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Hyperbolic Group C^*-Algebras and Free-Product C^*-Algebras as Compact Quantum Metric Spaces

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Abstract. Let ℓ be a length function on a group G, and let M_ℓ denote the operator of pointwise multiplication by ℓ on $\ell^2(G)$. Following Connes, M_ℓ can be used as a "Dirac" operator for $C_r^*(G)$. It defines a Lipschitz seminorm on $C_r^*(G)$, which defines a metric on the state space of $C_r^*(G)$. We show that if G is a hyperbolic group and if ℓ is a word-length function on G, then the topology from this metric coincides with the weak-$*$ topology (our definition of a "compact quantum metric space"). We show that a convenient framework is that of filtered C^*-algebras which satisfy a suitable "Haagerup-type" condition. We also use this framework to prove an analogous fact for certain reduced free products of C^*-algebras.

Introduction

The group C^*-algebras of discrete groups provide a much-studied class of "compact non-commutative spaces" (that is, unital C^*-algebras). In [4] Connes showed that the "Dirac" operator of a spectral triple (*i.e.,* of an unbounded Fredholm module) over a unital C^*-algebra provides in a natural way a metric on the state space of the algebra. The class of examples most discussed in [4] consists of the group C^*-algebras of discrete groups, with the Dirac operator coming in a simple way from a word-length function on the group. In [12, 13] the second author pointed out that, motivated by what happens for ordinary compact metric spaces, it is natural to desire that for a spectral triple the topology from the metric on the state space coincides with the weak-$*$ topology (for which the state space is compact). This property was verified in [12] for certain examples. In [14] this property was taken as the defining property for a "compact quantum metric space".

In [15] the second author studied this property for Connes' original example of discrete groups with Dirac operators coming from word-length functions, but was able to verify this property only for the case when the group is $\mathbb{Z}^n$. This already took a long and interesting argument. We refer the reader to the introduction of [15] for a more extensive discussion of this whole matter.

In the present paper we verify the property for the case of hyperbolic discrete groups. In the course of studying this case we discovered that a natural setting was that of filtered C^*-algebras with faithful trace. Voiculescu had shown earlier [17] how

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to define an appropriate Dirac operator in that setting. In Section 1 we formulate in that setting a "Haagerup-type condition", which in Sections 2 and 3 we show is sufficient to imply that the metric from the Dirac operator gives the state space the weak-$*$ topology. Then in Section 4 we show that this Haagerup-type condition is satisfied in the case of hyperbolic groups. We mention that quite recently Antonescu and Christensen [1] showed that for non-Abelian free groups the metric on the state space gives the state space finite diameter. Their techniques are close to ours, but make explicit the relationship with Schur multipliers.

In Section 5 we show that the Haagerup-type condition fails for the groups $\mathbb{Z}^n$ for $n \geq 2$ with their standard length functions, and for groups which contain an amenable group of growth ≥ 4 for the length function in use. Since the approach used in the present paper is entirely different from that used in [15] to successfully treat $\mathbb{Z}^n$, this raises the interesting question of finding a unified approach which covers both cases. And there remains wide open the question of what happens for other classes of groups, such as the discrete Heisenberg group and other nilpotent discrete groups.

Finally, in Section 6 we show that the Haagerup-type condition is satisfied by the reduced free product of any two filtered C^*-algebras which satisfy the Haagerup-type condition. (Their filtrations give in a natural way a filtration on the free product.) This provides yet more examples of compact quantum metric spaces.

1 Filtered C^*-Algebras

We let A be a unital $*$-algebra over $\mathbb{C}$ which has a $*$-filtration $\{A_n\}$ by finite-dimensional subspaces. Just as in [17] this means that $A_m \subset A_n$ if $m < n$, $A = \bigcup_{n=0}^{\infty} A_n$, $A_n^* = A_n$ and $A_m A_n \subseteq A_{m+n}$, and $A_0 = \mathbb{C}1_A$. We assume further that we are given a faithful state, σ, on A, that is, a linear functional such that $\sigma(a^*a) > 0$ for all $a \in A$ unless $a = 0$, and $\sigma(1_A) = 1$. Let $\mathcal{H} = L^2(A, \sigma)$ denote the corresponding GNS Hilbert space. We assume that the left regular representation of A on $\mathcal{H}$ is by bounded operators, and we identify A with the corresponding algebra of operators on $\mathcal{H}$. We let $\|\cdot\|$ denote the operator norm of A. Our notation will not distinguish between a as an operator on $\mathcal{H}$ and a as a vector in $\mathcal{H}$, so the context must be examined to see which is intended. We let $\|a\|_2$ denote the norm of a as a vector in $\mathcal{H}$.

