in Section 6 where dx on $\mathbb{R}^d$ is replaced by $w_\alpha(dx)$.

If $k \in \mathbb{N}^d$, $\mathbb{N} = \{0, 1, \dots\}$, then $|k| = k_1 + \cdots + k_d$ is the length of k. The notation $X \lesssim Y$ will be used to indicate that $X \leq CY$ with a positive constant C independent of significant quantities. We shall write $X \simeq Y$ when $X \lesssim Y$ and $Y \lesssim X$ simultaneously. Given $1 \leq p \leq \infty$, p' denotes its adjoint $1/p + 1/p' = 1$.

2 Negative Powers of the Harmonic Oscillator

The multi-dimensional Hermite functions $h_k(x)$, $k \in \mathbb{N}^d$ are given by tensor products

$$h_k(x) = \prod_{i=1}^{d} h_{k_i}(x_i), \qquad x = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

where $h_{k_i}(x_i) = (\pi^{1/2} 2^{k_i} k_i!)^{-1/2} H_{k_i}(x_i) e^{-x_i^2/2}$, and H_n denote the Hermite polynomials of degree $n \in \mathbb{N}$; cf. [16, p. 60]. The system $\{h_k : k \in \mathbb{N}^d\}$ is a complete orthonormal system in $L^2(\mathbb{R}^d)$. It consists of eigenfunctions of the d-dimensional harmonic oscillator $\mathcal{H} = -\Delta + \|x\|^2$, $\mathcal{H}h_k = \lambda_k h_k$, $\lambda_k = 2|k| + d$. We shall denote by the same symbol the natural self-adjoint extension of $\mathcal{H}$, whose spectral resolution is given by the h_k and λ_k; see [25]. The integral kernel of the Hermite semigroup $\{e^{-t\mathcal{H}} : t > 0\}$ is known explicitly (see [26] for this symmetric form of the kernel),

$$G_t(x, y) = \sum_{n=0}^{\infty} e^{-(2n+d)t} \sum_{|k|=n} h_k(x)h_k(y)$$
$$= \big(2\pi \sinh(2t)\big)^{-d/2} \exp\Big(-\frac{1}{4}\big[\tanh(t)\|x + y\|^2 + \coth(t)\|x - y\|^2\big]\Big).$$

Given $\sigma > 0$, consider the negative power $\mathcal{H}^{-\sigma}$. In view of the spectral theorem, it is expressed on $L^2(\mathbb{R}^d)$ by the spectral series

$$\mathcal{H}^{-\sigma} f = \sum_{k\in\mathbb{N}^d} (2|k| + d)^{-\sigma} \langle f, h_k\rangle\, h_k. \tag{2.1}$$

Notice that $\mathcal{H}^{-\sigma}$ is a contraction on $L^2(\mathbb{R}^d)$ for any $\sigma > 0$.

Motivated by the formal identity

$$\mathcal{H}^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^{\infty} e^{-t\mathcal{H}} t^{\sigma-1}\, dt,$$

it is natural to introduce the potential kernel

$$\mathcal{K}^{\sigma}(x, y) = \frac{1}{\Gamma(\sigma)} \int_0^{\infty} G_t(x, y)t^{\sigma-1}\, dt. \tag{2.2}$$

It follows from the decay of $G_t(x, y)$ as $t \to \infty$ and $t \to 0^+$, that for $\sigma > d/2$ the integral in (2.2) is convergent for every $x, y \in \mathbb{R}^d$, while for $0 < \sigma \le d/2$ the integral converges provided that $x \ne y$.

Define the auxiliary convolution kernel $K^{\sigma}(x)$, $x \in \mathbb{R}^d\setminus\{0\}$ by

$$K^{\sigma}(x) = \exp(-\|x\|^2/8), \qquad \|x\| \ge 1,$$

and, for $\|x\| < 1$,

$$K^{\sigma}(x) = \begin{cases} 1, & \sigma > d/2, \\ \log(4/\|x\|), & \sigma = d/2, \\ \|x\|^{2\sigma-d}, & \sigma < d/2. \end{cases}$$

It is immediately seen that $K^{\sigma} \in L^1(\mathbb{R}^d)$ for all $\sigma > 0$. Moreover, if $\sigma > d/2$, then $K^{\sigma} \in L^r(\mathbb{R}^d)$ for each $1 \le r \le \infty$. If $\sigma = d/2$, then $K^{\sigma} \in L^r(\mathbb{R}^d)$ for $1 \le r < \infty$, while for $\sigma < d/2$ we have $K^{\sigma} \in L^r(\mathbb{R}^d)$ if and only if $r < d/(d - 2\sigma)$.

It was proved in [2] that $\mathcal{K}^{\sigma}(x, y)$ is controlled by $K^{\sigma}(x - y)$. To make this section self-contained we include a short proof of this result. An estimate of the integral

$$E_a(T) = \int_0^1 \zeta^{-a} \exp(-T\zeta^{-1})\, d\zeta, \qquad T > 0$$

is needed. Lemma 2.1 is a refinement of [25, Lemma 1.1], see also [18, Lemma 2.3].

Lemma 2.1 *Let $a \in \mathbb{R}$ be fixed. Then*

$$E_a(T) \lesssim \exp(-T/2), \qquad T \geq 1, \tag{2.3}$$

and for $0 < T < 1$

$$E_a(T) \simeq \begin{cases} 1, & a < 1, \\ \log(2/T), & a = 1, \\ T^{-a+1}, & a > 1. \end{cases}$$

Proof A change of the variable of the integration yields

$$E_a(T) = T^{-a+1} \int_T^\infty y^{a-2} \exp(-y)\, dy. \tag{2.4}$$

Now the estimate for $T \geq 1$ follows, since

$$T^{-a+1} \int_T^\infty y^{a-2} \exp(-y)\, dy \lesssim T^{-a+1} \exp(-3T/4) \lesssim \exp(-T/2).$$

Notice that (2.3) can be improved; in fact we have $E_a(T) \lesssim \exp(-T/(1+\varepsilon))$ for any fixed $\varepsilon > 0$.

The estimates for $0 < T < 1$ are verified by splitting the integration in (2.4) onto the intervals $(T, 1)$ and $(1, \infty)$. Then in the first resulting integral the exponential factor can be neglected, and the second integral is just a positive constant. This easily implies the desired bounds from above and below. ■

Proposition 2.2 ([2, Proposition 2]) *For each $\sigma > 0$, $0 < \mathcal{K}^\sigma(x, y) \lesssim K^\sigma(x - y)$.*

Proof The lower estimate is a consequence of the strict positivity of the kernel $G_t(x, y)$. To show the upper estimate we write

$$\Gamma(\sigma)\mathcal{K}^\sigma(x, y) = \int_0^1 G_t(x, y) t^{\sigma-1}\, dt + \int_1^\infty G_t(x, y) t^{\sigma-1}\, dt \equiv \mathcal{J}_0^\sigma(x, y) + \mathcal{J}_\infty^\sigma(x, y).$$

Then

$$\mathcal{J}_\infty^\sigma(x, y) \lesssim \int_1^\infty e^{-dt} \exp\Big(-\frac{1}{4}\|x - y\|^2\Big) t^{\sigma-1}\, dt \lesssim \exp\Big(-\frac{\|x - y\|^2}{4}\Big)$$

and

$$\mathcal{J}_0^\sigma(x, y) \lesssim \int_0^1 \exp\Big(-\frac{1}{4}\frac{\|x - y\|^2}{t}\Big) t^{\sigma - d/2 - 1}\, dt.$$

To treat the last integral we use Lemma 2.1 and then combine the obtained bounds of $\mathcal{J}_0^\sigma(x, y)$ and $\mathcal{J}_\infty^\sigma(x, y)$. The required estimate of $\mathcal{K}^\sigma(x, y)$ follows. ■

Consider the potential operator $\mathcal{I}^\sigma$,

$$\mathcal{I}^\sigma f(x) = \int_{\mathbb{R}^d} \mathcal{K}^\sigma(x,y) f(y)\, dy,$$

defined on the natural domain $\operatorname{Dom}\mathcal{I}^\sigma$ consisting of those functions f for which the above integral is convergent x-a.e. (heuristically, $\mathcal{I}^\sigma f = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t\mathcal{H}} f\, t^{\sigma-1} dt$). By Proposition 2.2 and the fact that $K^\sigma \in L^1(\mathbb{R}^d)$, we see that $L^p(\mathbb{R}^d) \subset \operatorname{Dom}\mathcal{I}^\sigma$, $1 \le p \le \infty$. It will be seen in a moment that $\mathcal{I}^\sigma = \mathcal{H}^{-\sigma}$ as operators on $L^2(\mathbb{R}^d)$ (see the proof of Corollary 2.4).

The following result has recently been proved by Bongioanni and Torrea. Here we also include a discussion of the case $\sigma \ge d/2$.

Theorem 2.3 ([2, Theorem 8]) *Let $\sigma > 0$ and $1 \le p \le \infty$, $1 \le q \le \infty$. If $\sigma \ge d/2$, then*

$$\|\mathcal{I}^\sigma f\|_q \lesssim \|f\|_p, \qquad f \in L^p(\mathbb{R}^d), \tag{2.5}$$

excluding the cases when $\sigma = d/2$ and either $p = \infty$, $q = 1$ or $p = 1$, $q = \infty$. If $0 < \sigma < d/2$, then (2.5) *holds if $\frac{1}{p} - \frac{2\sigma}{d} \le \frac{1}{q} < \frac{1}{p} + \frac{2\sigma}{d}$, with exclusion of the cases: $p = 1$ and $q = \frac{d}{d-2\sigma}$ (in which $\mathcal{I}^\sigma$ satisfies the weak type $(1, q)$ estimate), and $p = \frac{d}{2\sigma}$ and $q = \infty$. Moreover, in each of the cases of strong type (p, q), $p < \infty$ for any $k \in \mathbb{N}^d$ we have*

$$\langle \mathcal{I}^\sigma f, h_k \rangle = \lambda_k^{-\sigma} \langle f, h_k \rangle, \qquad f \in L^p(\mathbb{R}^d). \tag{2.6}$$

Proof Consider first the case $\sigma \ge d/2$. If $1 \le p \le q \le \infty$, since $0 < \mathcal{K}^\sigma(x,y) \lesssim K^\sigma(x-y)$, the proof of (2.5) reduces to checking a similar estimate with $\mathcal{I}^\sigma$ replaced by the convolution operator $T^\sigma : f \mapsto K^\sigma * f$. Recall that the classical Young's inequality has the form

$$\|g * f\|_q \le \|g\|_r \|f\|_p, \qquad \frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}, \quad 1 \le p, q, r \le \infty$$

(in particular it follows that if $g \in L^r$ and $f \in L^p$, then $g * f(x)$ is well defined x-a.e.). Taking $g = K^\sigma$ above shows that $T^\sigma : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ boundedly provided $\sigma > d/2$ and $1 \le p \le q \le \infty$; for $\sigma = d/2$ the case $p = 1$, $q = \infty$, is excluded. If $q < p$, then we argue as in the proof of [2, Theorem 8(iii)] to get $\|\mathcal{I}^\sigma f\|_q \lesssim \|f\|_\infty$ for $1 \le q < \infty$ with the exception of $q = 1$ when $\sigma = d/2$. Similarly, we proceed as in the proof of [2, Theorem 8(iv)] to get $\|\mathcal{I}^\sigma f\|_1 \lesssim \|f\|_p$ for $p < \infty$. Then (2.5) follows for $1 < q < p < \infty$ by interpolation.

If $0 < \sigma < d/2$ and $\frac{1}{q} > \frac{1}{p} - \frac{2\sigma}{d}$, then using Young's inequality is limited to $\frac{1}{q} \le \frac{1}{p}$. In the endpoint case when $\frac{1}{q} = \frac{1}{p} - \frac{2\sigma}{d}$, the desired conclusion follows from Theorem 1.1. This is because (see [25, (2.9)])

$$G_t(x,y) \le W_t(x-y),$$

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where W_t denotes the Gauss–Weierstrass kernel in $\mathbb{R}^d$, which implies $\mathcal{I}^\sigma f \lesssim I^{2\sigma} f$ for any nonnegative f. The case $\frac{1}{p} < \frac{1}{q} < \frac{1}{p} + \frac{2\sigma}{d}$ is more delicate and requires further arguments based on the estimate

$$\|x\|^{2\sigma} \int_{\mathbb{R}^d} \mathcal{K}^\sigma(x,y)\,dy \le C, \qquad x \in \mathbb{R}^d,$$

and an interpolation argument; we refer to [2] for details.

To verify (2.6) observe that for each fixed $k \in \mathbb{N}^d$ the mapping $f \mapsto \langle \mathcal{I}^\sigma f, h_k\rangle$ is a bounded linear functional on $L^p(\mathbb{R}^d)$. This is because

$$|\langle \mathcal{I}^\sigma f, h_k\rangle| \le \|\mathcal{I}^\sigma f\|_q \|h_k\|_{q'} \lesssim \|h_k\|_{q'} \|f\|_p.$$

Moreover, in the proof of Corollary 2.4 we check that this functional agrees, on the linear span of Hermite functions, which is dense in $L^p(\mathbb{R}^d)$, with the linear functional $f \mapsto \lambda_k^{-\sigma}\langle f, h_k\rangle$, which is also bounded on $L^p(\mathbb{R}^d)$. Hence both functionals coincide and (2.6) is justified. ■

It is worth mentioning that in the case $0 < \sigma < d/2$ of the above theorem, in some occurrences the constraint between p and q gives optimal ranges of p and q for (2.5) to hold. This happens when $p = 1$ or $p = \infty$ (then $1 \le q < \frac{d}{d-2\sigma}$ or $\frac{d}{2\sigma} < q \le \infty$ are optimal, respectively), and $q = 1$ or $q = \infty$ (then $1 \le p < \frac{d}{d-2\sigma}$ or $\frac{d}{2\sigma} < p \le \infty$ are sharp, respectively). The corresponding proofs can be found in [2].

Corollary 2.4 *Let $\sigma > 0$ and (p,q), $1 \le p < \infty$, $1 \le q \le \infty$, be such a pair that* (2.5) *holds. Then $\mathcal{H}^{-\sigma}$ extends to a bounded operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$. Moreover, denoting this extension by $\mathcal{H}^{-\sigma}_{pq}$, in each of the cases, for any $k \in \mathbb{N}^d$ we have*

$$\langle \mathcal{H}^{-\sigma}_{pq} f, h_k\rangle = \lambda_k^{-\sigma}\langle f, h_k\rangle, \qquad f \in L^p(\mathbb{R}^d). \tag{2.7}$$

In particular, the $L^p - L^q$ extension of $\mathcal{H}^{-\sigma}$ coincides with that of $\mathcal{I}^\sigma$.

