Canad. Math. Bull. Vol. **40** (4), 1997 pp. 448–455

STABLE INDEX PAIRS FOR DISCRETE DYNAMICAL SYSTEMS

TOMASZ KACZYNSKI AND MARIAN MROZEK

ABSTRACT. A new shorter proof of the existence of index pairs for discrete dynamical systems is given. Moreover, the index pairs defined in that proof are stable with respect to small perturbations of the generating map. The existence of stable index pairs was previously known in the case of diffeomorphisms and flows generated by smooth vector fields but it was an open question in the general discrete case.

1. **Introduction.** Index pairs constitute a basic tool in the construction of the Conley index, which is a topological invariant used in qualitative studies of dynamical systems. The original construction of the Conley index by Charles Conley and his students (*cf.* [1]) concerned flows but in the recent years it was generalized to discrete dynamical systems [6, 7] and discrete multivalued systems [2]. This opened the way to many new applications, in particular to a computer assisted proof of chaos in the Lorenz equations [3, 4, 5].

The Conley index is associated with an isolated invariant set, *i.e.* an invariant set which is maximal in some its compact neighborhood called an isolating neighborhood. The construction of the Conley index for a discrete dynamical system consists of two steps that differ by the techniques employed. The first step, based on pure set-theoretical topology, is to construct a pair of subsets (P_1, P_2) of the isolating neighborhood, called an index pair. The second step consists in extracting algebraic information from the topology of the index pair by means of algebraic topology tools and certain purely algebraic functors.

The fundamental fact in the Conley index theory is that both the isolating neighborhood and the Conley index are preserved under a small perturbation. This is almost straightforward for isolating neighborhoods but required a rather complicated proof until recently, because the index pairs need not be stable under perturbations in general.

In a recent paper [2], the definitions of isolating neighbourhood, index pair, and the Conley index, together with the proof of homotopy and additivity property of the index, were generalized for discrete multivalued dynamical systems.

The main motivation of that paper was to provide a theoretical background of numerical computation used by Mischaikow and Mrozek [3] in their computer assisted proof of chaos in the Lorenz system, where finitely represented multivalued maps appear as a

Received by the editors May 23, 1995; revised November 28, 1995.
The first author was supported by a grant from NSERC of Canada.
The second author was partially supported by KBN grant no. 0449/P3/94/06.
AMS subject classification: Primary: 54H20; secondary: 54C60, 34C35.

tool for discretisation. However, the consequences of that generalization surpassed the authors' initial expectations. The multivalued mapping approach not only is a convenient model for certain numerics but also permits to simplify certain proofs and to obtain new results concerning single-valued continuous maps.

The aim of this short report is twofold: First, we provide a new proof of the existence of index pairs for continuous maps which is shorter and, as we believe, more intuitive than the previous ones given in [6] and [2]. Second, the index pairs we get in the proof are stable under small perturbations of the map generating the dynamical system. The existence of stable index pairs was previously known in the case of diffeomorphisms and flows generated by C^1 vector fields, *cf.* [7] and [10], but is was an open question in the case of a general discrete dynamical system (*i.e.* iterates of a homeomorphism) and a fortiori, in the case of a discrete semidynamical system (*i.e.* positive iteratives of a continuous map).

We refer the reader to [9] for another interesting application of multivalued dynamical systems to single-valued ones. It is proved there that, in the case of a dynamical system on $\mathbb{R}^n$, there always exist index pairs $P = (P_1, P_2)$ such that P_i are finite polyhedra.

2. **Basic concepts.** In this section, we recall from [2] basic definitions.