We can view each A_n as a finite-dimensional, thus closed, subspace of $\mathcal{H}$. We let Q_n denote the orthogonal projection of $\mathcal{H}$ onto A_n. We then set $P_n = Q_n - Q_{n-1}$ for $n \geq 1$, and $P_0 = Q_0$. The P_n's are mutually orthogonal, and $\sum P_n = I_{\mathcal{H}}$ for the strong operator topology. For each $a \in A$ and each n we set $a_n = P_n(a)$, where here a is viewed as a vector. Then $a_n \in A_n$, but $a_n \notin A_{n-1}$ unless $a_n = 0$. Furthermore $a = \sum a_n$, with at most p non-zero terms in the sum if $a \in A_p$.

For the above situation we define, as in [17], an unbounded operator, D, on $\mathcal{H}$ by $D = \sum_{n=1}^{\infty} nP_n$. Notice that A is contained in the domain of D. The following lemma is part of Proposition 5.1d of [17]. We include the proof here since we will need a similar argument in Section 3.

Lemma 1.1 *For any $a \in A$ the operator $[D, a]$ has dense domain and is a bounded operator.*

Proof Clearly A is contained in the domain of $[D, a]$, and A is dense. Suppose that $a \in A_p$. Then for any given $m, n \geq 0$, if $P_m a P_n \neq 0$ then there is a $\xi \in A_n$ such that $a\xi \in A_m$. Since $A_p A_n \subseteq A_{p+n}$, it follows that $p + n \geq m$. On taking the adjoint, we see that $P_m a^* P_n \neq 0$, so that $p + m \geq n$. Thus $|m - n| \leq p$. Consequently,

$$a = \sum_{|m-n| \leq p} P_m a P_n,$$

converging in the strong operator topology. For each j with $|j| \leq p$ set

$$T_j = \sum P_m a P_{m-j}.$$

Because the range of the terms $P_m a P_{m-j}$ are orthogonal for fixed j, as are the "domains", we have

$$\|T_j\| = \sup_m \|P_m a P_{m-j}\| \leq \|a\|.$$

But for any $m, n \geq 0$ we have

$$[D, P_m a P_n] = (m - n) P_m a P_n.$$

In particular, $[D, P_m a P_{m-j}] = j P_m a P_n$. Thus $[D, T_j] = j T_j$. Since $a = \sum T_j$, we obtain

$$[D, a] = \sum_{|j| \leq p} j T_j. \qquad \blacksquare$$

Thus $(A, \mathcal{H}, D)$ is a spectral triple (or unbounded Fredholm module) as defined by Connes [4, 5]. We can then define a seminorm, L, on A by

$$L(a) = \|[D, a]\|.$$

From the proof of Lemma 1.1 we can see that L will be a Lipschitz seminorm on A in the sense [13] that $L(a) = 0$ exactly if $a \in \mathbb{C}1_A = A_0$.

As pointed out by Connes, for any spectral triple $(A, \mathcal{H}, D)$, with L defined as above, we can define a metric, ρ_L, on the state space $S(A)$ of A by

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : L(a) \leq 1\},$$

(which may be $+\infty$). As discussed in [12, 13, 14] it is natural to ask whether the topology on $S(A)$ determined by ρ_L agrees with the weak-$*$ topology, as happens for ordinary compact metric spaces (X, ρ) and the usual Lipschitz seminorm on $C(X)$. If so, then [13] we call L a "Lip-norm". We consider a unital (pre-) C^*-algebra equipped with a Lip-norm to be a compact quantum metric space.

Main Theorem 1.2 *Let A, σ and the $*$-filtration $\{A_n\}$ be as above, and let D and L be defined as above. If furthermore there is a constant, C, such that*

$$\|P_m a_k P_n\| \leq C\|a_k\|_2$$

for all $a \in A$ and integers m, n, k, then L is a Lip-norm.