Proof In view of Theorem 2.3 it suffices to show that $\mathcal{H}^{-\sigma} = \mathcal{I}^\sigma$ as operators on $L^2(\mathbb{R}^d)$. This follows by observing that both operators, being bounded on $L^2(\mathbb{R}^d)$, coincide on the linear span of Hermite functions, which is dense in $L^2(\mathbb{R}^d)$. Indeed, to check that $\mathcal{H}^{-\sigma} h_k = \mathcal{I}^\sigma h_k$, $k \in \mathbb{N}^d$, we write

$$\begin{aligned}
\Gamma(\sigma)\int_{\mathbb{R}^d} \mathcal{K}^\sigma(x,y) h_k(y)\,dy &= \int_{\mathbb{R}^d}\int_0^\infty G_t(x,y)t^{\sigma-1}\,dt\,h_k(y)\,dy \\
&= \int_0^\infty t^{\sigma-1} e^{-t\mathcal{H}} h_k(x)\,dt \\
&= \int_0^\infty t^{\sigma-1} e^{-t\lambda_k}\,dt\,h_k(x) = \Gamma(\sigma)\mathcal{H}^{-\sigma} h_k(x).
\end{aligned}$$

Application of Fubini's theorem in the second identity above was possible, since for any fixed $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d}\int_0^\infty G_t(x,y)t^{\sigma-1}|h_k(y)|\,dt\,dy = \int_{\mathbb{R}^d} \mathcal{K}^\sigma(x,y)|h_k(y)|\,dy < \infty.$$

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This is because $\mathcal{K}^\sigma(x,\cdot) \in L^1(\mathbb{R}^d)$ for any fixed $x \in \mathbb{R}^d$ and $h_k \in L^\infty(\mathbb{R}^d)$.

Considering (2.7), given $1 \le p < \infty$, the subspace $L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, hence the extension $\mathcal{H}_{pq}^{-\sigma}$ coincides with $\mathcal{I}^\sigma$ as a bounded operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$. Thus (2.7) follows from (2.6). ∎

It is worth pointing out that for $1 < q < \infty$ the assertion of Corollary 2.4 remains valid if in (2.1), the definition of $\mathcal{H}^{-\sigma}$, the multi-sequence $\{(2|k| + d)^{-\sigma}\}$ is replaced by another multi-sequence of similar behavior at infinity, for instance by $\{(|k|+1)^{-\sigma}\}$ (in either case it would be reasonable to refer to the resulting operator as to the fractional integral operator for Hermite function expansions; then accordingly λ_k in (2.7) must be replaced by $(|k|+1)$). Indeed, this is a simple consequence of a multiplier theorem for multi-dimensional Hermite function expansions; see [28, Theorem 4.2.1] or [8, Theorems 7.10–11], since the multiplier multi-sequence

$$\Big\{ \Big(\frac{2|k| + d}{|k| + 1} \Big)^{\sigma} \Big\}$$

defines a bounded operator on $L^q(\mathbb{R}^d)$ for each $1 < q < \infty$. We can also bypass multiplier theorems; see the remarks at the end of this section.

Theorem 2.3 extends the result of Gasper and Trebels [10, Theorem 3] in several directions. First of all, the result is multi-dimensional. Secondly, the restriction $\sigma < 1/2$ (in the case $d = 1$ discussed in [10]) is released. Finally, the constraint $\frac{1}{q} = \frac{1}{p} - 2\sigma$ (still in the case $d = 1$) happens to be unnecessary, and in addition the case $p = 1$ is admitted.

We take this opportunity to generalize Theorem 2.3 and give a two-weight extension of Theorem 2.3 in the spirit of the result by Stein and Weiss stated in Theorem 1.2. It is clear that the range of q that depends on p in Theorem 2.5 is not optimal. In fact, the optimality is already lost in estimating the potential kernel by any convolution kernel. Indeed, by Hörmander's theorem [12, Theorem 1], a nontrivial convolution operator cannot be bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ for $p > q$. But Theorem 2.3 shows that the Hermite potential operator satisfies such boundedness also for certain $p > q$.

Theorem 2.5 *Let $\sigma > 0$, $1 < p \le q < \infty$, $a < d/p'$, $b < d/q$, $a + b \ge 0$.*

(i) *If $\sigma \ge d/2$, then $\mathcal{I}^\sigma$ maps boundedly $L^p(\mathbb{R}^d, \|x\|^{ap})$ into $L^q(\mathbb{R}^d, \|x\|^{-bq})$.*

(ii) *If $\sigma < d/2$, then the same boundedness holds under the additional condition*

$$\frac{1}{q} \ge \frac{1}{p} - \frac{2\sigma - a - b}{d}. \tag{2.8}$$

Moreover, under the assumptions ensuring boundedness of $\mathcal{I}^\sigma$ from $L^p(\mathbb{R}^d, \|x\|^{ap})$ into $L^q(\mathbb{R}^d, \|x\|^{-bq})$,

$$\langle \mathcal{I}^\sigma f, h_k \rangle = \lambda_k^{-\sigma} \langle f, h_k \rangle, \qquad f \in L^p(\mathbb{R}^d, \|x\|^{ap}). \tag{2.9}$$

Note that Theorem 2.5 implicitly asserts the inclusion $L^p(\mathbb{R}^d, \|x\|^{ap}) \subset \operatorname{Dom} \mathcal{I}^\sigma$ in all the cases when weighted $L^p - L^q$ boundedness holds. The proof of Theorem 2.5 requires suitable weighted inequalities for convolutions. We shall use those obtained by Kerman [15], which we formulate below for easy reference.

Lemma 2.6 ([15, Theorem 3.1]) *Assume that the parameters p, q, r, a, b, η satisfy*

$$1 < p, q, r < \infty, \qquad \frac{1}{q} \le \frac{1}{p} + \frac{1}{r}, \tag{2.10}$$

$$\frac{1}{q} - \frac{1}{p} - \Big(\frac{a+b}{d} - 1\Big) = \frac{1}{r} + \frac{\eta}{d}, \tag{2.11}$$

$$a < \frac{d}{p'}, \qquad b < \frac{d}{q}, \qquad \eta < \frac{d}{r'}, \tag{2.12}$$

$$a + b \ge 0, \qquad a + \eta \ge 0, \qquad b + \eta \ge 0. \tag{2.13}$$

If $g \in L^r(\|x\|^{\eta r})$ and $f \in L^p(\|x\|^{ap})$, then $g * f(x)$ is well defined for a.e. $x \in \mathbb{R}^d$ and

$$\|g * f\|_{L^q(\|x\|^{-bq})} \le C\|g\|_{L^r(\|x\|^{\eta r})}\|f\|_{L^p(\|x\|^{ap})}$$

with a constant C independent of g and f.

Proof of Theorem 2.5 We first deal with case (i), that is, when $\sigma \ge d/2$. As in the proof of Theorem 2.3, it is enough to prove the statement with $\mathcal{I}^\sigma$ replaced by the convolution operator $T^\sigma\colon f \mapsto K^\sigma * f$. To get the desired boundedness of T^σ we will apply Lemma 2.6 with a suitable choice of r and η, so that, in particular, assumptions (2.10)–(2.13) are satisfied.

It is easy to check that the kernel K^σ is in $L^r(\|x\|^{\eta r})$ if and only if the right-hand side of (2.11) is positive. Notice that (2.10) is satisfied with any $1 < r < \infty$, since $p \le q$. Also, the first two inequalities of (2.12) hold by the assumptions, and together they imply that the left-hand side of (2.11) is positive:

$$\xi := \frac{1}{q} - \frac{1}{p} - \Big(\frac{a+b}{d} - 1\Big) > 0.$$

On the other hand, since by assumption $a + b \ge 0$, the first inequality in (2.13) holds, and it follows that the quantity ξ is in the interval $(0, 1)$ except for the singular case when $p = q$ and $a + b = 0$, which will be treated separately. Finally, the third inequality in (2.12) is equivalent to saying that the right-hand side of (2.11) is less than 1. To make use of Lemma 2.6 it remains to show that any admissible value of ξ can be attained by the right-hand side of (2.11), with $1 < r < \infty$ and η such that $a + \eta \ge 0$ and $b + \eta \ge 0$. If $a, b > 0$, then we simply take $\eta = 0$ and let $r = 1/\xi$. When $a \le 0$ we take $\eta = -a$, and we have

$$1 > \frac{1}{r} = \xi + \frac{a}{d} = \frac{1}{q} - \frac{1}{p} - \frac{b}{d} + 1 > \frac{1}{q} - \frac{1}{p} - \frac{1}{q} + 1 = \frac{1}{p'} > 0,$$

so the appropriate choice of r is again possible. Finally, if $b \le 0$, then we take $\eta = -b$ and can choose suitable r, since now

$$1 > \frac{1}{r} = \xi + \frac{b}{d} = \frac{1}{q} - \frac{1}{p} - \frac{a}{d} + 1 > \frac{1}{q} - \frac{1}{p} - \frac{1}{p'} + 1 = \frac{1}{q} > 0.$$

We see that Lemma 2.6 does the job except for the singular case distinguished above.

To cover the case when $p = q$ and $a + b = 0$ we observe that T^σ is controlled by the (centered) Hardy–Littlewood maximal operator M. This is because the convolution kernel is integrable, radial, and essentially radially decreasing; see [7, Proposition 2.7]. But for $1 < p < \infty$, M is bounded on $L^p(w)$ with any weight w in Muckenhoupt's class A_p. The conclusion follows by the fact that a power weight $w(x) = \|x\|^{ap}$ belongs to A_p if and only if $-d/p < a < -d/p'$. This completes the proof of case (i).

We now treat case (ii), when $0 < \sigma < d/2$. Since $\mathcal{I}^\sigma$ is dominated by a constant times $I^{2\sigma}$ (see the proof of Theorem 2.3), the case of equality in (2.8) is covered by Theorem 1.2. Therefore we may assume that $\xi > 1 - \frac{2\sigma}{d}$. But the kernel K^σ belongs to $L^r(\|x\|^{\eta r})$ if and only if the right-hand side of (2.11) is greater than $1 - 2\sigma/d$. In this position we repeat the reasoning of case (i).

For the proof of (2.9) we copy the argument leading to the proof of (2.6). One only has to know that $\|h_k\|_{L^{q'}(\|x\|^{bq'})} < \infty$, but this holds in view of the inequality $b > -d/q'$ following from the assumptions imposed on d, p, q, a, b. ■

Following an idea from [5] we can furnish more general results by considering multiplier operators of Laplace transform type (these multipliers are only slightly different from those discussed in [23]). Let ν be a Borel measure (possibly signed or even complex) on $(0, \infty)$ such that the multi-sequence

$$\nu(\lambda_k) := \int_0^\infty e^{-t\lambda_k}\, d\nu(t), \qquad k \in \mathbb{N}^d, \tag{2.14}$$

is well defined and bounded. We define the multiplier operator M^ν, as the bounded operator on $L^2(\mathbb{R}^d)$ given by

$$M^\nu f = \sum_{k\in\mathbb{N}^d} \nu(\lambda_k)\langle f, h_k\rangle h_k, \qquad f \in L^2(\mathbb{R}^d).$$

Formal calculations show that

$$\mathcal{K}^\nu(x, y) = \int_0^\infty G_t(x, y)\, d\nu(t)$$

should be associated as a kernel of M^ν. In the case when $d\nu(t) = \frac{1}{\Gamma(\sigma)} t^{\sigma-1}\, dt$, $\sigma > 0$, we have $\nu(\lambda_k) = (\lambda_k)^{-\sigma}$ and we recover the kernel $\mathcal{K}^\sigma(x, y)$ and the operator $\mathcal{I}^\sigma$. A careful checking of the arguments used in the proof of Proposition 2.2 reveals that if we take $d\nu(t) = \psi(t)\, dt$, where $\psi(t)$ satisfies $|\psi(t)| \lesssim t^{\sigma-1}$, $t \in (0, 1)$, and $|\psi(t)| \lesssim e^{(d-\varepsilon)t}$, $t \in [1, \infty)$ for some $\varepsilon > 0$, then the multi-sequence $\nu(\lambda_k)$ is bounded and $|\mathcal{K}^\nu(x, y)| \lesssim K^\sigma(x - y)$ (in particular, $\mathcal{K}^\nu(x, y)$ is well defined at least for $x \neq y$). Moreover, still under the above assumptions on ψ, the conclusions of Theorem 2.3 remain valid with $\mathcal{I}^\sigma$ replaced by $\mathcal{I}^\nu$, the integral operator with the kernel $\mathcal{K}^\nu(x, y)$. Also, Corollary 2.4 and Theorem 2.5 remain valid with $\mathcal{H}^{-\sigma}$, $(\lambda_k)^{-\sigma}$ and $\mathcal{I}^\sigma$ replaced by M^ν, $\nu(\lambda_k)$, and $\mathcal{I}^\nu$, respectively. In particular, the measure ν with density $\psi(t) = \frac{2^\sigma}{\Gamma(\sigma)} t^{\sigma-1} e^{(d-2)t}$ satisfies the assumptions stated above and produces the multi-sequence $\nu(\lambda_k) = (|k| + 1)^{-\sigma}$ (cf. the remarks following the proof of Corollary 2.4).

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3 Laguerre Function Expansions of Hermite Type

Let $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in (-1, \infty)^d$ be multi-indices. The Laguerre function φ_k^α on $\mathbb{R}_+^d$ is the tensor product

$$\varphi_k^\alpha(x) = \varphi_{k_1}^{\alpha_1}(x_1) \cdots \varphi_{k_d}^{\alpha_d}(x_d), \qquad x = (x_1, \dots, x_d) \in \mathbb{R}_+^d,$$

where $\varphi_{k_i}^{\alpha_i}$ are the one-dimensional Laguerre functions

$$\varphi_{k_i}^{\alpha_i}(x_i) = \left(\frac{2\Gamma(k_i + 1)}{\Gamma(k_i + \alpha_i + 1)} \right)^{1/2} L_{k_i}^{\alpha_i}(x_i^2) x_i^{\alpha_i + 1/2} e^{-x_i^2/2}, \qquad x_i > 0, \quad i = 1, \dots, d,$$

given $\alpha_i > -1$ and $k_i \in \mathbb{N}$, $L_{k_i}^{\alpha_i}$ denotes the Laguerre polynomial of degree k_i and order α_i; see [16, p. 76].

Each φ_k^α is an eigenfunction of the differential operator

$$L_\alpha^H = -\Delta + \|x\|^2 + \sum_{i=1}^d \frac{1}{x_i^2} \left(\alpha_i^2 - \frac{1}{4} \right),$$

the corresponding eigenvalue being $\lambda_k^\alpha = 4|k| + 2|\alpha| + 2d$, that is, $L_\alpha^H \varphi_k^\alpha = \lambda_k^\alpha \varphi_k^\alpha$. Here by $|\alpha|$ we mean $|\alpha| = \alpha_1 + \cdots + \alpha_d$ (thus $|\alpha|$ may be negative). The operator L_α^H is symmetric and positive in $L^2(\mathbb{R}_+^d)$, and the system $\{\varphi_k^\alpha : k \in \mathbb{N}^d\}$ is an orthonormal basis in $L^2(\mathbb{R}_+^d)$.

