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MKZ Type Operators Providing a Better Estimation on $[1/2, 1)$

M. Ali Özarslan and Oktay Duman

Abstract. In the present paper, we introduce a modification of the Meyer-König and Zeller (MKZ) operators which preserve the test functions $f_0(x) = 1$ and $f_2(x) = x^2$, and we show that this modification provides a better estimation than the classical MKZ operators on the interval $[\frac{1}{2}, 1)$ with respect to the modulus of continuity and the Lipschitz class functionals. Furthermore, we present the r-th order generalization of our operators and study their approximation properties.

1 Introduction

The Meyer-König and Zeller operators M_n, $(n \in \mathbb{N})$ [3], are defined by

$$M_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \quad m_{n,k}(x),\ f \in C[0,1) \text{ and } x \in [0,1),$$

where

$$m_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^n.$$

In order to give the monotonicity properties, Cheney and Sharma [1] modified these operators as follows:

$$M_n^*(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n+1,k}(x), \qquad f \in C[0,1) \text{ and } x \in [0,1).$$

The operators $M_n^*(f;x)$ preserve $f_i(x) = x^i$, $(i = 0, 1)$, *i.e.*, $M_n^*(f_0;x) = f_0(x) = 1$ and $M_n^*(f_1;x) = f_1(x) = x$, $n \in \mathbb{N}$. It is also known that

$$f_2(x) \le M_n^*(f_2;x) \le f_2(x) + \frac{x}{n} \text{ with } f_2(x) = x^2.$$

Then the second central moment for the operators $M_n^*(f;x)$ satisfies the following inequality:

$$M_n^*((y-x)^2;x) \le \frac{f_1(x)}{n} = \frac{x}{n}.$$

In this paper we present a modification of the operators $M_n^*(f;x)$ which preserve $f_0(x)$ and $f_2(x)$, and show that this modification provides a better estimation than the operators $M_n^*(f;x)$ on the interval $[\frac{1}{2}, 1)$. We also deal with the r-th order generalization of our operators and discuss their approximation properties.

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2 Construction of the Operators

We consider the following operators

$$R_n(f;x) = \sum_{k=0}^{\infty} f\left(\sqrt{\frac{k(k-1)}{(n+k)(n+k-1)}}\right) m_{n+1,k}(x), \quad n \in \mathbb{N}, \tag{2.1}$$

where $x \in [0,1)$ and $f \in C[0,1)$. It is obvious that the operators R_n are positive and linear. Also we have

$$R_n(f_0;x) = f_0(x) \text{ and } R_n(f_2;x) = f_2(x). \tag{2.2}$$

To obtain our main results we require the following lemma.

Lemma 2.1 *For all $n \in \mathbb{N}$ and $x \in [0,1)$, we have*

$$x - \frac{1-x}{n} \le R_n(f_1;x) \le x + \frac{x(1-x)}{n}. \tag{2.3}$$

Proof Since

$$\sqrt{\frac{k(k-1)}{(n+k)(n+k-1)}} \le \frac{k}{n+k-1}, \quad k \in \mathbb{N}_0 = \{0,1,2,\dots\},$$

we get, for all $n \in \mathbb{N}$, that

$$\begin{aligned} R_n(f_1;x) &= \sum_{k=0}^{\infty} \sqrt{\frac{k(k-1)}{(n+k)(n+k-1)}}\, m_{n+1,k}(x) \\ &\le \sum_{k=1}^{\infty} \frac{k}{n+k-1}\binom{n+k}{k} x^k(1-x)^{n+1} \\ &= \sum_{k=1}^{\infty} \frac{((n+k-1)+1)(n+k-1)!}{(n+k-1)n!(k-1)!} x^k(1-x)^{n+1} \\ &= x + \frac{x(1-x)}{n}. \end{aligned} \tag{2.4}$$

On the other hand, using the fact that

$$\sqrt{\frac{k(k-1)}{(n+k)(n+k-1)}} \ge \frac{k-1}{n+k}, \quad k \in \mathbb{N},$$

we obtain, for all $n \in \mathbb{N}$,

$$\begin{aligned} R_n(f_1;x) &\ge \sum_{k=1}^{\infty} \frac{k-1}{n+k}\binom{n+k}{k} x^k(1-x)^{n+1} \\ &= x - \sum_{k=1}^{\infty} \frac{(n+k-1)!}{n!k!} x^k(1-x)^{n+1} \\ &= x - \frac{1-x}{n}. \end{aligned} \tag{2.5}$$

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So (2.4) and (2.5) complete the proof. ■

Then, using (2.2) and (2.3) we have the following Korovkin type approximation theorem for the operators R_n given by (2.1) (see, for instance, [5]).

Theorem 2.2 *For $x \in [0, a]$, $(a \in (0, 1))$, and for $f \in C[0, a]$, we have*

$$\lim_n \|R_n(f) - f\| = 0,$$

where the symbol $\|\cdot\|$ denotes the usual supremum norm on $[0, a]$.