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As we will see at the end of Section 3, the key condition involving C stated just above is closely related to the Haagerup inequality. We will call a condition of this kind a "Haagerup-type condition".

Necessary and sufficient conditions for a Lipschitz seminorm on a pre-C^*-algebra to be a Lip-norm are given in [12] (in a more general context). For our present purposes it is convenient to reformulate these conditions slightly.

Proposition 1.3 *Let L be a Lipschitz seminorm on a unital pre-C^*-algebra A, and let σ be a state of A. Then L is a Lip-norm if and only if*

$$\{a \in A : L(a) \leq 1 \text{ and } \sigma(a) = 0\}$$

is a norm-totally-bounded subset of A.

Proof We apply Theorem 1.8 of [12]. Let $E = \{a \in A : L(a) \leq 1 \text{ and } \sigma(a) = 0\}$. Suppose first that E is totally bounded. As in theorem 1.8 of [12] let $\mathcal{L}_1 = \{a \in A : L(a) \leq 1\}$, and let $\tilde{A} = A/\mathbb{C}1_A$ with the quotient norm. Let $\tilde{\mathcal{L}}_1$ denote the image of $\mathcal{L}_1$ in $\tilde{A}$. For any $a \in \mathcal{L}_1$ the element $a - \sigma(a)1_A$ is in E. Thus the image of E in $\tilde{A}$ coincides with $\tilde{\mathcal{L}}_1$. Thus if E is totally bounded then so is $\tilde{\mathcal{L}}_1$. But this is exactly the condition in Theorem 1.8 of [12] for L to be a Lip-norm. Conversely, if L is a Lip-norm so that $\tilde{\mathcal{L}}_1$ is totally bounded, then a simple 2ε-argument shows that E is totally bounded. ■

2 The Action of the One-Parameter Group

In this section we consider a Hilbert space $\mathcal{L}$ with a sequence $\{P_n\}$ of mutually orthogonal projections whose sum is $I_{\mathcal{H}}$, much as above. We set $D = \sum nP_n$, and for each $t \in \mathbb{R}$ we let $U_t = e^{itD} = \sum e^{itn}P_n$. We let α_t denote the inner automorphism of $\mathcal{B}(\mathcal{H})$ defined by $\alpha_t(T) = U_t T U_t^*$. Because the spectrum of D consists of integers, we can view α as an action of the circle group $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. In general the function $t \mapsto \alpha_t(T)$ will not be norm-continuous. But it is always strong-operator continuous. Thus for any finite measure μ on $\mathbb{T}$ and any $T \in \mathcal{B}(\mathcal{H})$ we can define $\alpha_\mu(T)$ by

$$(\alpha_\mu(T))\xi = \int_{\mathbb{T}} \alpha_t(T)\xi \, d\mu(t)$$

for each $\xi \in \mathcal{H}$. Then $\|\alpha_\mu(T)\| \leq \|T\|\|\mu\|_1$, where $\|\mu\|_1$ is the total-variation norm. Notice then that for any $m, n \geq 0$ we have

$$\begin{aligned} P_m\alpha_\mu(T)P_n &= \int e^{imt}P_m T P_n e^{-int} \, d\mu(t) \\ &= \hat{\mu}(n-m)P_m T P_n, \end{aligned}$$

where $\hat{\mu}$ is the Fourier transform of μ. In particular, if $[D, T]$ is a bounded operator, then

$$\begin{aligned} P_m\alpha_\mu([D, T])P_n &= \hat{\mu}(n-m)P_m[D, T]P_n \\ &= (m-n)\hat{\mu}(n-m)P_m T P_n. \end{aligned}$$

For any integer $N \geq 0$ let $\varphi_N \in \ell^2(\mathbb{Z})$ be defined by $\varphi_N(k) = -1/k$ if $|k| > N$ and 0 otherwise. Then the inverse Fourier transform, $\check{\varphi}_N$, of φ_N is in $L^2(\mathbb{T})$, and so in $L^1(\mathbb{T})$. Thus as the measure μ above we can use $\check{\varphi}_N(t)dt$. With some abuse of notation, we denote the corresponding operator by α_{φ_N}. For any $T \in \mathcal{B}(\mathcal{H})$ for which $[D, T]$ is bounded we set

$$T^{(N)} = \alpha_{\varphi_N}([D, T]).$$

Then for any $m, n \geq 0$ we have, as above,

$$\begin{aligned} P_m T^{(N)} P_n &= (m-n)\varphi_N(n-m) P_m T P_n \\ &= \begin{cases} 0 & \text{if } |m-n| \leq N \\ P_m T P_n & \text{if } |m-n| > N. \end{cases} \end{aligned}$$