As defined in [18, p. 402], L_α^H has a self-adjoint extension $\mathcal{L}_\alpha^H$ whose spectral decomposition is given by the φ_k^α and λ_k^α. The heat-diffusion semigroup $\{e^{-t\mathcal{L}_\alpha^H}\}_{t>0}$ generated by $\mathcal{L}_\alpha^H$,

$$e^{-t\mathcal{L}_\alpha^H} f = \sum_{n=0}^\infty e^{-t(4n+2|\alpha|+2d)} \sum_{|k|=n} \langle f, \varphi_k^\alpha \rangle \varphi_k^\alpha, \qquad f \in L^2(\mathbb{R}_+^d),$$

is a strongly continuous semigroup of contractions on $L^2(\mathbb{R}_+^d)$. We have the integral representation

$$e^{-t\mathcal{L}_\alpha^H} f(x) = \int_{\mathbb{R}_+^d} G_t^{\alpha,H}(x, y) f(y)\, dy, \qquad x \in \mathbb{R}_+^d,$$

where

$$G_t^{\alpha,H}(x, y) = \sum_{n=0}^\infty e^{-t(4n+2|\alpha|+2d)} \sum_{|k|=n} \varphi_k^\alpha(x) \varphi_k^\alpha(y), \qquad x, y \in \mathbb{R}_+^d.$$

It is known (cf. [16, (4.17.6)]) that

$$G_t^{\alpha,H}(x, y) = (\sinh 2t)^{-d} \exp\left(-\frac{1}{2} \coth(2t) \big(\|x\|^2 + \|y\|^2 \big) \right) \prod_{i=1}^d \sqrt{x_i y_i}\, I_{\alpha_i}\left(\frac{x_i y_i}{\sinh 2t} \right).$$

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Here I_ν denotes the modified Bessel function of the first kind and order ν. Considered on the positive half-line, it is real, positive, and smooth for any $\nu > -1$.

It was observed in the proof of [18, Proposition 2.1] that given $\alpha \in [-1/2, \infty)^d$, there exists a constant C_α such that

$$G_t^{\alpha,H}(x,y) \le C_\alpha G_t^{\alpha_o,H}(x,y), \qquad t>0, \quad x,y \in \mathbb{R}_+^d, \tag{3.1}$$

with $\alpha_o = (-1/2, \dots, -1/2)$. This was based on the asymptotics (cf. [16, (5.16.4), (5.16.5)])

$$I_\nu(z) \simeq z^\nu, \quad z \to 0^+; \qquad I_\nu(z) \simeq z^{-1/2} e^z, \quad z \to \infty \tag{3.2}$$

(more information on C_α can be obtained from properties of the function $\nu \mapsto I_\nu(x)$, $x \in \mathbb{R}_+$; see [18]). Moreover, we have (cf. [18, (A.2)])

$$G_t^{\alpha_o,H}(x,y) = \sum_{\varepsilon \in \mathcal{E}} G_t(\varepsilon x, y), \qquad x,y \in \mathbb{R}_+^d, \tag{3.3}$$

where $\mathcal{E} = \{(\varepsilon_1, \dots, \varepsilon_d) : \varepsilon_i = \pm 1\}$ and $\varepsilon x = (\varepsilon_1 x_1, \dots, \varepsilon_d x_d)$.

Given $\sigma > 0$, consider the operator $(\mathcal{L}_\alpha^H)^{-\sigma}$ defined on $L^2(\mathbb{R}_+^d)$ by the spectral series

$$(\mathcal{L}_\alpha^H)^{-\sigma} f = \sum_{k \in \mathbb{N}^d} (\lambda_k^\alpha)^{-\sigma} \langle f, \varphi_k^\alpha \rangle \, \varphi_k^\alpha.$$

Observe that $(\mathcal{L}_\alpha^H)^{-\sigma}$ is a contraction on $L^2(\mathbb{R}_+^d)$ if $\alpha \in [-1/2, \infty)^d$.

We next define the potential kernel

$$\mathcal{K}_H^{\alpha,\sigma}(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty G_t^{\alpha,H}(x,y) t^{\sigma-1}\, dt, \qquad x,y \in \mathbb{R}_+^d, \tag{3.4}$$

and the potential operator (that coincides with $(\mathcal{L}_\alpha^H)^{-\sigma}$ on $L^2(\mathbb{R}_+^d)$)

$$\mathcal{I}_H^{\alpha,\sigma} f(x) = \int_{\mathbb{R}_+^d} \mathcal{K}_H^{\alpha,\sigma}(x,y) f(y)\, dy, \qquad x \in \mathbb{R}_+^d, \tag{3.5}$$

and note that (3.1) and (3.3) lead immediately to

$$\mathcal{K}_H^{\alpha,\sigma}(x,y) \le C_\alpha \mathcal{K}_H^{\alpha_o,\sigma}(x,y) = C_\alpha \sum_{\varepsilon \in \mathcal{E}} \mathcal{K}^\sigma(\varepsilon x, y), \qquad x,y \in \mathbb{R}_+^d. \tag{3.6}$$

Consequently,

$$\mathcal{I}_H^{\alpha,\sigma} f(x) \le C_\alpha \sum_{\varepsilon \in \mathcal{E}} \mathcal{I}^\sigma f(\varepsilon x), \qquad x \in \mathbb{R}_+^d, \tag{3.7}$$

for any $f \ge 0$ defined on $\mathbb{R}_+^d$, where on the right-hand side of (3.7) the function f is understood as the extension of f onto $\mathbb{R}^d$ obtained by setting 0 outside $\mathbb{R}_+^d$. Note at this point that the integral in (3.4) is convergent for every $x, y \in \mathbb{R}^d_+$ when $\sigma > d/2$, while for $0 < \sigma \le d/2$ the integral converges provided $x \ne y$ (this is a consequence of the former statement concerning convergence of the integral in (2.2), or it can be seen directly from the decay of $G_t^{\alpha,H}(x, y)$ when $t \to \infty$ and $t \to 0^+$). Similarly, $L^p(\mathbb{R}^d_+) \subset \operatorname{Dom} \mathcal{I}_H^{\alpha,\sigma}$, $1 \le p \le \infty$ due to (3.7) and the former statement concerning $\operatorname{Dom} \mathcal{I}^\sigma$. Here $\operatorname{Dom} \mathcal{I}_H^{\alpha,\sigma}$ denotes the natural domain of $\mathcal{I}_H^{\alpha,\sigma}$ consisting of those functions f for which the integral in (3.5) is convergent x-a.e.

Thus, in view of Theorem 2.3, we obtain the following result.

Theorem 3.1 *Let $\alpha \in [-1/2, \infty)^d$ and $\sigma > 0$. If $1 \le p \le \infty$, $1 \le q \le \infty$ and (p, q) is one of the pairs specified in Theorem 2.3 for which (2.5) holds, then*

$$\|\mathcal{I}_H^{\alpha,\sigma} f\|_q \lesssim \|f\|_p, \qquad f \in L^p(\mathbb{R}^d_+), \tag{3.8}$$

is also true. Moreover, for each such pair (p, q) with $p < \infty$, for any $k \in \mathbb{N}^d$ we have

$$\langle \mathcal{I}_H^{\alpha,\sigma} f, \varphi_k^\alpha \rangle = (\lambda_k^\alpha)^{-\sigma} \langle f, \varphi_k^\alpha \rangle, \qquad f \in L^p(\mathbb{R}^d_+). \tag{3.9}$$

Proof Only (3.9) requires an explanation. It is proved in the same way as (2.6). The important fact to be used is that $\|\varphi_k^\alpha\|_{q'} < \infty$ for $1 \le q' \le \infty$, and this is indeed assured by the assumption imposed on α. ∎

Furthermore, we also obtain the following corollary.

Corollary 3.2 *Let $\alpha \in [-1/2, \infty)^d$, $\sigma > 0$ and (p, q), $p < \infty$ be such a pair that (3.8) holds. Then $(\mathcal{L}_\alpha^H)^{-\sigma}$ extends to a bounded operator from $L^p(\mathbb{R}^d_+)$ to $L^q(\mathbb{R}^d_+)$. Moreover, denoting this extension by $(\mathcal{L}_\alpha^H)^{-\sigma}_{pq}$, in each of the cases for any $k \in \mathbb{N}^d$ we have*

$$\big\langle (\mathcal{L}_\alpha^H)^{-\sigma}_{pq} f, \varphi_k^\alpha \big\rangle = (\lambda_k^\alpha)^{-\sigma} \langle f, \varphi_k^\alpha \rangle, \qquad f \in L^p(\mathbb{R}^d_+).$$

In particular, the $L^p - L^q$ extension of $(\mathcal{L}_\alpha^H)^{-\sigma}$ coincides with that of $\mathcal{I}_H^{\alpha,\sigma}$.

Proof The arguments are parallel to those used in the proof of Corollary 2.4. Application of Fubini's theorem in the relevant place is possible, since, for any fixed $x \in \mathbb{R}^d_+$,

$$\int_{\mathbb{R}^d_+} \int_0^\infty G_t^{\alpha,H}(x, y) t^{\sigma-1} |\varphi_k^\alpha(y)| \, dt \, dy = \int_{\mathbb{R}^d_+} \mathcal{K}_H^{\alpha,\sigma}(x, y) |\varphi_k^\alpha(y)| \, dy < \infty.$$

This is because $\mathcal{K}_H^{\alpha,\sigma}(x, \cdot\,) \in L^1(\mathbb{R}^d_+)$ for any $x \in \mathbb{R}^d_+$ fixed, and $\varphi_k^\alpha \in L^\infty(\mathbb{R}^d_+)$. ∎

A weighted analogue of Theorem 3.1 is obtained by combining (3.7) with Theorem 2.5.

Theorem 3.3 *Let $\alpha \in [-1/2, \infty)^d$, $\sigma > 0$, $1 < p \le q < \infty$, $a < d/p'$, $b < d/q$, $a + b \ge 0$.*

(i) *If $\sigma \ge d/2$, then $\mathcal{I}_H^{\alpha,\sigma}$ maps boundedly $L^p(\mathbb{R}^d_+, \|x\|^{ap})$ into $L^q(\mathbb{R}^d_+, \|x\|^{-bq})$.*

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(ii) *If $\sigma < d/2$, then the same boundedness holds under the additional condition* (2.8). *Moreover, under the assumptions ensuring boundedness of $\mathcal{I}_H^{\alpha,\sigma}$ from $L^p(\mathbb{R}_+^d, \|x\|^{ap})$ into $L^q(\mathbb{R}_+^d, \|x\|^{-bq})$,*

$$\langle \mathcal{I}_H^{\alpha,\sigma} f, \varphi_k^\alpha \rangle = (\lambda_k^\alpha)^{-\sigma} \langle f, \varphi_k^\alpha \rangle, \qquad f \in L^p(\mathbb{R}_+^d, \|x\|^{ap}).$$

Note that Theorem 3.3 implicitly asserts the inclusion $L^p(\mathbb{R}_+^d, \|x\|^{ap}) \subset \operatorname{Dom} \mathcal{I}_H^{\alpha,\sigma}$ in all the cases when weighted $L^p - L^q$ boundedness holds.

Analogously to the Hermite case (see the remarks at the end of Section 2) we can slightly enhance the theory by considering multiplier operators of Laplace transform type given by (2.14), now with λ_k replaced by λ_k^α. Accordingly, we also consider the operators $M_H^{\alpha,\nu}$, $\mathcal{I}_H^{\alpha,\nu}$, and the kernel $\mathcal{K}_H^{\alpha,\nu}$. Then (assuming that all the relevant integrals against $d\nu$ are convergent) we have

$$\begin{aligned} (3.10) \qquad |\mathcal{K}_H^{\alpha,\nu}(x,y)| &\le \mathcal{K}_H^{\alpha,|\nu|}(x,y) \le C_\alpha \mathcal{K}_H^{\alpha_o,|\nu|}(x,y) \\ &= C_\alpha \sum_{\varepsilon\in\mathcal{E}} \mathcal{K}^{|\nu|}(\varepsilon x, y), \qquad x,y \in \mathbb{R}_+^d, \end{aligned}$$

an analogue of (3.6); $|\nu|$ denotes the total variation of ν. Assuming $d\nu(t) = \psi(t)\,dt$, where $\psi(t)$ satisfies $|\psi(t)| \lesssim t^{\sigma-1}$, $t \in (0,1)$, and $|\psi(t)| \lesssim e^{(2d+2|\alpha|-\varepsilon)t}$, $t \in [1,\infty)$ for some $\varepsilon > 0$, (3.10) implies that the statements of Theorem 3.1, Corollary 3.2 and Theorem 3.3 remain valid with $\mathcal{I}_H^{\alpha,\sigma}$, $(\lambda_k^\alpha)^{-\sigma}$, and $(\mathcal{L}_\alpha^H)^{-\sigma}$ replaced by $\mathcal{I}_H^{\alpha,\nu}$, $\nu(\lambda_k^\alpha)$, and $M_H^{\alpha,\nu}$, respectively. Finally note that taking the measure ν with density $\psi(t) = \frac{4^\sigma}{\Gamma(\sigma)} t^{\sigma-1} e^{(2d+2|\alpha|-4)t}$ produces the multi-sequence $\nu(\lambda_k^\alpha) = (|k|+1)^{-\sigma}$. We point out that for $1 < q < \infty$ the modified assertion of Corollary 3.2 with the multi-sequence $\{(\lambda_k^\alpha)^{-\sigma}\}$ replaced by $\{(|k|+1)^{-\sigma}\}$ or by another multi-sequence with similar behavior at infinity is also a simple consequence of a multiplier theorem, this time for Laguerre expansions of Hermite type; see [28, Theorem 6.4.3] or [8, Theorem 7.12]. This is because the multiplier multi-sequence $\{(\frac{4|k|+2|\alpha|+2d}{|k|+1})^\sigma\}$ generates a bounded operator on each $L^q(\mathbb{R}_+^d)$, $1 < q < \infty$.