Let us recall that a mapping $F: X \longrightarrow \mathcal{P}(Y)$, where X, Y are metric spaces and $\mathcal{P}(Y)$ is the set of all subsets of Y, is called *upper semicontinuous* (*usc*) if $F^{-1}(A) := \{x \in X : F(x) \cap A \neq \emptyset\}$ is closed for any closed $A \subset Y$ or, equivalently, if the set $\{x \in X : F(x) \subset U\}$ is open for any open $U \subset Y$. If $A \subset X$, we denote by $F(A)$ the union $\bigcup\{F(x) : x \in A\} \subset Y$ and not a subset of $\mathcal{P}(Y)$. Given a positive integer n, F^n denotes the n-th superposition of F defined recursively by $F^n(x) := F\big(F^{n-1}(x)\big)$. The *graph of* F is the set $G(F) := \{(x, y) \in X \times Y : y \in F(x)\}$. Let us recall that any usc mapping with compact values has a closed graph and it sends compact sets to compact sets. If $F: X \longrightarrow \mathcal{P}(Y)$ is usc then the set $D(F) := \{x \in X : F(x) \neq \emptyset\}$ (called the *effective domain of* F) is closed.

Let now (X, d) be a given locally compact metric space. If $A \subset X$, we denote the boundary of A by $\operatorname{bd} A$, its interior by $\operatorname{int} A$, and we let $B_\varepsilon(A) := \{x \in X : d(x, A) < \varepsilon\}, \varepsilon > 0$. We denote the sets of all integers, nonnegative integers, and nonpositive integers by $\mathbb{Z}, \mathbb{Z}^+$, and $\mathbb{Z}^-$, respectively. By an *interval* we mean an interval in $\mathbb{Z}$, *i.e.* an intersection of a closed real interval with $\mathbb{Z}$.

DEFINITION 2.1. An usc mapping $F: X \times \mathbb{Z} \longrightarrow \mathcal{P}(X)$ with compact values is called a *discrete multivalued dynamical system* (*dmds*) if the following conditions are satisfied:

(i) For all $x \in X$, $F(x, 0) = \{x\}$;

(ii) For all $n, m \in \mathbb{Z}$ with $nm \geq 0$ and all $x \in X$, $F\big(F(x, n), m\big) = F(x, n + m)$;

(iii) For all $x, y \in X$, $y \in F(x, -1) \Longleftrightarrow x \in F(y, 1)$.

We use the notation $F^n(x) := F(x, n)$. Note that F^n coincides with a superposition of $F^1: X \longrightarrow \mathcal{P}(X)$ or its inverse $(F^1)^{-1}$. This justifies that we will call F^1 the *generator* of the dmds F. We will usually denote the generator simply by F and identify it with the dmds. This will cause no misunderstanding unless a value of F is considered but in that case the meaning will be clear from the number of arguments.

https://doi.org/10.4153/CMB-1997-053-2 Published online by Cambridge University Press

We do not assume that the values of F are non-empty. Thus, the definition of dmds extends, to the multivalued case, not only the definition of a discrete dynamical system $f: X \times \mathbb{Z} \longrightarrow X$ (generated by a homeomorphism) but also the definition of a discrete semidynamical system $f: X \times \mathbb{Z}^+ \longrightarrow X$ (generated by a continuous map) since one may define negative-time values by the property (iii). More precisely, an usc map $F: X \longrightarrow P(X)$ with compact values generates a dmds if and only if it is *proper*, *i.e.* $F^{-1}(K)$ is compact for any compact $K \subset X$. If X is compact (and problems are often reduced to that case) then any continuous map $f: X \longrightarrow X$ generates a dmds by $F(x, 1) = \{f(x)\}, F(x, -1) = f^{-1}(x)$, for $x \in X$.

DEFINITION 2.2. Let I be an interval in $\mathbb{Z}$ with $0 \in I$. A single valued mapping $\sigma: I \longrightarrow X$ is a *solution for F through* $x \in X$ if $\sigma(n+1) \in F\big(\sigma(n)\big)$ for all $n, n+1 \in I$, and $\sigma(0) = x$.

Note that if $\sigma: I \longrightarrow X$ is a solution for F then $\sigma(n) \in F^n\big(\sigma(0)\big)$ for all $n \in I$ (The proof is straightforward by induction on m and k, where $I = [-k, m], k, m \in \mathbb{Z}^+$). The existence of a solution through x forces $F^n(x)$ to be nonempty for $n \in I$. Note that if $f: X \longrightarrow X$ is continuous and proper, and $F(x) := \overline{B_\sigma\big(F(x)\big)}$ then the definition of a solution $\sigma: \mathbb{Z} \longrightarrow X$ for F coincides with the definition of a δ-pseudo trajectory of f, *cf.* [8].