Thus

$$T^{(N)} = \sum_{|m-n|>N} P_m T P_n.$$

Furthermore,

$$\|T^{(N)}\| \leq 2\pi \|\varphi_N\|_2 \|[D, T]\|,$$

since $\|\check{\varphi}_N\|_1 \leq \sqrt{2\pi}\|\check{\varphi}_N\|_2 = 2\pi\|\varphi_N\|_2$. Notice that $\|\varphi_N\|_2 \to 0$ as $N \to +\infty$.

3 The Proof of the Main Theorem

We resume the notation of Section 1. According to Proposition 1.3 we must show that, under the hypotheses of the Main Theorem, the set

$$E = \{a \in A : \|[D, a]\| \leq 1 \text{ and } \sigma(a) = 0\}$$

is totally bounded in A for the operator norm. Given $a \in A$, we set $a_n = P_n(a)$ as in Section 1, so that $a = \sum a_n$. The condition that $\sigma(a) = 0$ is then just the condition that $a_0 = 0$.

Let $\varepsilon > 0$ be given. We now show that E can be covered by a finite number of 3ε-balls. For φ_N's as in the previous section, choose N large enough that $2\pi\|\varphi_N\|_2 < \varepsilon$. For $a \in E$ define $a^{(N)}$ as in the previous section by $a^{(N)} = \alpha_{\varphi_N}([D, a])$. Then from the discussion there we have $\|a^{(N)}\| < \varepsilon$. Set $a^N = a - a^{(N)}$, so that $\|a - a^N\| < \varepsilon$. Since as above

$$a^{(N)} = \sum_{|m-n|>N} P_m a P_n,$$

we have

$$a^N = \sum_{|m-n|\leq N} P_m a P_n,$$

which converges in the strong operator topology. Note that in general $a^N \notin A$.

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Let 1_A be viewed as a vector in $L^2(A, \sigma)$, so that $\|1_A\|_2 = 1$ and $D(1_A) = 0$. Then for any $a \in A$ we have

$$[D, a](1_A) = D(a) = \sum na_n.$$

Since the a_n's are mutually orthogonal, it follows that for $a \in E$ we have

$$\sum n^2 \|a_n\|_2^2 \leq \|[D, a]\|^2 \leq 1.$$

Then from the Cauchy–Schwarz inequality we see that for any integer $K \geq 0$ we have

$$\begin{aligned} \sum_{n>K} \|a_n\|_2 &= \sum_{n>K} (n^{-1})(n\|a_n\|_2) \\ &\leq \Big(\sum_{n>K} n^{-2}\Big)^{1/2} \Big(\sum n^2 \|a_n\|_2^2\Big)^{1/2} \\ &\leq \Big(\sum_{n>K} n^{-2}\Big)^{1/2}. \end{aligned}$$

We now choose K large enough that

$$\Big(\sum_{n>K} n^{-2}\Big)^{1/2} < \varepsilon(C(2N+1))^{-1}.$$

For each $a \in A$ set $\hat{a}_K = \sum_{k \leq K} a_k$ and $\tilde{a}_K = \sum_{k>K} a_k$, so that $a = \hat{a}_K + \tilde{a}_K$. Then

$$a = a^N + a^{(N)} = \hat{a}_K^N + \tilde{a}_K^N + a^{(N)},$$

where $\hat{a}_K^N = (\hat{a}_K)^N$ and similarly for $\tilde{a}_K^N$. For $a \in E$ we have chosen N so that $\|a^{(N)}\| < \varepsilon$. We show next that $\{\hat{a}_K^N : a \in E\}$ is totally bounded. Then we will show that because of our choice of K we have $\|\tilde{a}_K^N\| < \varepsilon$ for any $a \in E$. It will follow immediately that E can be covered by a finite number of 3ε-balls, as desired.