4 Laguerre Function Expansions of Convolution Type

In this section we shall work on the space $\mathbb{R}_+^d$, $d \ge 1$, equipped with the measure

$$\mu_\alpha(dx) = x_1^{2\alpha_1+1} \cdots x_d^{2\alpha_d+1}\,dx.$$

Given multi-indices $k \in \mathbb{N}^d$ and $\alpha \in (-1,\infty)^d$, the Laguerre functions ℓ_k^α are

$$\ell_k^\alpha(x) = \ell_{k_1}^{\alpha_1}(x_1) \cdots \ell_{k_d}^{\alpha_d}(x_d), \qquad x = (x_1,\dots,x_d) \in \mathbb{R}_+^d,$$

where $\ell_{k_i}^{\alpha_i}$ are the one-dimensional Laguerre functions

$$\ell_{k_i}^{\alpha_i}(x_i) = \left(\frac{2\Gamma(k_i+1)}{\Gamma(k_i+\alpha_i+1)}\right)^{1/2} L_{k_i}^{\alpha_i}(x_i^2) e^{-x_i^2/2}, \qquad x_i > 0, \quad i = 1,\dots,d.$$

Each ℓ_k^α is an eigenfunction of the differential operator

$$L_\alpha = -\Delta + \|x\|^2 - \sum_{i=1}^{d} \frac{2\alpha_i + 1}{x_i} \frac{\partial}{\partial x_i}$$

with the corresponding eigenvalue $\lambda_k^\alpha = 4|k| + 2|\alpha| + 2d$, that is, $L_\alpha \ell_k^\alpha = \lambda_k^\alpha \ell_k^\alpha$. The operator L_α is symmetric and positive in $L^2(d\mu_\alpha)$, and the system $\{\ell_k^\alpha : k \in \mathbb{N}^d\}$ is an orthonormal basis in $L^2(d\mu_\alpha)$.

Let $\mathcal{L}_\alpha$ denote the self-adjoint extension of L_α as defined in [19, p. 646], whose spectral decomposition is given by the ℓ_k^α and λ_k^α. The heat-diffusion semigroup $\{e^{-t\mathcal{L}_\alpha} : t > 0\}$ generated by $\mathcal{L}_\alpha$ is a strongly continuous semigroup of contractions on $L^2(d\mu_\alpha)$. By the spectral theorem,

$$e^{-t\mathcal{L}_\alpha} f = \sum_{n=0}^{\infty} e^{-t(4n+2|\alpha|+2d)} \sum_{|k|=n} \langle f, \ell_k^\alpha \rangle_{d\mu_\alpha} \ell_k^\alpha, \qquad f \in L^2(d\mu_\alpha).$$

We have the integral representation

$$e^{-t\mathcal{L}_\alpha} f(x) = \int_{\mathbb{R}_+^d} G_t^\alpha(x, y) f(y)\, d\mu_\alpha(y), \qquad x \in \mathbb{R}_+^d,$$

where the heat kernel is given by

$$G_t^\alpha(x, y) = \sum_{n=0}^{\infty} e^{-t(4n+2|\alpha|+2d)} \sum_{|k|=n} \ell_k^\alpha(x) \ell_k^\alpha(y).$$

As in the previous settings, we define the potential kernel

$$\mathcal{K}^{\alpha,\sigma}(x, y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty G_t^\alpha(x, y) t^{\sigma-1}\, dt, \qquad x, y \in \mathbb{R}_+^d, \tag{4.1}$$

and the potential operator (to be equal to $(\mathcal{L}_\alpha)^{-\sigma}$ on $L^2(\mathbb{R}_+^d, d\mu_\alpha)$)

$$\mathcal{I}^{\alpha,\sigma} f(x) = \int_{\mathbb{R}_+^d} \mathcal{K}^{\alpha,\sigma}(x, y) f(y)\, d\mu_\alpha(y).$$

Let S_α be the multiplication operator

$$S_\alpha f(x) = f(x) \prod_{i=1}^{d} x_i^{\alpha_i + 1/2}, \qquad x \in \mathbb{R}_+^d.$$

Then S_α is an isometric isomorphism of $L^2(\mathbb{R}_+^d, d\mu_\alpha)$ onto $L^2(\mathbb{R}_+^d, dx)$, which intertwines the differential operators L_α^H and L_α, $L_\alpha^H \circ S_\alpha = S_\alpha \circ L_\alpha$. Moreover, $\varphi_k^\alpha = S_\alpha \ell_k^\alpha$, $k \in \mathbb{N}^d$, and the eigenvalues in both settings coincide. Therefore,

$$G_t^\alpha(x, y) = G_t^{\alpha,H}(x, y) \prod_{i=1}^{d} (x_i y_i)^{-\alpha_i - 1/2}, \tag{4.2}$$

$$\mathcal{K}^{\alpha,\sigma}(x, y) = \mathcal{K}_H^{\alpha,\sigma}(x, y) \prod_{i=1}^{d} (x_i y_i)^{-\alpha_i - 1/2} \tag{4.3}$$

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and hence

$$\mathcal{J}^{\alpha,\sigma} f(x) = \Big(\prod_{i=1}^{d} x_i^{-\alpha_i - 1/2} \Big) \mathcal{J}_H^{\alpha,\sigma} \big(S_\alpha f \big)(x). \tag{4.4}$$

It follows from (4.2) that comments concerning convergence of the integral in (4.1) are exactly the same as those describing convergence of the integral in (3.4). Moreover, (4.4) forces the following relation between the natural domains of $\mathcal{J}^{\alpha,\sigma}$ and $\mathcal{J}_H^{\alpha,\sigma}$: $f \in \operatorname{Dom} \mathcal{J}^{\alpha,\sigma}$ if and only if $S_\alpha^{-1} f \in \operatorname{Dom} \mathcal{J}_H^{\alpha,\sigma}$. Further, (4.4) gives the following interplay between weighted $L^p - L^q$ estimates for $\mathcal{J}^{\alpha,\sigma}$ and $\mathcal{J}_H^{\alpha,\sigma}$. Given two weights U and V on $\mathbb{R}_+^d$, the inequality

$$\|\mathcal{J}^{\alpha,\sigma} f\|_{L^q(V d\mu_\alpha)} \le C \|f\|_{L^p(U d\mu_\alpha)}, \qquad f \in L^p(U d\mu_\alpha),$$

is equivalent to

$$\|\mathcal{J}_H^{\alpha,\sigma} f\|_{L^q(\widetilde{V})} \le C \|f\|_{L^p(\widetilde{U})}, \qquad f \in L^p(\widetilde{U}),$$

where

$$\widetilde{U}(x) = U(x) \prod_{i=1}^{d} x_i^{(2\alpha_i+1)(1-\frac{p}{2})}, \qquad \widetilde{V}(x) = V(x) \prod_{i=1}^{d} x_i^{(2\alpha_i+1)(1-\frac{q}{2})}.$$

Consequences of this equivalence regarding $L^p - L^q$ estimates are commented on in Section 7.

Our main results for the Laguerre system $\{\ell_k^\alpha\}$ read as follows (notice that now the role of the dimension is played by the quantity $2|\alpha| + 2d$).

Theorem 4.1 *Assume that $\alpha \in [-1/2, \infty)^d$. Let $\sigma > 0$ and $1 \le p < \infty$, $1 \le q < \infty$. If $\sigma \ge |\alpha| + d$, then*

$$\|\mathcal{J}^{\alpha,\sigma} f\|_{L^q(d\mu_\alpha)} \lesssim \|f\|_{L^p(d\mu_\alpha)}, \qquad f \in L^p(\mathbb{R}_+^d, d\mu_\alpha). \tag{4.5}$$

If $0 < \sigma < |\alpha| + d$, then (4.5) holds under the additional condition

$$\frac{1}{p} - \frac{\sigma}{|\alpha| + d} \le \frac{1}{q} < \frac{1}{p} + \frac{\sigma}{|\alpha| + d},$$

with exclusion of the case when $p = 1$ and $q = \frac{|\alpha|+d}{|\alpha|+d-\sigma}$. Moreover, under the assumptions ensuring (4.5),

$$\langle \mathcal{J}^{\alpha,\sigma} f, \ell_k^\alpha \rangle_{d\mu_\alpha} = (\lambda_k^\alpha)^{-\sigma} \langle f, \ell_k^\alpha \rangle_{d\mu_\alpha}, \qquad f \in L^p(\mathbb{R}_+^d, d\mu_\alpha). \tag{4.6}$$

Considering the weighted setting, we also prove the following theorem.

Theorem 4.2 *Assume that $\alpha \in [-1/2, \infty)^d$. Let $\sigma > 0$, $1 < p \le q < \infty$, and*

$$a < (2|\alpha| + 2d)/p', \qquad b < (2|\alpha| + 2d)/q, \qquad a + b \ge 0.$$

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(i) *If $\sigma \geq |\alpha| + d$, then $\mathcal{I}^{\alpha,\sigma}$ maps $L^p(\mathbb{R}^d_+, \|x\|^{ap} d\mu_\alpha)$ into $L^q(\mathbb{R}^d_+, \|x\|^{-bq} d\mu_\alpha)$ boundedly.*

(ii) *If $\sigma < |\alpha| + d$, then the same boundedness holds under the additional condition*

$$\frac{1}{q} \geq \frac{1}{p} - \frac{2\sigma - a - b}{2|\alpha| + 2d}.$$

Moreover, under the assumptions ensuring boundedness of $\mathcal{I}^{\alpha,\sigma}$ from $L^p(\mathbb{R}^d_+, \|x\|^{ap} d\mu_\alpha)$ into $L^q(\mathbb{R}^d_+, \|x\|^{-bq} d\mu_\alpha)$,

$$(4.7) \qquad \langle \mathcal{I}^{\alpha,\sigma} f, \ell_k^\alpha \rangle_{d\mu_\alpha} = (\lambda_k^\alpha)^{-\sigma} \langle f, \ell_k^\alpha \rangle_{d\mu_\alpha}, \qquad f \in L^p(\mathbb{R}^d_+, \|x\|^{ap} d\mu_\alpha).$$

The proofs will be given in the next section after elaborating necessary tools. Note that Theorem 4.1 implicitly asserts the inclusion $L^p(\mathbb{R}^d_+, d\mu_\alpha) \subset \operatorname{Dom} \mathcal{I}^{\alpha,\sigma}$, $1 \leq p < \infty$. Similarly, Theorem 4.2 asserts the inclusion $L^p(\mathbb{R}^d_+, \|x\|^{ap} d\mu_\alpha) \subset \operatorname{Dom} \mathcal{I}^{\alpha,\sigma}$ in all the cases when weighted $L^p - L^q$ boundedness holds. Notice also that for $\alpha_o = (-1/2, \dots, -1/2)$ Theorems 4.2 and 3.3 coincide, as it should be, since for $\alpha = \alpha_o$ both settings coincide. An analogous remark concerns Theorems 4.1 and 3.1.

Corollary 4.3 *Let $\alpha \in [-1/2, \infty)^d$, $\sigma > 0$, $1 \leq p < \infty$, $1 \leq q < \infty$, and (p, q) be such a pair that (4.5) holds. Then the operator $(\mathcal{L}_\alpha)^{-\sigma}$, defined on $L^2(\mathbb{R}^d_+, d\mu_\alpha)$ by means of the spectral theorem, extends to a bounded operator from $L^p(\mathbb{R}^d_+, d\mu_\alpha)$ to $L^q(\mathbb{R}^d_+, d\mu_\alpha)$. Moreover, denoting this extension by $(\mathcal{L}_\alpha)^{-\sigma}_{pq}$, for any $k \in \mathbb{N}^d$ we have*

$$(4.8) \qquad \left\langle (\mathcal{L}_\alpha)^{-\sigma}_{pq} f, \ell_k^\alpha \right\rangle_{d\mu_\alpha} = (\lambda_k^\alpha)^{-\sigma} \langle f, \ell_k^\alpha \rangle_{d\mu_\alpha}, \qquad f \in L^p(\mathbb{R}^d_+, d\mu_\alpha).$$

In particular, the $L^p - L^q$ extension of $(\mathcal{L}_\alpha)^{-\sigma}$ coincides with that of $\mathcal{I}^{\alpha,\sigma}$.

Proof We use Theorem 4.1 and the arguments from the proof of Corollary 2.4. We check that $(\mathcal{L}_\alpha)^{-\sigma} = \mathcal{I}^{\alpha,\sigma}$ in $L^2(\mathbb{R}^d_+, d\mu_\alpha)$ by verifying that both operators, being bounded on $L^2(\mathbb{R}^d_+, d\mu_\alpha)$, coincide on the linear span of Laguerre functions ℓ_k^α, which is dense in $L^2(\mathbb{R}^d_+, d\mu_\alpha)$. Note that to apply Fubini's theorem it is enough to know that $\mathcal{K}^{\alpha,\sigma}(x, \cdot) \in L^1(\mathbb{R}^d_+, d\mu_\alpha)$ for any fixed $x \in \mathbb{R}^d_+$ and that $\ell_k^\alpha \in L^\infty(\mathbb{R}^d_+)$. The latter fact is obvious. To justify the first one observe that due to (4.3) and (3.6) it suffices to show that

$$\int_{\mathbb{R}^d_+} \mathcal{K}^\sigma(x, y) \prod_{i=1}^d y_i^{\alpha_i + 1/2} \, dy < \infty$$

for any fixed $x \in \mathbb{R}^d$. This, however, easily follows by Proposition 2.2.

Considering (4.8), given $1 \leq p < \infty$, the subspace $L^2(\mathbb{R}^d_+, d\mu_\alpha) \cap L^p(\mathbb{R}^d_+, d\mu_\alpha)$ is dense in $L^p(\mathbb{R}^d_+, d\mu_\alpha)$, hence the extension $(\mathcal{L}_\alpha)^{-\sigma}_{pq}$ coincides with $\mathcal{I}^{\alpha,\sigma}$ as a bounded operator from $L^p(\mathbb{R}^d_+, d\mu_\alpha)$ to $L^q(\mathbb{R}^d_+, d\mu_\alpha)$, and therefore (4.8) follows from (4.6). ■

Similarly as in the settings of Hermite expansions and Laguerre expansions of Hermite type, the following observation is in order. The assertion of Corollary 4.3 remains valid, at least in the case when $d = 1$, $q \neq 1$ and $\alpha \geq 0$, and the sequence

$(\lambda_k^\alpha)^{-\sigma}$ in the definition of $(\mathcal{L}_\alpha)^{-\sigma}$ is replaced by $(k+1)^{-\sigma}$ (then accordingly λ_k^α in (4.8) must be replaced by $k+1$) or by any other sequence of similar behavior at infinity. This time this is a consequence of a multiplier theorem for one-dimensional Laguerre expansions of convolution type; see [27, Theorem 1.1] (actually that theorem admits a weighted setting with power weights involved).