Given a subset $N \subset X$, we introduce the following notation:

$\operatorname{inv}^+ N := \{x \in N : \text{there exists a solution } \sigma: \mathbb{Z}^+ \longrightarrow N \text{ for } F \text{ through } x\}$

$\operatorname{inv}^- N := \{x \in N : \text{there exists a solution } \sigma: \mathbb{Z}^- \longrightarrow N \text{ for } F \text{ through } x\}$

$\operatorname{inv} N := \{x \in N : \text{there exists a solution } \sigma: \mathbb{Z} \longrightarrow N \text{ for } F \text{ through } x\}$

By (i) we have: $\operatorname{inv} N = \operatorname{inv}^+ N \cap \operatorname{inv}^- N$. It was proved in [2] that the sets $\operatorname{inv}^{(\pm)} N$ are compact for any compact N.

Let $\operatorname{diam}_N F := \sup\{\operatorname{diam} F(x) : x \in N\}$ and $\operatorname{dist}(A, B) := \min\{d(x, y) : x \in A, y \in B\}, A, B \subset X$.

DEFINITION 2.3. A compact subset $N \subset X$ is called

(a) an *isolating neighbourhood for F* if

$$B_{\operatorname{diam}_N F}(\operatorname{inv} N) \subset \operatorname{int} N \tag{2.1}$$

or equivalently

$$\operatorname{dist}(\operatorname{inv} N, \operatorname{bd} N) > \operatorname{diam}_N F$$

(b) an *isolating block for F* if

$$B_{\operatorname{diam}_N F}\big(F^{-1}(N) \cap N \cap F(N)\big) \subset \operatorname{int} N \tag{2.2}$$

or equivalently

$$\operatorname{dist}\big(F^{-1}(N) \cap N \cap F(N), \operatorname{bd} N\big) > \operatorname{diam}_N F$$

A straightforward verification shows that (2.2) implies (2.1), *i.e.* every isolating block is an isolating neighbourhood but not necessarily vice versa. The importance of the notion of isolating block lies in the fact that it may be verified even if the set $\operatorname{inv} N$ is not known, which is usually the case.

Notice that when F is single valued then $\operatorname{diam}_N F = 0$ and conditions (2.1), (2.2) reduce to standard definitions of the isolating neighbourhood and isolating block.

DEFINITION 2.4. Let N be an isolating neighbourhood for F. A pair $P = (P_1, P_2)$ of compact subsets $P_2 \subset P_1 \subset N$ is called an *index pair* if the following conditions are satisfied:

(a) $F(P_i) \cap N \subset P_i$, $i = 1, 2$;
(b) $F(P_1 \setminus P_2) \subset N$;
(c) $\operatorname{inv} N \subset \operatorname{int}(P_1 \setminus P_2)$

The following result was proved in [2]:

THEOREM 2.5. *Let F be a* dmds, *N an isolating neighbourhood for F and W a neighbourhood of* $\operatorname{inv} N$. *Then there exists an index pair P for N with $P_1 \setminus P_2 \subset W$.*

In the next section, a new proof of the above theorem will be provided in the case when F is generated by a continuous map. We shall need two lemmas from [2] on parametrised families of dmds. For the sake of completeness we shall also recall their proofs.

Let $\Lambda \subset R$ be a compact interval and $F: \Lambda \times X \times \mathbb{Z} \longrightarrow P(X)$ an usc mapping with compact values such that, for each $\lambda \in \Lambda, F_\lambda: X \times \mathbb{Z} \longrightarrow P(X)$ given by $F_\lambda(x, n) := F(\lambda, x, n)$ is a dmds. Given a compact subset $N \subset X$ and $\lambda \in \Lambda$, the sets $\operatorname{inv}^{(\pm)} N$ with respect to F_λ are denoted by $\operatorname{inv}^{(\pm)}(N, \lambda)$.