3 Rates of Convergence

In this section, we compute the rates of convergence of the operators $R_n(f; x)$ to $f(x)$ by means of the modulus of continuity and the elements of Lipschitz class. In this case we show that our estimations are more powerful than the classical MKZ operators on the interval $[\frac{1}{2}, 1)$.

Let $f \in C[0, 1)$. The modulus of continuity of f, denoted by $w(f, \delta)$, is defined to be

$$w(f, \delta) = \sup_{\substack{|y-x| \le \delta \\ x,y \in [0,1)}} |f(y) - f(x)|.$$

Then it is known that the necessary and sufficient condition for a function $f \in C[0, 1)$ is $\lim_{\delta \to 0} w(f, \delta) = 0$; and also, for any $\delta > 0$ and each $x, y \in [0, 1)$,

$$|f(y) - f(x)| \le w(f, \delta)\Big(\frac{|y - x|}{\delta} + 1\Big). \tag{3.1}$$

Now using this terminology we first have the following.

Theorem 3.1 *For every $f \in C[0, 1)$ and $x \in [0, 1)$, and for any $\delta > 0$, we have*

$$|R_n(f; x) - f(x)| \le w(f, \delta)\Big\{1 + \frac{1}{\delta}\sqrt{\frac{2x(1 - x)}{n}}\Big\}.$$

Proof Using monotonicity and linearity of the operators R_n and considering (3.1) we may write that

$$\begin{aligned} |R_n(f; x) - f(x)| &\le R_n(|f(y) - f(x)|; x) \\ &\le w(f, \delta)\Big\{1 + \frac{1}{\delta}R_n(|y - x|; x)\Big\} \\ &\le w(f, \delta)\Big\{1 + \frac{1}{\delta}\sqrt{R_n((y - x)^2; x)}\Big\}. \end{aligned} \tag{3.2}$$

On the other hand, since

$$\begin{aligned} R_n((y-x)^2;x) &= R_n(f_2;x) - 2xR_n(f_1;x) + x^2R_n(f_0;x) \\ &= x^2 - 2xR_n(f_1;x) + x^2 \\ &\le 2x|R_n(f_1;x) - f_1(x)|, \end{aligned}$$

it follows from Lemma 2.1 that

$$R_n((y-x)^2;x) \le \frac{2x(1-x)}{n}. \tag{3.3}$$

Then using (3.3) in (3.2) we complete the proof. ∎

In the classical case, for the Meyer-König and Zeller operators M_n^* (see, for instance, [3]), it is known that

$$|M_n^*(f;x) - f(x)| \le w(f,\delta)\Big\{1 + \frac{1}{\delta}\sqrt{\frac{x}{n}}\Big\}, \quad \text{for } f \in C[0,1) \text{ and } \delta > 0. \tag{3.4}$$

Now we claim that our rate of convergence in Theorem 3.1 is better than the estimate (3.4) whenever $x \in [\frac{1}{2}, 1)$. Indeed, since $1 - x \le \frac{1}{2}$ for all $x \in [\frac{1}{2}, 1)$, we have

$$\sqrt{\frac{2x(1-x)}{n}} \le \sqrt{\frac{x}{n}} \quad \text{for } x \in [\tfrac{1}{2}, 1) \text{ and } n \in \mathbb{N}, \tag{3.5}$$

which corrects our claim.

We can also compute the rate of convergence of the operators R_n by means of the elements of the Lipschitz class $\mathrm{Lip}_M(\alpha)$, $(\alpha \in (0,1])$. To get this, we recall that a function $f \in C[0,1)$ belongs to $\mathrm{Lip}_M(\alpha)$ if the inequality

$$|f(y) - f(x)| \le M|y-x|^\alpha, \quad (x, y \in [0,1)) \tag{3.6}$$

holds.

Theorem 3.2 *For every $f \in \mathrm{Lip}_M(\alpha)$ and $x \in [0,1)$, we have*

$$|R_n(f;x) - f(x)| \le M\Big\{\frac{2x(1-x)}{n}\Big\}^{\frac{\alpha}{2}}.$$

Proof Since $f \in \mathrm{Lip}_M(\alpha)$ and $x \in [0,1)$, using inequality (3.6) and then applying the Hölder inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$ we get

$$\begin{aligned} |R_n(f;x) - f(x)| &\le R_n(|f(y) - f(x)|;x) \\ &\le MR_n(|y-x|^\alpha;x) \\ &\le M\{R_n((y-x)^2;x)\}^{\frac{\alpha}{2}} \\ &\le M\Big\{\frac{2x(1-x)}{n}\Big\}^{\frac{\alpha}{2}}, \end{aligned}$$

whence the result. ∎

Notice that as in the proof of Theorem 3.2, since $M_n^*((y-x)^2;x) \le \frac{x}{n}$, the classical MKZ operators satisfy, for every $f \in \mathrm{Lip}_M(\alpha)$, that

$$|M_n^*(f;x) - f(x)| \le M\Big\{\frac{x}{n}\Big\}^{\frac{\alpha}{2}}. \tag{3.7}$$

So, it follows from (3.6) that the above claim also holds for Theorem 3.2, *i.e.*, the rate of convergence of the operators R_n by means of the Lipschitz functionals is better than the ordinary error estimation given by (3.7) whenever $x \in [\frac{1}{2}, 1)$.