Theorem 4.2 extends the result of Gasper, Stempak, and Trebels [9, Theorem 1.1] in several directions. To make appropriate comments we first state an equivalent form of the theorem from [9]. In terms of one-dimensional $\{\ell_k^\alpha\}$-expansions it reads as follows. Let

$$\alpha \geq 0, \quad 1 < p \leq q < \infty, \quad 0 < \sigma < \alpha + 1, \quad a < \frac{2\alpha+2}{p'}, \quad b < \frac{2\alpha+2}{q},$$
$$a + b \geq 0, \quad \frac{1}{q} = \frac{1}{p} - \frac{2\sigma - a - b}{2\alpha+2}.$$

Then the operator I_σ, defined initially by the series

$$I_\sigma f = \sum_{k=0}^{\infty} (k+1)^{-\sigma} \langle f, \ell_k^\alpha \rangle_{d\mu_\alpha} \ell_k^\alpha$$

on the space spanned by the ℓ_k^α, $k \geq 0$ (hence, in fact, the series terminates), extends to a bounded operator from $L^p(\mathbb{R}_+, x^{ap} d\mu_\alpha)$ to $L^q(\mathbb{R}_+, x^{-bq} d\mu_\alpha)$. Thus Theorem 4.2, being first of all multi-dimensional, in dimension one releases the restriction $\sigma < \alpha + 1$, enlarges the range of α parameter from $[0, \infty)$ to $[-1/2, \infty)$, and finally, shows that the constraint $\frac{1}{q} = \frac{1}{p} - \frac{2\sigma-a-b}{2\alpha+2}$ is unnecessary (the possibility of replacing λ_k^α by $k+1$, at least when $\alpha \geq 0$, is possible due to the weighted multiplier theorem mentioned above).

5 Convexity Principle

We start by showing a convexity principle for the heat kernel $G_t^\alpha(x, y)$, the potential kernel $\mathcal{K}^{\alpha,\sigma}(x, y)$, and the potential operator $\mathcal{I}^{\alpha,\sigma}$.

Proposition 5.1 *Let $\beta, \gamma \in (-1, \infty)^d$, $\beta \neq \gamma$, $\sigma_1, \sigma_2 > 0$. Further, let*

$$\alpha = \lambda\beta + (1-\lambda)\gamma, \qquad \sigma = \lambda\sigma_1 + (1-\lambda)\sigma_2$$

for some $\lambda \in (0, 1)$. Then

$$G_t^\alpha(x, y) \simeq \big(G_t^\beta(x, y)\big)^\lambda \big(G_t^\gamma(x, y)\big)^{1-\lambda}, \qquad x, y \in \mathbb{R}_+^d, \tag{5.1}$$

$$\mathcal{K}^{\alpha,\sigma}(x, y) \lesssim \big(\mathcal{K}^{\beta,\sigma_1}(x, y)\big)^\lambda \big(\mathcal{K}^{\gamma,\sigma_2}(x, y)\big)^{1-\lambda}. \tag{5.2}$$

Moreover, for $f = f_1^\lambda f_2^{1-\lambda}$ with $f_1, f_2 \geq 0$ we have

$$\mathcal{I}^{\alpha,\sigma} f(x) \lesssim \big(\mathcal{I}^{\beta,\sigma_1} f_1(x)\big)^\lambda \big(\mathcal{I}^{\gamma,\sigma_2} f_2(x)\big)^{1-\lambda}, \qquad x \in \mathbb{R}_+^d. \tag{5.3}$$

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Proof The first relation is a direct consequence of the explicit formula for $G_t^\alpha(x,y)$, the asymptotics (3.2) and the continuity of the Bessel functions involved. The estimate for potential kernels follows from (5.1) and Hölder's inequality:

$$\begin{aligned}
\mathcal{K}^{\alpha,\sigma}(x,y) &\simeq \int_0^\infty \big(G_t^\beta(x,y)\big)^\lambda \big(G_t^\gamma(x,y)\big)^{1-\lambda} t^{\lambda(\sigma_1-1)} t^{(1-\lambda)(\sigma_2-1)}\,dt \\
&\leq \Big(\int_0^\infty G_t^\beta(x,y) t^{\sigma_1-1}\,dt\Big)^\lambda \Big(\int_0^\infty G_t^\gamma(x,y) t^{\sigma_2-1}\,dt\Big)^{1-\lambda} \\
&\simeq \big(\mathcal{K}^{\beta,\sigma_1}(x,y)\big)^\lambda \big(\mathcal{K}^{\gamma,\sigma_2}(x,y)\big)^{1-\lambda}.
\end{aligned}$$

Finally, to justify (5.3) we first observe that

$$w_\alpha(x) = (w_\beta(x))^\lambda (w_\gamma(x))^{1-\lambda},$$

where w_α denotes the density of the measure μ_α. Then we use (5.2) and again Hölder's inequality to get

$$\begin{aligned}
\mathcal{I}^{\alpha,\sigma} f(x) &= \int_{\mathbb{R}_+^d} \mathcal{K}^{\alpha,\sigma}(x,y) f(y) w_\alpha(y)\,dy \\
&\lesssim \int_{\mathbb{R}_+^d} \big(\mathcal{K}^{\beta,\sigma_1}(x,y)\big)^\lambda \big(\mathcal{K}^{\gamma,\sigma_2}(x,y)\big)^{1-\lambda} \big(f_1(y)\big)^\lambda \big(f_2(y)\big)^{1-\lambda} \\
&\quad \times \big(w_\beta(y)\big)^\lambda \big(w_\gamma(y)\big)^{1-\lambda}\,dy \\
&\leq \Big(\int_{\mathbb{R}_+^d} \mathcal{K}^{\beta,\sigma_1}(x,y) f_1(y) w_\beta(y)\,dy\Big)^\lambda \Big(\int_{\mathbb{R}_+^d} \mathcal{K}^{\gamma,\sigma_2}(x,y) f_2(y) w_\gamma(y)\,dy\Big)^{1-\lambda} \\
&= \big(\mathcal{I}^{\beta,\sigma_1} f_1(x)\big)^\lambda \big(\mathcal{I}^{\gamma,\sigma_2} f_2(x)\big)^{1-\lambda}. \qquad \blacksquare
\end{aligned}$$

We now state the convexity principle that concerns $L^p - L^q$ mapping properties of the potential operators.

Theorem 5.2 *Let $\beta,\gamma \in (-1,\infty)^d$, $\beta \neq \gamma$, $\sigma_1,\sigma_2 > 0$, and $1 \leq p,q < \infty$. Further, let U_1,U_2,V_1,V_2 be strictly positive (up to a set of null measure) weights on $\mathbb{R}_+^d$ and put*

$$\alpha = \lambda\beta + (1-\lambda)\gamma, \qquad \sigma = \lambda\sigma_1 + (1-\lambda)\sigma_2, \qquad U = U_1^\lambda U_2^{1-\lambda}, \qquad V = V_1^\lambda V_2^{1-\lambda}$$

for a fixed $\lambda \in (0,1)$. Then boundedness of the operators

$$\mathcal{I}^{\beta,\sigma_1} \colon L^p(U_1 d\mu_\beta) \longrightarrow L^q(V_1 d\mu_\beta), \qquad \mathcal{I}^{\gamma,\sigma_2} \colon L^p(U_2 d\mu_\gamma) \longrightarrow L^q(V_2 d\mu_\gamma),$$

implies boundedness of the operator

$$\mathcal{I}^{\alpha,\sigma} \colon L^p(U\,d\mu_\alpha) \longrightarrow L^q(V\,d\mu_\alpha).$$

Proof To obtain the norm estimates it is enough, by the positivity of the potential kernels, to consider only nonnegative functions f. By (5.3), Hölder's inequality, and the assumed boundedness of $\mathcal{I}^{\beta,\sigma_1}$ and $\mathcal{I}^{\gamma,\sigma_2}$, it follows that for $f = f_1^{\lambda} f_2^{1-\lambda}$

$$\begin{aligned}
&\|\mathcal{I}^{\alpha,\sigma} f\|^q_{L^q(V d\mu_\alpha)} \\
&\quad = \int_{\mathbb{R}^d_+} \big(\mathcal{I}^{\alpha,\sigma} f(x)\big)^q V(x) w_\alpha(x)\, dx \\
&\quad \lesssim \int_{\mathbb{R}^d_+} \big(\mathcal{I}^{\beta,\sigma_1} f_1(x)\big)^{\lambda q} \big(V_1(x) w_\beta(x)\big)^{\lambda} \big(\mathcal{I}^{\gamma,\sigma_2} f_2(x)\big)^{(1-\lambda)q} \big(V_2(x) w_\gamma(x)\big)^{1-\lambda}\, dx \\
&\quad \le \left(\int_{\mathbb{R}^d_+} \big(\mathcal{I}^{\beta,\sigma_1} f_1(x)\big)^q V_1(x) w_\beta(x)\, dx \right)^{\lambda} \left(\int_{\mathbb{R}^d_+} \big(\mathcal{I}^{\gamma,\sigma_2} f_2(x)\big)^q V_2(x) w_\gamma(x)\, dx \right)^{1-\lambda} \\
&\quad \lesssim \left(\int_{\mathbb{R}^d_+} \big(f_1(x)\big)^p U_1(x) w_\beta(x)\, dx \right)^{\lambda q/p} \left(\int_{\mathbb{R}^d_+} \big(f_2(x)\big)^p U_2(x) w_\gamma(x)\, dx \right)^{(1-\lambda)q/p}.
\end{aligned}$$

With the choice

$$f_1(x) = f(x) \left(\frac{U(x) w_\alpha(x)}{U_1(x) w_\beta(x)} \right)^{1/p}, \qquad f_2(x) = f(x) \left(\frac{U(x) w_\alpha(x)}{U_2(x) w_\gamma(x)} \right)^{1/p},$$

we get the desired estimate $\|\mathcal{I}^{\alpha,\sigma} f\|_{L^q(V d\mu_\alpha)} \lesssim \|f\|_{L^p(U d\mu_\alpha)}$. ∎

The importance of the convexity principles comes from the fact that they allow us to obtain weighted mapping properties of the Laguerre potential operators from those for Hermite potential operators, which are much easier to analyze. More precisely, we shall use a transference method relating various objects and results between Hermite and Laguerre expansions to obtain the relevant results in the Laguerre setting, but with the type multi-index restricted to a discrete set of half-integer multi-indices. Then the convexity principles will enable us to interpolate those "half-integer results" to cover general α.

A detailed description of the transference between the settings of Hermite polynomial expansions and Laguerre polynomial expansions of half-integer order can be found in [11]. In this situation, a completely parallel transference method is valid for the Hermite function setting and the Laguerre function setting of convolution type and half-integer order. Let $n = (n_1, \dots, n_d) \in (\mathbb{N}\backslash\{0\})^d$ be a multi-index and let $y^i = (y^i_1, \dots, y^i_{n_i}) \in \mathbb{R}^{n_i}$, $i = 1, \dots, d$. Define the transformation $\phi\colon \mathbb{R}^{|n|} \to \mathbb{R}^d_+$ by

$$\phi(y^1, \dots, y^d) = (\|y^1\|, \dots, \|y^d\|).$$

Various objects in the Hermite setting in $\mathbb{R}^{|n|}$ and in the Laguerre setting in $\mathbb{R}^d_+$ with the type multi-index α such that

$$\alpha_i = \frac{n_i}{2} - 1, \qquad i = 1, \dots, d \tag{5.4}$$

are connected by means of ϕ. In particular, we have the following proposition.

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Proposition 5.3 *Let n and α be as above, and let $t, \sigma > 0$ be fixed. Then for any f in the space spanned by the ℓ_k^α we have*

$$(e^{-t\mathcal{L}_\alpha} f) \circ \phi(\bar{x}) = e^{-t\mathcal{H}}(f \circ \phi)(\bar{x}), \qquad \bar{x} \in \mathbb{R}^{|n|},$$
$$((\mathcal{L}_\alpha)^{-\sigma} f) \circ \phi(\bar{x}) = \mathcal{H}^{-\sigma}(f \circ \phi)(\bar{x}), \qquad \bar{x} \in \mathbb{R}^{|n|},$$

where $\mathcal{L}_\alpha$ is the Laguerre operator in $\mathbb{R}_+^d$, and $\mathcal{H}$ is the Hermite operator in $\mathbb{R}^{|n|}$.

Proof It suffices to take $f = \ell_k^\alpha$ and use the identity

$$\ell_k^\alpha \circ \phi(\bar{x}) = \sum a_r h_{2r}(\bar{x}), \qquad \bar{x} \in \mathbb{R}^{|n|},$$

where the summation runs over all $r = (r^1, \dots, r^d) \in \mathbb{N}^{|n|}$ such that $|r^i| = k_i$, $i = 1, \dots, d$ (see [11, Lemma 1.1, Proposition 3.1]), and a_r are some constants. ■

Since $(\mathcal{L}_\alpha)^{-\sigma}\ell_k^\alpha = \mathcal{I}^{\alpha,\sigma}\ell_k^\alpha$ and $\mathcal{H}^{-\sigma}h_r = \mathcal{I}^\sigma h_r$ (see the proofs of Corollaries 4.3 and 2.4), we get the following corollary.

Corollary 5.4 *Assume that $n \in (\mathbb{N}\backslash\{0\})^d$ is related to α as in (5.4). Given $\sigma > 0$, let $\mathcal{I}^{\alpha,\sigma}$ be the Laguerre potential operator in $\mathbb{R}_+^d$, and let $\mathcal{I}^\sigma$ be the Hermite potential operator in $\mathbb{R}^{|n|}$. Then*

$$(\mathcal{I}^{\alpha,\sigma} f) \circ \phi(\bar{x}) = \mathcal{I}^\sigma(f \circ \phi)(\bar{x}), \qquad \bar{x} \in \mathbb{R}^{|n|},$$

for any f in the space spanned by the ℓ_k^α.

The relations of Proposition 5.3 and Corollary 5.4 hold in fact for more general functions f. This can be seen from connections between heat and potential kernels in both settings.

Proposition 5.5 *Assume that n and α are related by (5.4). Let $G_t^\alpha(x, y)$, $\mathcal{K}^{\alpha,\sigma}(x, y)$ be the Laguerre heat and potential kernels in $\mathbb{R}_+^d$, and let $G_t(\bar{x}, \bar{y})$, $\mathcal{K}^\sigma(\bar{x}, \bar{y})$ be the Hermite heat and potential kernels in $\mathbb{R}^{|n|}$. Then, for all $\bar{x} \in \mathbb{R}^{|n|}$, $y \in \mathbb{R}_+^d$,*

$$G_t^\alpha(\phi(\bar{x}), y) = \int_{S_{n_1-1}} \cdots \int_{S_{n_d-1}} G_t\Big(\bar{x}, \big(y_1\xi^1, \dots, y_d\xi^d\big)\Big)\, d\sigma_1(\xi^1) \cdots d\sigma_d(\xi^d),$$

hence also

$$\mathcal{K}^{\alpha,\sigma}(\phi(\bar{x}), y) = \int_{S_{n_1-1}} \cdots \int_{S_{n_d-1}} \mathcal{K}^\sigma\Big(\bar{x}, \big(y_1\xi^1, \dots, y_d\xi^d\big)\Big)\, d\sigma_1(\xi^1) \cdots d\sigma_d(\xi^d),$$

where S_{n_i-1} is the unit sphere in $\mathbb{R}^{n_i}$ and σ_i is the surface measure on S_{n_i-1}.