LEMMA 2.6. *Let $N \subset X$ be compact. Then the mappings $\lambda \longrightarrow \operatorname{inv}^+(N, \lambda), \lambda \longrightarrow \operatorname{inv}^-(N, \lambda)$, and $\lambda \longrightarrow \operatorname{inv}(N, \lambda), \lambda \in \Lambda$, are* usc.

PROOF. We prove the assertion for the first mapping, since the other two proofs are by extending the same argument to negative integers. Suppose that $\lambda \longrightarrow \operatorname{inv}^+(N, \lambda)$ is not usc at $\lambda_0 \in \Lambda$. Then there exists an open U and a sequence $\lambda_n \longrightarrow \lambda_0$ such that $\operatorname{inv}^+(N, \lambda_0) \subseteq U$ but $\operatorname{inv}^+(N, \lambda_n) \cap N \setminus U \neq \emptyset$. Let $x_n \in \operatorname{inv}^+(N, \lambda_n) \cap (N \setminus U)$. Since $N \setminus U$ is compact, we may assume that $x_n \longrightarrow x \in N \setminus U$. In order to achieve a contradiction, we have to show that $x \in \operatorname{inv}^+(N, \lambda_0)$. Indeed, let $\sigma_n: \mathbb{Z}^+ \longrightarrow N$ be a solution for F_{λ_n} with $\sigma_n(0) = x_n$. Then $\sigma_n(k) \subset \operatorname{inv}^+(N, \lambda_n) \subseteq N \setminus U$ for all $k = 1, 2, \ldots$. We construct a solution $\sigma: \mathbb{Z}^+ \longrightarrow N \setminus U$ for F_λ by induction on k. Let $\sigma(0) = \lim_n \sigma_n(0) = x$. Let $\sigma(k)$ be constructed for a given k, so that $\sigma(k) = \lim_i \sigma_{n_i}(k)$, where $\{\sigma_{n_i}(k)\}_i$ is a subsequence of $\{\sigma_n(k)\}_n$ convergent in $N \setminus U$. Passing again to a subsequence, we may assume that $\{\sigma_{n_i}(k+1)\}_i$ is convergent. Define $\sigma(k+1)$ to be its limit. Since $\sigma_n(k+1) \in F\big(\lambda_n(k)\big)$ for all n, the closed graph property of F implies that $\sigma(k+1) \in F\big(\lambda, \sigma(k)\big)$. ∎

LEMMA 2.7. *Let $\lambda_0 \in \Lambda$ and let N be an isolating neighbourhood for F_{λ_0}. Then N is an isolating neighbourhood for F_λ for all λ sufficiently close to λ_0.*

PROOF. By the compactness of N, the condition (2.2) implies that

$$B_{\operatorname{diam}_N F_{\lambda_0} + 3\varepsilon}\big(\operatorname{inv}(N, \lambda_0)\big) \subset \operatorname{inv} N$$

for some $\varepsilon > 0$. Since F is usc, $F_\lambda(x) \subset B_\varepsilon\big(F_{\lambda_0}(x)\big)$ for all λ close to λ_0 and all $x \in N$. Again by compactness of N,

$$\operatorname{diam}_N F_\lambda < \operatorname{diam}_N F_{\lambda_0} + 2\varepsilon$$

for all λ close to λ_0. By Lemma 2.6 $\operatorname{inv}(N,\lambda) \subset B_\varepsilon\big(\operatorname{inv}(N,\lambda_0)\big)$ for all λ close to λ_0 and we get

$$\begin{aligned} B_{\operatorname{diam}_N F_\lambda}\big(\operatorname{inv}(N,\lambda)\big) &\subset B_{\operatorname{diam}_N F_{\lambda_0}+2\varepsilon}\Big(B_\varepsilon\big(\operatorname{inv}(N,\lambda_0)\big)\Big) \\ &= B_{\operatorname{diam}_N} F_{\lambda_0+3\varepsilon}\big(\operatorname{inv}(N,\lambda_0)\big) \\ &\subset \operatorname{int} N. \end{aligned}$$ ∎

3**. Existence of stable index pairs.** Another way of stating Lemma 2.7 is by saying that isolating neighbourhoods are stable with respect to small perturbations of generators of dmds. That would not be true about index pairs, as pointed out in [7] and the goal of this paper is to show that there exist ones which are stable. Let us start from the following simple but important observation.