4 A Generalization of the r-th Order of the Operators R_n

Let $C^{[r]}[0,1)$, $(r = 0,1,2,\dots)$ denote the set of all functions f having the continuous r-th derivative $f^{(r)}$ with $f^{(0)}(x) := f(x)$ on the interval $[0,1)$ (see [4, 2]). Throughout this section we set

$$a_{n,k} := \sqrt{\frac{k(k-1)}{(n+k)(n+k-1)}}, \quad n \in \mathbb{N} \text{ and } k \in \mathbb{N}_0.$$

Now we consider the following r-th order generalization of the positive linear operators R_n defined by (2.1):

$$R_{n,r}(f;x) = \sum_{k=0}^{\infty}\sum_{i=0}^{r} f^{(i)}(a_{n,k})\frac{(x-a_{n,k})^i}{i!} m_{n+1,k}(x), \tag{4.1}$$

where $f \in C^{[r]}[0,1)$, $r = 0,1,2,\dots$, and $n \in \mathbb{N}$. Observe that $R_{n,0} = R_n$.

Now we have the following:

Theorem 4.1 *For all $f \in C^{[r]}[0,1)$ such that $f^{(r)} \in \mathrm{Lip}_M(\alpha)$, and for every $x \in [0,1)$ we have*

$$|R_{n,r}(f;x) - f(x)| \le \frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)\Big|R_n\big(|t-x|^{r+\alpha};x\big)\Big|,$$

where $B(\alpha,r)$ is the beta function and $n,r \in \mathbb{N}$.

Proof By (4.1) one can write that

$$f(x) - R_{n,r}(f;x) = \sum_{k=0}^{\infty}\Big\{f(x) - \sum_{i=0}^{r} f^{(i)}(a_{n,k})\frac{(x-a_{n,k})^i}{i!}\Big\} m_{n+1,k}(x). \tag{4.2}$$

It is known from Taylor's formula that

$$\begin{aligned} f(x) - \sum_{i=0}^{r} f^{(i)}(a_{n,k})\frac{(x-a_{n,k})^i}{i!} &= \frac{(x-a_{n,k})^i}{(r-1)!} \\ &\times \int_0^1 (1-t)^{r-1}\big\{f^{(r)}(a_{n,k}+t(x-a_{n,k})) - f^{(r)}(a_{n,k})\big\}\,dt. \end{aligned} \tag{4.3}$$

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Since $f^{(r)} \in \mathrm{Lip}_M(\alpha)$,

$$\left| f^{(r)}(a_{n,k} + t(x - a_{n,k})) - f^{(r)}(a_{n,k}) \right| \leq M\, t^{\alpha} |x - a_{n,k}|^{\alpha}. \tag{4.4}$$

Using (4.4) and the usual definition of the beta integral in (4.3) we conclude that

$$\left| f(x) - \sum_{i=0}^{r} f^{(i)}(a_{n,k}) \frac{(x - a_{n,k})^i}{i!} \right| \leq \frac{M}{(r-1)!} \frac{\alpha}{\alpha + r} B(\alpha, r) |x - a_{n,k}|^{r+\alpha}. \tag{4.5}$$

Taking into consideration (4.2) and (4.5) we get the result. ■

For the uniform convergence of the operators $R_{n,r}$ we obtain the following.

Theorem 4.2 *For every $f \in C^{[r]}[0, a]$, $(0 < a < 1)$, such that $f^{(r)} \in \mathrm{Lip}_M(\alpha)$, we have*

$$\lim_n \|R_{n,r}(f) - f\| = 0, \quad (r \in \mathbb{N}_0).$$

Proof From Theorem 2.2, it is clear that $\lim_n \|R_n(g)\| = 0$, where $g(t) = |t - x|^{r+\alpha}$. So the proof follows immediately from Theorem 4.1. ■

Finally, taking into consideration Theorem 3.1 one can deduce the following result from Theorem 4.1.

Corollary 1 *For all $f \in C^{[r]}[0, 1)$ such that $f^{(r)} \in \mathrm{Lip}_M(\alpha)$, and for every $x \in [0, 1)$ and $\delta > 0$, we have*

$$|R_{n,r}(f; x) - f(x)| \leq \frac{M}{(r-1)!} \frac{\alpha}{\alpha + r} B(\alpha, r) w(g, \delta) \left\{ 1 + \frac{1}{\delta} \sqrt{\frac{2x(1-x)}{n}} \right\},$$

where $g(t) = |t - x|^{r+\alpha}$.

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Department of Mathematics,
Eastern Mediterranean University
Gazimagusa
Mersin 10
Turkey
e-mail: mehmetali.ozarslan@emu.edu.tr

Department of Mathematics
TOBB University of Economics and Technology
Sögütözü 06530
Ankara
Turkey
e-mail: oduman@etu.edu.tr