Proof Let f be a linear combination of the ℓ_k^α. Integrating in poly-polar coordinates in $\mathbb{R}^{|n|}$ associated with the factorization $\mathbb{R}^{|n|} = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}$ we get, for each $\bar{x} \in \mathbb{R}^{|n|}$,

$$\begin{aligned} e^{-t\mathcal{H}}&(f \circ \phi)(\bar{x}) \\ &= \int_{\mathbb{R}^{|n|}} G_t(\bar{x}, \bar{y}) f \circ \phi(\bar{y})\, d\bar{y} \\ &= \int_{\mathbb{R}_+^d} \Big[\int_{S_{n_1-1}} \cdots \int_{S_{n_d-1}} G_t\Big(\bar{x}, \big(y_1\xi^1, \dots, y_d\xi^d\big)\Big)\, d\sigma_1(\xi^1) \cdots d\sigma_d(\xi^d)\Big] f(y)\, d\mu_\alpha(y). \end{aligned}$$

On the other hand, by Proposition 5.3 the last expression is equal to

$$(e^{-t\mathcal{L}_\alpha} f) \circ \phi(\bar{x}) = \int_{\mathbb{R}_+^d} G_t^\alpha\big(\phi(\bar{x}), y\big) f(y)\, d\mu_\alpha(y).$$

Thus the desired identity for the heat kernels follows by their continuity and the density in $L^2(d\mu_\alpha)$ of the subspace spanned by the ℓ_k^α. The identity for the potential kernels is an easy consequence of the previous one. ■

Corollary 5.6 *Let $\alpha, n, \mathcal{I}^\sigma, \mathcal{I}^{\alpha,\sigma}$ be as in Corollary* 5.4 *and let f be a function in $\mathbb{R}_+^d$. If $f \circ \phi \in \operatorname{Dom} \mathcal{I}^\sigma$, then $f \in \operatorname{Dom} \mathcal{I}^{\alpha,\sigma}$, and for almost all $\bar{x} \in \mathbb{R}^{|n|}$,*

$$(\mathcal{I}^{\alpha,\sigma} f) \circ \phi(\bar{x}) = \mathcal{I}^\sigma(f \circ \phi)(\bar{x}).$$

Proof Assuming that $f \circ \phi$ is in the domain of $\mathcal{I}^\sigma$ and $\bar{x}$ is not in an exceptional set of null measure, we write the convergent integral defining $\mathcal{I}^\sigma(f \circ \phi)(\bar{x})$ as an iterated integral in the poly-polar coordinates emerging from the factorization $\mathbb{R}^{|n|} = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}$. Then the pointwise connection from Proposition 5.5 between the potential kernels can be plugged in, and this leads precisely to the desired conclusions. ■

An important feature of the transference method is that norm estimates in the two settings in question are also related. The following lemma makes this precise (compare with [11, Lemma 2.2]).

Lemma 5.7 *Let n and α be as in* (5.4). *Assume that U, V are nonnegative weights in $\mathbb{R}_+^d$, $1 \le p, q < \infty$, and f is a fixed function in $L^p(U d\mu_\alpha)$. Suppose that $T, \widetilde{T}$ are operators defined on $L^p(U d\mu_\alpha)$ and $L^p(\mathbb{R}^{|n|}, U \circ \phi)$, respectively, satisfying $(Tf) \circ \phi = \widetilde{T}(f \circ \phi)$. If*

$$\|\widetilde{T}(f \circ \phi)\|_{L^q(\mathbb{R}^{|n|}, V \circ \phi)} \le C \|f \circ \phi\|_{L^p(\mathbb{R}^{|n|}, U \circ \phi)},$$

then also

$$\|Tf\|_{L^q(V d\mu_\alpha)} \le C(\mathcal{C}_{dn})^{1/p - 1/q} \|f\|_{L^p(U d\mu_\alpha)}$$

with the same constant C and the constant $\mathcal{C}_{dn} = \prod_{i=1}^d \sigma_i(S_{n_i-1})$.

Corollary 5.6 and Lemma 5.7 imply the following corollary.

Corollary 5.8 *Fix $\sigma > 0$. Let n and α satisfy* (5.4), *and let U, V be nonnegative weights in $\mathbb{R}^d_+$. If the Hermite potential operator*

$$\mathcal{I}^\sigma : L^p(\mathbb{R}^{|n|}, U \circ \phi) \longrightarrow L^q(\mathbb{R}^{|n|}, V \circ \phi)$$

is bounded, then the Laguerre potential operator

$$\mathcal{I}^{\alpha,\sigma} : L^p(U\,d\mu_\alpha) \longrightarrow L^q(V\,d\mu_\alpha)$$

is also bounded.

Combining this with Theorems 2.3 and 2.5, we get the following proposition.

Proposition 5.9 *The statements of Theorems* 4.1 *and* 4.2 *are true for the discrete set of half-integer type multi-indices α,* i.e., *for α of the form* (5.4).

Now the convexity principle comes into play, and together with Proposition 5.9 it allows us to justify Theorems 4.1 and 4.2. We give a proof of Theorem 4.2 only. Proving Theorem 4.1 relies on exactly the same arguments and is in fact simpler due to the absence of weights.

Proof of Theorem 4.2 We shall iterate Theorem 5.2 d times, applying it gradually for successive coordinate axes. Fix $\sigma > 0$. In the first step we deal with the first axis. Let α be of the form $\alpha = (\alpha_1, \alpha_{2,d})$, where $\alpha_1 \in [-1/2, \infty)$ is arbitrary and $\alpha_{2,d} \in [-1/2, \infty)^{d-1}$ is half-integer. In addition, we may assume that α_1 is not a half-integer (otherwise Proposition 5.9 does the job). Then α is a convex combination of two half-integer multi-indices, say

$$\alpha = \lambda\beta + (1-\lambda)\gamma, \quad \text{where} \quad \beta = (\beta_1, \alpha_{2,d}), \quad \gamma = (\gamma_1, \alpha_{2,d}),$$

for some half-integers $\beta_1, \gamma_1 \geq -1/2$ and certain $\lambda \in (0,1)$. Take

$$\sigma_\beta = \sigma\frac{|\beta| + d}{|\alpha| + d}, \qquad \sigma_\gamma = \sigma\frac{|\gamma| + d}{|\alpha| + d}$$

satisfying $\sigma = \lambda\sigma_\beta + (1-\lambda)\sigma_\gamma$. Further, given a and b, take

$$a_\beta = a\frac{|\beta| + d}{|\alpha| + d}, \qquad b_\beta = b\frac{|\beta| + d}{|\alpha| + d}, \qquad a_\gamma = a\frac{|\gamma| + d}{|\alpha| + d}, \qquad b_\gamma = b\frac{|\gamma| + d}{|\alpha| + d},$$

and notice that $a = \lambda a_\beta + (1-\lambda)a_\gamma$ and $b = \lambda b_\beta + (1-\lambda)b_\gamma$. Now the crucial observation is that the assumptions of Theorem 4.2 are satisfied with the parameters (d, p, q, α, a, b) if and only if they are satisfied with the parameters $(d, p, q, \beta, a_\beta, b_\beta)$ and simultaneously with the parameters $(d, p, q, \gamma, a_\gamma, b_\gamma)$. Moreover, splitting Theorem 4.2 into cases (i) and (ii) with parameters (d, p, q, α, a, b) coincides with the two splittings determined by the parameters $(d, p, q, \beta, a_\beta, b_\beta)$ and $(d, p, q, \gamma, a_\gamma, b_\gamma)$. Thus, in virtue of Theorem 5.2 and Proposition 5.9, we infer the desired boundedness of $\mathcal{I}^{\alpha,\sigma}$.

In the second step we let α be of the form $\alpha = (\alpha_1, \alpha_2, \alpha_{3,d})$, where $\alpha_1, \alpha_2 \in [-1/2, \infty)$ are arbitrary and $\alpha_{3,d} \in [-1/2, \infty)^{d-2}$ is a half-integer (we may assume that α_2 is not a half-integer). Then α is a convex combination,

$$\alpha = \lambda\beta + (1-\lambda)\gamma, \quad \text{with} \quad \beta = (\alpha_1, \beta_2, \alpha_{3,d}), \quad \gamma = (\alpha_1, \gamma_2, \alpha_{3,d}),$$

for some half-integers $\beta_2, \gamma_2 \geq -1/2$ and certain $\lambda \in (0,1)$. Choosing $\sigma_\beta, \sigma_\gamma$, $a_\beta, b_\beta, a_\gamma, b_\gamma$ by means of the formulas from the previous step, and using the already proved partial result, we get suitable boundedness properties of $\mathcal{I}^{\sigma_\beta,\beta}$ and $\mathcal{I}^{\sigma_\gamma,\gamma}$. Then Theorem 5.2 gives the required boundedness properties of $\mathcal{I}^{\alpha,\sigma}$.

In the next steps we repeat the procedure until the last coordinate of α is reached, in each step using a partial result obtained in the preceding step. After d steps the theorem is proved.

Finally, proving (4.7) we only need to know that $\|\ell_k^\alpha\|_{L^{q'}(\|x\|^{bq'}d\mu_\alpha)} < \infty$, but this holds in view of the inequality $b > -2(|\alpha| + d)/q'$ following from the assumptions imposed on d, p, q, α, a, b. ■

We finish this section by noting that weak type estimates in the setting of Hermite function expansions and of Laguerre expansions of convolution type are also related. In fact statements analogous to Lemma 5.7 and Corollary 5.8 are true for weak type inequalities. This allows us to transfer the weak type $(1,q)$ result from Theorem 2.3 to the Laguerre settings of half-integer orders α. Unfortunately, there seems to be no tool similar to the convexity principle that would enable us to interpolate the weak type estimates to all intermediate α.

6 Negative Powers of the Dunkl Harmonic Oscillator

In this section we take the opportunity to continue the study of spectral properties of the Dunkl harmonic oscillator in the context of a finite reflection group acting on $\mathbb{R}^d$ and isomorphic to $\mathbb{Z}_2^d = \{0,1\}^d$. This completes in some sense the investigation we began in [20] and continued in [21]. We now recall the most relevant ingredients of the setting, kindly referring the reader to [21] for a more detailed description. We keep the notation used in [20, 21], but to avoid a possible collision with notation of this paper, in several places we use the letter D to indicate the Dunkl setting.

The Dunkl harmonic oscillator

$$L_\alpha^D = -\Delta_\alpha + \|x\|^2$$

is a differential-difference operator in $\mathbb{R}^d$, where Δ_α denotes the Dunkl Laplacian in the context of a finite reflection group on $\mathbb{R}^d$ isomorphic to $\mathbb{Z}_2^d$. Here $\alpha \in [-1/2, \infty)^d$ plays the role of the multiplicity function. L_α^D considered initially on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ as the natural domain is symmetric and positive in $L^2(\mathbb{R}^d, w_\alpha)$, where

$$w_\alpha(x) = \prod_{j=1}^d |x_j|^{2\alpha_j+1}, \qquad x \in \mathbb{R}^d$$

(notice that w_α restricted to $\mathbb{R}^d_+$ is precisely the density of the measure μ_α related to the Laguerre setting from Sections 4 and 5). The associated generalized Hermite functions are tensor products

$$h_k^\alpha(x) = h_{k_1}^{\alpha_1}(x_1)\cdots h_{k_d}^{\alpha_d}(x_d), \qquad x \in \mathbb{R}^d, \quad k \in \mathbb{N}^d,$$

where $h_{k_i}^{\alpha_i}$ is defined by

$$h_{2k_i}^{\alpha_i}(x_i) = d_{2k_i,\alpha_i} e^{-x_i^2/2} L_{k_i}^{\alpha_i}(x_i^2), \qquad h_{2k_i+1}^{\alpha_i}(x_i) = d_{2k_i+1,\alpha_i} e^{-x_i^2/2} x_i L_{k_i}^{\alpha_i+1}(x_i^2),$$

with

$$d_{2k_i,\alpha_i} = (-1)^{k_i}\left(\frac{\Gamma(k_i+1)}{\Gamma(k_i+\alpha_i+1)}\right)^{1/2}, \qquad d_{2k_i+1,\alpha_i} = (-1)^{k_i}\left(\frac{\Gamma(k_i+1)}{\Gamma(k_i+\alpha_i+2)}\right)^{1/2}.$$

The system $\{h_k^\alpha : k \in \mathbb{N}^d\}$ is orthonormal and complete in $L^2(\mathbb{R}^d, w_\alpha)$, and $L_\alpha^D h_k^\alpha = (2|k| + 2|\alpha| + 2d) h_k^\alpha$.

Let $\mathcal{L}_\alpha^D$ be the self-adjoint extension of L_α^D, as defined in [20, p. 542] or [21, p. 3]. The negative power $(\mathcal{L}_\alpha^D)^{-\sigma}$, $\sigma > 0$ is given on $L^2(\mathbb{R}^d, w_\alpha)$ by the spectral series

$$\left(\mathcal{L}_\alpha^D\right)^{-\sigma} f = \sum_{k\in\mathbb{N}^d} (2|k| + 2|\alpha| + 2d)^{-\sigma} \langle f, h_k^\alpha\rangle_{dw_\alpha} h_k^\alpha.$$

Here $\langle\cdot,\cdot\rangle_{dw_\alpha}$ denotes the canonical inner product in $L^2(\mathbb{R}^d, w_\alpha)$. Clearly, $(\mathcal{L}_\alpha^D)^{-\sigma}$ is a contraction in $L^2(\mathbb{R}^d, w_\alpha)$. For $\alpha_o = (-1/2, \dots, -1/2)$ one recovers the setting of the classic harmonic oscillator: Δ_{α_o} becomes the Euclidean Laplacian, $w_{\alpha_o} \equiv 1$, and $h_k^{\alpha_o}$ are the usual Hermite functions.