PROPOSITION 3.1. *Let $F\colon X \longrightarrow P(X)$ be a generator of a* dmds*, N an isolating neighbourhood for F, and P an index pair for N and F. If $G\colon X \longrightarrow P(X)$ is an* usc *proper map which is a selector of F, i.e. $G(x) \subset F(x)$ for all $x \in X$, then N is an isolating neighbourhood for G,* $\operatorname{inv}^{\pm}(N,G) \subset \operatorname{inv}^{\pm}(N,F)$ *and P also is an index pair for G.*

PROOF. The proof is a routine verification. ∎

THEOREM 3.2. *Let $f\colon X \longrightarrow X$ be a continuous proper map, N an isolating neighbourhood for f and W an open neighbourhood of* $\operatorname{inv} N$*. Then there exists an index pair P for N with $P_1 \setminus P_2 \subset W$ which is stable under small* usc *perturbations of f, i.e. there exists $\varepsilon > 0$ such that if $G\colon X \longrightarrow P(X)$ is an* usc *proper map with the property*

$$G(x) \subset B_\varepsilon\big(f(x)\big), \quad \text{for all } x \in X, \tag{3.1}$$

then P also is an index pair for G.

PROOF. Define a family of dmds on generators by

$$F_\lambda(x) := \overline{B_\lambda\big(f(x)\big)}, \quad x \in X,\ \lambda \geq 0. \tag{3.2}$$

By Lemma 2.6 and Lemma 2.7, there exists $\tau > 0$ such that N is an isolating neighbourhood for F_λ and $\operatorname{inv}(N,\lambda) \subset W$ provided $0 \leq \lambda \leq \tau$. Define

$$\begin{aligned} P_1 &:= \operatorname{inv}^-(N,\tau) \\ P_2 &:= P_1 \setminus \operatorname{int}\ \operatorname{inv}^+(N,\tau). \end{aligned}$$

Note that $\overline{P_1 \setminus P_2} = \operatorname{inv}(N,\tau) \subset W$. We shall verify below that $P := (P_1, P_2)$ is an index pair for all F_λ with $0 \leq \lambda < \tau$. In particular, it is an index pair for $F_0 = \{f\}$. Moreover, if $G\colon X \longrightarrow P(X)$ is an usc map satisfying (3.1) for $\varepsilon < \tau, \tau$ found above, then G is a selector of F_λ for $\varepsilon \leq \lambda < \tau$ and the conclusion follows from Proposition 3.1.

P is an index pair for F_λ provided $0 \leq \lambda < \tau$:

a) $F_\lambda(P_i) \cap N \subset P_i$, $qi = 1, 2$:

https://doi.org/10.4153/CMB-1997-053-2 Published online by Cambridge University Press

Let $x \in P_1$ and let $\sigma: \mathbb{Z}^- \longrightarrow N$ be a solution for F_τ through x. If $y \in F_\lambda(P_1) \cap N \subset F_\tau(P_1) \cap N$ then $\sigma': \mathbb{Z}^- \longrightarrow N$, $\sigma'(0) := y$, $\sigma'(n) := \sigma(n+1)$, $n < 0$, is a solution for F_τ through y. Thus $y \in P_1$.

Let now $x \in P_2$ and $y \in F_\lambda(x) \cap N \subset F_\tau(x) \cap N$. Since we already know that $y \in P_1$, it remains to show that $y \notin \operatorname{int} \operatorname{inv}^+(N,\tau)$. Suppose the contrary and let $B_\varepsilon(y) \subset \operatorname{inv}^+(N,\tau)$, $\varepsilon > 0$. Since f is continuous, there is $\delta > 0$ such that $d\big(F_\tau(x), F_\tau(x')\big) < \varepsilon$ provided $d(x,x') < \delta$. Let $x' \in B_\delta(x)$, $y' \in F_\tau(x')$. Then $y' \in B_\varepsilon(y)$, so there exists a solution $\sigma: \mathbb{Z}^+ \longrightarrow N$ for F_τ through y'. Then $\sigma': \mathbb{Z}^+ \longrightarrow N, \sigma'(0) := x', \sigma'(n) := \sigma(n-1)$, $n > 0$, is a solution for F_τ through x'. Thus $B_\delta(x) \subset \operatorname{inv}^+(N,\tau)$ which contradicts that $x \in P_2$.

b) $F_\lambda(P_1 \setminus P_2) \subset N$:

This is straightforward since $F_\lambda(P_1 \setminus P_2) \subset F_\tau(P_1 \setminus P_2)$ and $P_1 \setminus P_2 \subset \operatorname{inv}(N,\tau)$.

c) $\operatorname{inv}(N,\lambda) \subset \operatorname{int}(P_1 \setminus P_2)$:

Since $\operatorname{int}(P_1 \setminus P_2) = \operatorname{int} \operatorname{inv}(N,\tau)$, we need to prove that

$$0 \leq \lambda < \tau \Rightarrow \operatorname{inv}^{\pm}(N,\lambda) \subset \operatorname{int}_N \operatorname{inv}^{\pm}(N,\tau) \tag{3.3}$$

continuous, there exists $\delta > 0$ such that, for any $x' \in N$ with $d(x,x') < \delta$, we have $d\big(f(x), f(x')\big) < \tau - \lambda$. Therefore

$$F_\lambda(x) = \overline{B_\lambda\big(f(x)\big)} \subset \overline{B_\lambda\Big(B_{\tau-\lambda}\big(f(x')\big)\Big)} = F_\tau(x').$$

If $\sigma: \mathbb{Z}^+ \longrightarrow N$ is a solution for F_λ through x then we define a solution $\sigma': \mathbb{Z}^+ \longrightarrow N$ for F through x' by $\sigma'(0) := x'$ and $\sigma'(n) := \sigma(n)$, $n \geq 1$. This shows that $B_\sigma\big(\operatorname{inv}^+(N,\lambda)\big) \subset \operatorname{inv}^+(N,\tau)$ and (3.4) follows for $\operatorname{inv}^+(N,\lambda)$. Let now $x \in \operatorname{inv}^-(N,\lambda)$ and let $\sigma: \mathbb{Z}^- \longrightarrow N$ be a solution for F_λ through x. If $x' \in B_{\tau-\lambda}(x) \cap N$ then

$$x' \in \overline{B_{\tau-\lambda}\Big(F_\lambda\big(\sigma(-1)\big)\Big)} = F_\tau\big(\sigma(-1)\big),$$

therefore we may define a solution $\sigma': \mathbb{Z}^- \longrightarrow N$ for F_τ through x' by $\sigma'(0) := x'$ and $\sigma'(n) := \sigma(n), n < 0$. This shows that $B_{\tau-\lambda}\big(\operatorname{inv}^-(N,\lambda)\big) \subset \operatorname{inv}^-(N,\tau)$ and completes the proof of (3.4). ∎

REMARKS. 1. The arguments in the proof remain correct if we replace a single-valued map $f: X \longrightarrow X$ by a map $F: X \longrightarrow \mathcal{P}(X)$ with compact values which is continuous (*i.e.* both usc and lsc or, equivalently, continuous with respect to the Hausdorff distance between compact sets). That hypothesis is still more restrictive than the hypothesis of Theorem 2.6 in [2] but a shorter proof based on a different idea makes stating the theorem this way worthwhile.