The semigroup $\{e^{-t\mathcal{L}_\alpha^D} : t > 0\}$ generated by $\mathcal{L}_\alpha^D$ has the integral representation

$$e^{-t\mathcal{L}_\alpha^D} f(x) = \int_{\mathbb{R}^d} G_t^{\alpha,D}(x,y) f(y)\, dw_\alpha(y), \qquad x \in \mathbb{R}^d,$$

where the heat kernel is given by

$$G_t^{\alpha,D}(x,y) = \sum_{n=0}^{\infty} e^{-t(2n+2|\alpha|+2d)} \sum_{|k|=n} h_k^\alpha(x) h_k^\alpha(y).$$

The oscillating series defining $G_t^{\alpha,D}(x,y)$ can be summed, and the resulting expression involves the modified Bessel function I_ν, see [20] or [21]. Here, however, we shall not need that formula explicitly. To obtain $L^p - L^q$ estimates for $(\mathcal{L}_\alpha^D)^{-\sigma}$ we will make use of the estimate (cf. [20, p. 545])

$$G_t^{\alpha,D}(x,y) \lesssim G_t^\alpha(|x|,|y|), \qquad x, y \in \mathbb{R}^d, \quad t > 0, \tag{6.1}$$

where $|x| = (|x_1|, \dots, |x_d|)$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, and the kernel $G_t^\alpha(x, y)$ is understood to be given by (see (4.2))

$$G_t^\alpha(x, y) = (\sinh 2t)^{-d-|\alpha|} \exp\Big(-\frac{1}{2}\coth(2t)\big(\|x\|^2 + \|y\|^2\big)\Big) \prod_{i=1}^{d} \Big(\frac{x_i y_i}{\sinh 2t}\Big)^{-\alpha_i} I_{\alpha_i}\Big(\frac{x_i y_i}{\sinh 2t}\Big).$$

The following comment is in order at this point. The modified Bessel function of the first kind and order ν is defined by

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}.$$

Here we consider the function $z \mapsto z^\nu$ and thus also the Bessel function $I_\nu(z)$, as an analytic function defined on $\mathbb{C}\backslash\{ix : x \le 0\}$ (usually I_ν is considered as a function on $\mathbb{C}$ cut along the half-line $(-\infty, 0]$). Hence $z^{-\nu} I_\nu(z)$ is readily extended to an entire function, and $s^{-\nu} I_\nu(s)$, as a function on $\mathbb{R}$, is real, even, positive, and smooth for any $\nu > -1$; see [16, Chapter 5]. In particular we see that the function $G_t^\alpha(|x|, |y|)$ is well defined by the above formula also at points where $x_i = 0$ or $y_i = 0$ for some $i = 1, \dots, d$. For future reference it is convenient to call such points *critical* ($x \in \mathbb{R}^d$ is critical if and only if $x_i = 0$ for some $i = 1, \dots, d$). Notice that critical points in $\mathbb{R}^d$ form a set of null measure.

We now define the potential kernel

$$\mathcal{K}_D^{\alpha,\sigma}(x, y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty G_t^{\alpha,D}(x, y) t^{\sigma-1}\, dt, \qquad x, y \in \mathbb{R}^d,$$

and the potential operator (to be equal to $(\mathcal{L}_\alpha^D)^{-\sigma}$ on $L^2(\mathbb{R}^d)$)

$$\mathcal{I}_D^{\alpha,\sigma} f(x) = \int_{\mathbb{R}^d} \mathcal{K}_D^{\alpha,\sigma}(x, y) f(y)\, dw_\alpha(y)$$

with the natural domain $\operatorname{Dom} \mathcal{I}_D^{\alpha,\sigma}$. Since by (6.1)

$$\mathcal{K}_D^{\alpha,\sigma}(x, y) \lesssim \mathcal{K}^{\alpha,\sigma}(|x|, |y|), \qquad x, y \in \mathbb{R}^d \tag{6.2}$$

(for the critical points $\mathcal{K}^{\alpha,\sigma}(|x|, |y|)$ is defined by integrating $G_t^\alpha(|x|, |y|)$), it follows from remarks concerning analogous properties of the Laguerre potential kernel $\mathcal{K}^{\alpha,\sigma}(x, y)$ that $\mathcal{K}_D^{\alpha,\sigma}(x, y)$ is well defined (the relevant integral is convergent) whenever $|x| \neq |y|$ and x, y are not critical. The fact that $\mathcal{K}_D^{\alpha;\sigma}(x, y)$ is well defined at the critical points, still for $|x| \neq |y|$, can be easily seen by analysis of the kernel $G_t^\alpha(|x|, |y|)$, taking into account the product structure of the kernel, the above comment concerning non-critical points, and the fact that the function $s \mapsto s^{-\nu} I_\nu(s)$, $\nu > -1$ has a finite value at $s = 0$. We remark that a more detailed analysis reveals that $\mathcal{K}_D^{\alpha;\sigma}(x, y)$ is well defined for all $x, y \in \mathbb{R}^d$ provided that $\sigma > d + |\alpha|$.

As consequences of Theorems 4.1 and 4.2 we get the following results in the Dunkl setting.

Theorem 6.1 *Assume that* $\alpha \in [-1/2, \infty)^d$. *Let* $\sigma > 0$ *and* $1 \le p < \infty$, $1 \le q < \infty$. *If* $\sigma \ge |\alpha| + d$, *then*

$$\|\mathcal{I}_D^{\alpha,\sigma} f\|_{L^q(dw_\alpha)} \lesssim \|f\|_{L^p(dw_\alpha)}, \qquad f \in L^p(\mathbb{R}^d, dw_\alpha). \tag{6.3}$$

If $0 < \sigma < |\alpha| + d$, *then* (6.3) *holds under the additional condition*

$$\frac{1}{p} - \frac{\sigma}{|\alpha| + d} \le \frac{1}{q} < \frac{1}{p} + \frac{\sigma}{|\alpha| + d},$$

with exclusion of the case when $p = 1$ *and* $q = \frac{|\alpha|+d}{|\alpha|+d-\sigma}$. *Moreover, under the assumptions ensuring* (6.3),

$$\langle \mathcal{I}_D^{\alpha,\sigma} f, h_k^\alpha \rangle_{dw_\alpha} = (2|k| + 2|\alpha| + 2d)^{-\sigma} \langle f, h_k^\alpha \rangle_{dw_\alpha}, \qquad f \in L^p(\mathbb{R}^d, dw_\alpha).$$

Theorem 6.2 *Assume that* $\alpha \in [-1/2, \infty)^d$. *Let* $\sigma > 0$, $1 < p \le q < \infty$ *and*

$$a < (2|\alpha| + 2d)/p', \qquad b < (2|\alpha| + 2d)/q, \qquad a + b \ge 0.$$

(i) *If* $\sigma \ge |\alpha| + d$, *then* $\mathcal{I}_D^{\alpha,\sigma}$ *maps* $L^p(\mathbb{R}^d, \|x\|^{ap} dw_\alpha)$ *into* $L^q(\mathbb{R}^d, \|x\|^{-bq} dw_\alpha)$ *boundedly.*

(ii) *If* $\sigma < |\alpha| + d$, *then the same boundedness holds under the additional condition*

$$\frac{1}{q} \ge \frac{1}{p} - \frac{2\sigma - a - b}{2|\alpha| + 2d}.$$

Moreover, under the assumptions ensuring boundedness of $\mathcal{I}_D^{\alpha,\sigma}$ *from* $L^p(\mathbb{R}^d, \|x\|^{ap} dw_\alpha)$ *into* $L^q(\mathbb{R}^d, \|x\|^{-bq} dw_\alpha)$,

$$\langle \mathcal{I}_D^{\alpha,\sigma} f, h_k^\alpha \rangle_{dw_\alpha} = (2|k| + 2|\alpha| + 2d)^{-\sigma} \langle f, h_k^\alpha \rangle_{dw_\alpha}, \qquad f \in L^p(\mathbb{R}^d, \|x\|^{ap} dw_\alpha). \tag{6.4}$$

Note that Theorem 6.1 implicitly asserts the inclusion $L^p(\mathbb{R}^d, dw_\alpha) \subset \operatorname{Dom} \mathcal{I}_D^{\alpha,\sigma}$, $1 \le p < \infty$. Similarly, Theorem 6.2 asserts the inclusion $L^p(\mathbb{R}^d, \|x\|^{ap} dw_\alpha) \subset \operatorname{Dom} \mathcal{I}_D^{\alpha,\sigma}$ in all the cases when weighted $L^p - L^q$ boundedness holds. Notice also that for $\alpha_o = (-1/2, \dots, -1/2)$ Theorems 6.2 and 2.5 coincide, as it should be, since for $\alpha = \alpha_o$ both settings coincide. Thus the present result generalizes Theorem 2.5. Moreover, Theorem 6.2 can be regarded as an extension of Theorem 4.2, since the Laguerre potential operator $\mathcal{I}^{\alpha,\sigma}$ applied to a function f on $\mathbb{R}^d_+$ corresponds to $\mathcal{I}_D^{\alpha,\sigma}$ applied to the $\mathbb{Z}_2^d$-symmetric (reflection invariant) extension of f to $\mathbb{R}^d$. Analogous comments concern relations between Theorem 6.1 and Theorems 2.3 and 4.1.

We now give a proof of Theorem 6.2. Proving Theorem 6.1 relies on exactly the same arguments and Theorem 4.1, hence we omit the details.

Proof of Theorem 6.2 It is sufficient to consider only nonnegative functions on $\mathbb{R}^d$; let f be such a function. Given $\varepsilon \in \mathbb{Z}_2^d$, denote by f_ε the ε-reflection of f, that is,

$$f_\varepsilon(y) = f\big((-1)^{\varepsilon_1} y_1, \dots, (-1)^{\varepsilon_d} y_d\big), \qquad y \in \mathbb{R}^d.$$

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Noticing that the density w_α is reflection-invariant and using (6.2) we get

$$\begin{aligned}\mathcal{I}_D^{\alpha,\sigma} f(x) &\lesssim \int_{\mathbb{R}^d} \mathcal{K}^{\alpha,\sigma}(|x|,|y|) f(y)\, dw_\alpha(y) \\ &= \sum_{\varepsilon\in\mathbb{Z}_2^d} \int_{\mathbb{R}_+^d} \mathcal{K}^{\alpha,\sigma}(|x|,y) f_\varepsilon(y)\, d\mu_\alpha(y) = \sum_{\varepsilon\in\mathbb{Z}_2^d} \mathcal{I}^{\alpha,\sigma} f_\varepsilon(|x|),\end{aligned}$$

where $x \in \mathbb{R}^d$ is assumed to be non-critical and f_ε in the last occurrence is understood as the restriction of f_ε to $\mathbb{R}_+^d$. Since power weights in $\mathbb{R}^d$ as well as the function $\mathcal{I}^{\alpha,\sigma} f_\varepsilon(|\cdot|)$ are reflection-invariant, it follows that

$$\|\mathcal{I}_D^{\alpha,\sigma} f\|_{L^q(\mathbb{R}^d, \|x\|^{-bq} dw_\alpha)} \lesssim \sum_{\varepsilon\in\mathbb{Z}_2^d} \|\mathcal{I}^{\alpha,\sigma} f_\varepsilon\|_{L^q(\mathbb{R}_+^d, \|x\|^{-bq} d\mu_\alpha)}.$$

Observing now that

$$\sum_{\varepsilon\in\mathbb{Z}_2^d} \|f_\varepsilon\|_{L^p(\mathbb{R}_+^d, \|x\|^{ap} d\mu_\alpha)} \simeq \|f\|_{L^p(\mathbb{R}^d, \|x\|^{ap} dw_\alpha)},$$

we see that the first part of Theorem 6.2 is readily deduced from Theorem 4.2.

To prove (6.4) we argue as in the other settings. Then we only need to know that $\|h_k^\alpha\|_{L^{q'}(\|x\|^{bq'} dw_\alpha)} < \infty$, but this follows from the explicit form of the functions h_k^α and the assumptions imposed on d, p, q, α, a, b. ■

As a consequence of Theorem 6.1 we state the following corollary.

Corollary 6.3 *Let $\alpha \in [-1/2,\infty)^d$, $\sigma > 0$, $1 \le p < \infty$, $1 \le q < \infty$, and (p,q) be such a pair that (6.3) holds. Then $(\mathcal{L}_\alpha^D)^{-\sigma}$ extends to a bounded operator from $L^p(\mathbb{R}^d, dw_\alpha)$ to $L^q(\mathbb{R}^d, dw_\alpha)$. Moreover, denoting this extension by $(\mathcal{L}_\alpha^D)_{pq}^{-\sigma}$ for any $k \in \mathbb{N}^d$ we have*

$$\langle (\mathcal{L}_\alpha^D)_{pq}^{-\sigma} f, h_k^\alpha \rangle_{dw_\alpha} = (2|k| + 2|\alpha| + 2d)^{-\sigma} \langle f, h_k^\alpha \rangle_{dw_\alpha}, \qquad f \in L^p(\mathbb{R}^d, dw_\alpha).$$

In particular, the $L^p - L^q$ extension of $(\mathcal{L}_\alpha^D)^{-\sigma}$ coincides with that of $\mathcal{I}_D^{\alpha,\sigma}$.

Proof We repeat the reasoning from the proof of Corollary 2.4. The only point that requires further comment is the application of Fubini's theorem. To see that it is possible it is enough to verify that for $x \in \mathbb{R}^d$ and each $k \in \mathbb{N}^d$

$$\int_{\mathbb{R}^d} \mathcal{K}_D^{\alpha,\sigma}(x,y) |h_k^\alpha(y)|\, dw_\alpha(y) < \infty. \tag{6.5}$$

Actually, here we can exclude the set of critical x (which is of null measure), since it is sufficient to verify the identity $(\mathcal{L}_\alpha^D)^{-\sigma} h_k^\alpha = \mathcal{I}_D^{\alpha,\sigma} h_k^\alpha$ in the L^2 sense only. But when x is not a critical point, (6.5) holds because $h_k^\alpha \in L^\infty(\mathbb{R}^d)$ and $\mathcal{K}_D^{\alpha,\sigma}(x,\cdot) \in L^1(\mathbb{R}^d, dw_\alpha)$ (the first fact is obvious by the definition of h_k^α, and the second one follows taking into account (6.2) and that $\mathcal{K}^{\alpha,\sigma}(|x|,\cdot) \in L^1(\mathbb{R}_+^d, d\mu_\alpha)$, as verified in the proof of Corollary 4.3).

We remark that $(\mathcal{L}_\alpha^D)^{-\sigma} h_k^\alpha(x) = \mathcal{I}_D^{\alpha,\sigma} h_k^\alpha(x)$ holds also for all critical x, but showing this requires some extra arguments. ■

7 Final Observations and Remarks

It is interesting to observe that in dimension one the weighted results from Theorem 4.2 imply new weighted results for the potentials $\mathcal{I}_H^{\alpha,\sigma}$, which are different from those in Theorem 3.3. Here are the details. Recall that the operator S_α given by $S_\alpha f(x) = x^{\alpha+1/2} f(x)$ intertwines the potential operators in both settings

$$\mathcal{I}_H^{\alpha,\sigma} \circ S_\alpha = S_\alpha \circ \mathcal{I}^{\alpha,\sigma},$$

and the $L^p(U d\mu_\alpha) \to L^q(V d\mu_\alpha)$ estimate for $\mathcal{I}^{\alpha,\sigma}$ is equivalent to the $L^p(\widetilde{U}) \to L^q(\widetilde{V})$ estimate for $\mathcal{I}_H^{\alpha,\sigma}$, where

$$\widetilde{U}(x) = U(x)x^{-(\alpha+1/2)p+2\alpha+1}, \qquad \widetilde{V}(x) = V(x)x^{-(\alpha+1/2)q+2\alpha+1}.$$

This allows us to translate the results of Theorem 4.2 (specified to $d = 1$) to the setting of Laguerre expansions of Hermite type directly. Thus we get the following proposition.