2. The conclusion about the stability of P remains valid even if we consider a general usc proper map $F: X \longrightarrow \mathcal{P}(X)$ and

$$G(x) \subset B_\varepsilon\big(F(x)\big), x \in X.$$

https://doi.org/10.4153/CMB-1997-053-2 Published online by Cambridge University Press

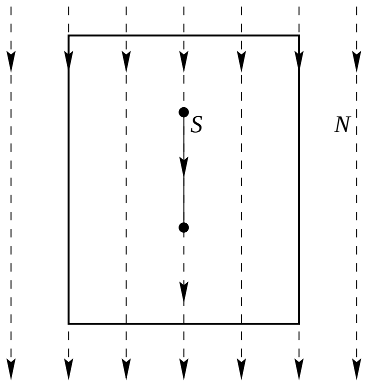

FIGURE 1

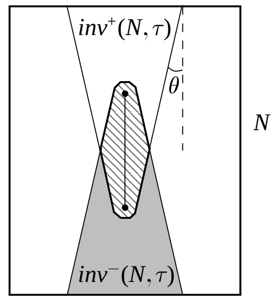

FIGURE 2

Indeed, on may define $F_\lambda(x) = \overline{B_\lambda\big(F(x)\big)}$ as in (3.2) and use Theorem 2.6 in [2] to conclude the existence of an index pair P for F_λ with $0 \leq \lambda < \tau$. Then one may refer to Proposition 3.1, as previously.

3. By the arguments in the proof and by Lemma 2.6, we obtain an additional information:

$$\operatorname{inv}(N, f) = \bigcap_{\lambda > 0} \operatorname{inv}(N, \lambda).$$

EXAMPLE. Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a time-one map of a downward flow with two stationary points and a connecting trajectory as in Figure 1. The set $S = \operatorname{inv} N$ consists of the two stationary points and the connecting interval. We assume that f is downward with a constant speed v, *i.e.* $f(x, y) = (x, y - v)$, on outside of some small neighbourhood of S.

If $F_\tau(x, y) = \overline{B_\tau\big(f(x, y)\big)}$, then $\operatorname{inv}^{\pm}(N, \tau)$ are two cones with "rounded vertices" as on Figure 2. The angle θ of the slope of each cone far from S is given by $\sin\theta = \tau / v$.

REFERENCES

1. C. Conley, *On a generalization of the Morse index*. In: Ordinary Differential Equations, 1971 NRL-MRC Conference, (ed. L. Weiss), Academic Press, New York, 1972, 27–33.

2. T. Kaczynski and M. Mrozek, *Conley index for discrete multivalued dynamical systems*, Topology Appl. **65**(1995), 83–96.

3. K. Mischaikow and M. Mrozek, *Chaos in the Lorenz equations: a computer assisted proof*, Bull. Amer. Math. Soc. (1) **33**(1995), 66–72.

4. ———, *Chaos in Lorenz equations: a computer assisted proof, Part II: details*, preprint.

5. K. Mischaikow, M. Mrozek and A. Szymczak, *Chaos in Lorenz equations: a computer assisted proof, Part III: the classic case*, in preparation.

6. M. Mrozek, *Leray functor and cohomological index for discrete dynamical systems*, Trans. Amer. Math. Soc. **318**(1990), 149–178.

7. J. W. Robbin and D. Salamon, *Dynamical systems, shape theory and the Conley index*, Ergodic Theory Dynamical Systems **8**(1988), 375–393.

8. H. Steinlein and H. O. Walter, *Hyperbolic sets, transversal homoclinic trajectories, and symbolic dynamics for C^1-maps in Banach spaces*, J. Dynamics Differential Equations (3) **2**(1990), 325–365.

9. A. Szymczak, *The Conley index and symbolic dynamics*, Topology **35**(1996), 287–299.

10. F. W. Wilson and J. A. Yorke, *Lyapunov functions and isolating blocks*, J. Differential Equations **13**(1973), 106–123.

Département de mathématiques et d'informatique
Sherbrooke, Québec
J1K 2R1
kaczyn@.usherb.ca

Instytut Informatyki
Uniwersytet Jagielloński
PL-30-072 Kraków
Poland
mrozek@.uj.edu.pl