Proposition 7.1 *Assume that $d = 1$ and $\alpha \geq -1/2$. Let $\sigma > 0$, $1 < p \leq q < \infty$ and*

$$A < \tfrac{1}{p'} + \alpha + \tfrac{1}{2}, \qquad B < \tfrac{1}{q} + \alpha + \tfrac{1}{2}, \qquad A + B \geq (2\alpha + 1)\big(\tfrac{1}{p} - \tfrac{1}{q}\big).$$

(i) *If $\sigma \geq \alpha + 1$, then $\mathcal{I}_H^{\alpha,\sigma}$ maps boundedly $L^p(\mathbb{R}_+, x^{Ap})$ into $L^q(\mathbb{R}_+, x^{-Bq})$.*
(ii) *If $\sigma < \alpha + 1$, then the same boundedness is true under the additional condition*

$$\tfrac{1}{q} \geq \tfrac{1}{p} + A + B - 2\sigma.$$

Notice that here the unweighted case $A = B = 0$ is not admitted when $\alpha > -1/2$ and $p < q$, whereas it is covered by Theorem 3.3. Also, the splittings into cases (i) and (ii) in Proposition 7.1 and Theorem 3.3 are different. In fact, by the abovementioned equivalence, Theorems 3.3 and 4.2 are independent in the sense that neither of them follows from the other one. In particular, at least in dimension one, Theorem 3.3 implies some weighted $L^p - L^q$ estimates for $\mathcal{I}^{\alpha,\sigma}$ that are not covered by Theorem 4.2. This indicates that Theorem 4.2 is not optimal in the sense of admissible power weights.

In [14] Kanjin and Sato proved a fractional integration theorem for one-dimensional expansions with respect to the so-called standard system of Laguerre functions $\{\mathcal{L}_k^\alpha\}_{k\geq 0}$, where

$$\mathcal{L}_k^\alpha(x) = \Big(\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\Big)^{1/2} L_k^\alpha(x) x^{\alpha/2} e^{-x/2}.$$

This system is orthonormal and complete in $L^2(\mathbb{R}_+)$. The fractional integral operator I_σ^α considered in [14] is defined by

$$I_\sigma^\alpha f \sim \sum_{k=1}^{\infty} \frac{1}{k^\sigma} a_k^\alpha(f) \mathcal{L}_k^\alpha, \tag{7.1}$$

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where $a_k^\alpha(f) = \int_0^\infty f(x)\mathcal{L}_k^\alpha(x)\,dx$ are the Fourier–Laguerre coefficients of f (provided they exist), and the sign $\sim$ means that the coefficients of a function on the left-hand side of (7.1) coincide with those appearing on the right-hand side.

The main result of [14] says that, given $0 < \sigma < 1$ and $\alpha > -1$, the inequality

$$\|I_\sigma^\alpha f\|_q \lesssim \|f\|_p$$

holds provided that $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \sigma$, with the additional restriction $(1 + \frac{\alpha}{2})^{-1} < p < q < -2/\alpha$ in the case when $-1 < \alpha < 0$ (this restriction in particular guarantees existence of the coefficients $a_k^\alpha(f)$ and $a_k^\alpha(g)$ for every $f \in L^p(\mathbb{R}_+)$ and every $g \in L^q(\mathbb{R}_+)$). We shall briefly see that for $\alpha \geq -1/2$ our present results extend in several directions the result of Kanjin and Sato. To this end we always assume that $d = 1$.

Each $\mathcal{L}_k^\alpha$ is an eigenfunction of the differential operator

$$\mathbb{L}_\alpha = -x\frac{d^2}{dx^2} - \frac{d}{dx} + \frac{x}{4} + \frac{\alpha^2}{4x},$$

which is formally symmetric and positive in $L^2(\mathbb{R}_+)$, and the corresponding eigenvalue is $k + \alpha/2 + 1/2$. Moreover, $\mathbb{L}_\alpha$ has the natural self-adjoint extension in $L^2(\mathbb{R}_+)$ for which the spectral decomposition is given by the $\mathcal{L}_k^\alpha$. We consider the potential operator

$$\mathcal{I}_S^{\alpha,\sigma} f(x) = \int_0^\infty \mathcal{K}_S^{\alpha,\sigma}(x,y) f(y)\,dy,$$

where, as in the other settings, the potential kernel is obtained by integrating the associated heat kernel

$$\mathcal{K}_S^{\alpha,\sigma}(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty G_t^{\alpha,S}(x,y) t^{\sigma-1}\,dt.$$

Since $\varphi_k^\alpha(x) = \sqrt{2x}\mathcal{L}_k^\alpha(x^2)$ and the eigenvalues in both settings coincide up to the constant factor 4, it follows that $G_t^{\alpha,H}(x,y) = 2\sqrt{xy}G_{4t}^{\alpha,S}(x^2,y^2)$ and, consequently, $\mathcal{K}_H^{\alpha,\sigma}(x,y) = 2^{-2\sigma+1}\sqrt{xy}\mathcal{K}_S^{\alpha,\sigma}(x^2,y^2)$. This shows that the potential operators in the settings of standard Laguerre expansions and Laguerre expansions of Hermite type are related, namely $P \circ \mathcal{I}_S^{\alpha,\sigma} = 2^{2\sigma}\mathcal{I}_H^{\alpha,\sigma} \circ P$, where the linking transformation is given by $Pf(x) = \sqrt{x}f(x^2)$, $x > 0$. We now see that the $L^p(U) \to L^q(V)$ estimate for $\mathcal{I}_S^{\alpha,\sigma}$ is equivalent to the $L^p(\widehat{U}) \to L^q(\widehat{V})$ estimate for $\mathcal{I}_H^{\alpha,\sigma}$, where

$$\widehat{U}(x) = U(x^2)x^{1-p/2}, \qquad \widehat{V}(x) = V(x^2)x^{1-q/2}.$$

This allows us to conclude immediately the following result from Proposition 7.1.

Proposition 7.2 *Assume that $d = 1$ and $\alpha \geq -1/2$. Let $\sigma > 0$, $1 < p \leq q < \infty$, and*

$$A < \tfrac{1}{p'} + \tfrac{\alpha}{2}, \qquad B < \tfrac{1}{q} + \tfrac{\alpha}{2}, \qquad A + B \geq \alpha\left(\tfrac{1}{p} - \tfrac{1}{q}\right).$$

(i) *If* $\sigma \geq \alpha + 1$, *then* $\mathcal{I}_S^{\alpha,\sigma}$ *maps boundedly* $L^p(\mathbb{R}_+, x^{Ap})$ *into* $L^q(\mathbb{R}_+, x^{-Bq})$.
(ii) *If* $\sigma < \alpha + 1$, *then the same boundedness is true under the additional condition*

$$\frac{1}{q} \geq \frac{1}{p} + A + B - \sigma. \tag{7.2}$$

Taking into account the fact that the kernel $\mathcal{K}_S^{\alpha,\sigma}(x, y)$ is decreasing with respect to the index $\alpha \geq 0$ (this follows from the analogous monotonicity of the involved Bessel function I_α; see for instance the proof of [18, Proposition 2.1]), we get the following corollary.

Corollary 7.3 *Assume that* $d = 1$ *and* $\alpha \geq 0$. *Let* $\sigma > 0$, $1 < p \leq q < \infty$, *and*

$$A < \frac{1}{p'}, \qquad B < \frac{1}{q}, \qquad A + B \geq 0.$$

(i) *If* $\sigma \geq 1$, *then* $\mathcal{I}_S^{\alpha,\sigma}$ *maps boundedly* $L^p(\mathbb{R}_+, x^{Ap})$ *into* $L^q(\mathbb{R}_+, x^{-Bq})$.
(ii) *If* $\sigma < 1$, *then the same boundedness is true under condition* (7.2).

Now, with the aid of a suitable multiplier theorem (for example, the result of Długosz [6] for integer α combined with Kanjin's transplantation theorem [13] gives sufficiently general multiplier theorem for all $\alpha > -1$), the aforementioned result of Kanjin and Sato can be recovered as a special case of Proposition 7.2 for $\alpha \in [-1/2, 0]$ or Corollary 7.3 when $\alpha \geq 0$. On the other hand, it is perhaps interesting to note that this result does not follow from Theorem 3.3. We remark that the result of Kanjin and Sato for the full range of $\alpha \in (-1, \infty)$ can be recovered by specifying Proposition 7.2 to $\alpha = 0 = A = B$, using a multiplier theorem, and then applying Kanjin's transplantation theorem.

Another comment concerns the very recent paper by Bongioanni and Torrea [3]. The authors obtain there, among many other results, power weighted L^p-boundedness of potential operators related to one-dimensional Laguerre function systems; see [3, Theorem 7, Proposition 2] and remarks closing [3, Section 5]. Our present results contain those of [3] on mapping properties of Laguerre potentials as special cases, at least when $\alpha \geq -1/2$. Indeed, for such α [3, Theorem 7] follows by specifying Proposition 7.2 to $p = q$ and $B = -A$.

Still another comment explains, to some extent, why $L^p - L^q$ boundedness of the Hermite potential operator $\mathcal{I}^\sigma$ should be expected for all $p, q \in [1, \infty]$, provided that σ is large. This "phenomenon" should be linked with the ultracontractivity property of the Hermite semigroup $\{e^{-t\mathcal{H}}\}_{t>0}$: given $p, q \in [1, \infty]$, $e^{-t\mathcal{H}}$ maps $L^p(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$ boundedly. Thus we can expect a similar property for the average $\mathcal{H}^{-\sigma} = \Gamma(\sigma)^{-1} \int_0^\infty e^{-t\mathcal{H}} t^{\sigma-1}\, dt$, of course having a hope for a good control of the corresponding operator norms of $e^{-t\mathcal{H}}$. This indeed happens. For $t \geq 1$ a crude estimate based on bounds of L^p-norms of the Hermite functions gives $\|e^{-t\mathcal{H}}\|_{L^p \to L^q} \leq C_{pq} e^{-td}$, $t \geq 1$, $p, q \in [1, \infty]$ (see [25, Remark 2.11]), hence $\int_1^\infty \|e^{-t\mathcal{H}}\|_{L^p \to L^q} t^{\sigma-1}\, dt < \infty$ follows for $\sigma > 0$. On the other hand, we will show that for $p, q \in [1, \infty]$,

$$\|e^{-t\mathcal{H}}\|_{L^p \to L^q} \leq C_{pq} t^{-d/2}, \qquad 0 < t \leq 1, \tag{7.3}$$

hence $\int_0^1 \|e^{-t\mathcal{H}}\|_{L^p\to L^q} t^{\sigma-1}\,dt < \infty$ for $\sigma > d/2$, and thus $\|\mathcal{H}^{-\sigma}\|_{L^p\to L^q} < \infty$ for $\sigma > d/2$. To verify (7.3) it is sufficient to consider only the cases $p = \infty, q = 1$ and $p = 1, q = \infty$. In the cases $p = q = 1$ and $p = q = \infty$, (7.3) is trivially satisfied ($\{e^{-t\mathcal{H}}\}_{t>0}$ is a semigroup of contractions on $L^p(\mathbb{R}^d)$, $1 \le p \le \infty$; see [25, Remark 2.10]), and then interpolation does the job. Checking (7.3) is easy for $p = 1, q = \infty$: when estimating the relevant operator norm it suffices to use the estimate $G_t(x, y) \le Ct^{-d/2}$. Proving (7.3) for $p = \infty, q = 1$ reduces to checking that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \exp\Big(-\frac{1}{8}t\|x+y\|^2 - \frac{1}{4t}\|x-y\|^2\Big)\,dx\,dy \le C, \qquad 0 < t \le 1.$$

Since the variables may be separated, it is sufficient to consider the case $d = 1$. Then the desired estimate follows by the change of variables $u = x + y$, $v = x - y$.

Finally, we comment on relations between the classical Bessel potentials in $\mathbb{R}^d$ and the potential operators studied in this paper and point out important consequences of these relations. The Bessel potentials J^σ are associated with the powers $(\mathrm{Id} - \Delta)^{-\sigma/2}$ analogous to the way the Riesz potentials I^σ correspond to $(-\Delta)^{-\sigma/2}$; see [1]. For $\sigma > 0$ they are given as convolutions $J^\sigma = \mathcal{G}^\sigma * f$ with the radial kernel

$$\mathcal{G}^\sigma(x) = K_{\frac{d-\sigma}{2}}\big(\|x\|\big)\,\|x\|^{\frac{\sigma-d}{2}}, \qquad x \in \mathbb{R}^d,$$

where K_β is the modified Bessel function of the third kind and order β, usually referred to as McDonald's function; cf. [16, Chapter 5]. The kernels $\mathcal{G}^\sigma$ share many properties of the Riesz kernels $\|x\|^{\sigma-d}$ (this in particular concerns positiveness), but, in contrast with the Riesz kernels, they decay exponentially at infinity. This last feature makes a significant difference in mapping properties of the two kinds of potentials. By basic properties of McDonald's function (see [1, Ch. 2, Sec. 3]) it is easily seen that $\mathcal{G}^{2\sigma}$ behaves very similarly to the majorization kernel K^σ used in the Hermite setting in Section 2. In fact, given $\sigma > 0$, K^σ can be controlled pointwise by $\mathcal{G}^{2\sigma}$; see [1, Ch. 2, Sec. 4]. Thus, for nonnegative functions f in $\mathbb{R}^d$,

$$\mathcal{I}^\sigma f(x) \lesssim J^{2\sigma} f(x) \lesssim I^{2\sigma} f(x), \qquad x \in \mathbb{R}^d,$$

and therefore any weighted $L^p - L^q$ boundedness of $J^{2\sigma}$ (or $I^{2\sigma}$) is inherited by the Hermite potential operator $\mathcal{I}^\sigma$. Clearly, using the methods presented in this paper, there are consequences of this concerning (a little less explicitly) the Laguerre potentials $\mathcal{I}_H^{\alpha,\sigma}$, $\mathcal{I}^{\alpha,\sigma}$, and $\mathcal{I}_S^{\alpha,\sigma}$, and the Dunkl potentials.

Note that in a similar way, instead of the negative power $\mathcal{H}^{-\sigma}$, one could wish to consider $(I + \mathcal{H}^2)^{-\sigma/2}$. It is interesting to mention (as it was pointed out in [5]) that the corresponding multiplier multi-sequence $(1 + \lambda_k^2)^{-\sigma/2}$ can be realized as a Laplace transform type multiplier for the measure ν with density $\psi(t) = c_\sigma e^{-t(\sigma+1)/2} J_{(\sigma-1)/2} t^{(\sigma-1)/2}$, where c_σ is a constant and J_β is the Bessel function of the first kind and order β. This density satisfies the growth properties listed in the remark formulated at the end of Section 2, hence consequences of this fact are the same as in the remark. Similar observations apply to the operators $(I + (\mathcal{L}_\alpha^H)^2)^{-\sigma/2}$, $(I + (\mathcal{L}_\alpha)^2)^{-\sigma/2}$, and $(I + (\mathcal{L}_\alpha^D)^2)^{-\sigma/2}$.

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