For any $a \in E$ we have

$$\|\hat{a}_K\|_2 \leq \sum_{k \leq K} \|a_k\|_2 \leq \Big(\sum_{k=1}^{\infty} n^{-2}\Big)^{1/2}.$$

Thus $\{\hat{a}_K : a \in E\}$ is a bounded subset of the finite dimensional vector space A_K. The map $a \mapsto \hat{a}^N$ is linear, and so when restricted to A_K it must carry $\{\hat{a}_K : a \in E\}$ to a bounded subset of a finite-dimensional subspace of $\mathcal{B}(\mathcal{H})$. Thus $\{\hat{a}_K^N : a \in E\}$ is totally bounded, as needed. (This is the only place in this proof where we use the assumption that the A_n's are finite dimensional. Without that assumption this proof only shows that the metric on $S(A)$ gives $S(A)$ finite diameter.)

We now show that $\|\tilde{a}_K^N\| < \varepsilon$ for $a \in E$. It is convenient to first show the following slightly more general fact:

Lemma 3.1 *With notation as above, for any $a \in A$ we have*

$$\|a^N\| \le (2N+1)C \sum_{k=0}^{\infty} \|a_k\|_2.$$

Proof For each integer j with $|j| \le N$ set,

$$T_j = \sum_m P_m a P_{m-j}.$$

As in the proof of Lemma 1.1 we have

$$\|T_j\| = \sup_m \|P_m a P_{m-j}\|.$$

For each integer m we have, by hypothesis,

$$\|P_m a P_{m-j}\| \le \sum_k \|P_m a_k P_{m-j}\| \le C \sum \|a_k\|_2,$$

so that $\|T_j\| \le C \sum \|a_k\|_2$. Since $a^N = \sum_{|m-n| \le N} P_m a P_n = \sum_{|j| \le N} T_j$, we obtain the asserted fact. ∎

Now for any $a \in E$, because $\tilde{a}_K = \sum_{k>K} a_k$, the above proposition gives

$$\begin{aligned} \|\tilde{a}_K^N\| &\le (2N+1)C \sum_{k>K} \|a_k\|_2 \\ &\le (2N+1)C \Big(\sum_{k>K} (k^{-2}) \Big)^{1/2} < \varepsilon \end{aligned}$$

by our choice of K, as needed. This concludes the proof of Main Theorem 1.2.

We show next that from our Haagerup-type condition we can obtain a Haagerup inequality in its more usual form. Let $a \in A$, and let the a_k's be its components as above. For any k and for $|j| \le k$ set $T_j = \sum P_m a_k P_{m-j}$, much as above. Then, as above,

$$\|T_j\| = \sup_m \|P_m a_k P_{m-j}\| \le C\|a_k\|_2.$$

Since, as above, $a_k = \sum_{|j| \le k} T_j$, we obtain the following analog of the third line of the proof of Lemma 1.4 of [9], which we record for later use:

Lemma 3.2 *With notation as above, we have*

$$\|a_k\| \le C(2k+1)\|a_k\|_2.$$

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Then from the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\|a\| &\leq \sum_k \|a_k\| \leq \sum_k C(2k+1)\|a_k\|_2 \\ &= C\sum_k (1/(k+1))(k+1)(2k+1)\|a_k\|_2 \\ &\leq C\Big(\sum_{p\geq 1} 1/p^2\Big)^{1/2}\Big(\sum (2k^2+3k+1)^2\|a_k\|_2^2\Big)^{1/2}.\end{aligned}$$

If we note that $2k^2+3k+1 \leq 2(k+1)^2$ for $k \geq 0$, and set $C' = 2C\big(\sum_{p\geq 1} 1/p^2\big)^{1/2}$, we obtain the following inequality, which is similar to the usual form [5] for the Haagerup inequality for groups:

Proposition 3.3 *For any $a \in A$ we have*

$$\|a\| \leq C'\Big(\sum (1+k)^4\|a_k\|_2^2\Big)^{1/2}.$$

We now obtain a related inequality which we will need shortly.

Proposition 3.4 *There is a constant, C'', such that for any integer p and any $a \in A_p$ we have*

$$\|a\| \leq C''(p+1)^{3/2}\|a\|_2.$$

Proof We use Lemma 3.2 to calculate that

$$\begin{aligned}\|a\| &\leq \sum_0^p \|a_k\| \leq C\Big(\sum_0^p (2k+1)\|a_k\|_2\Big) \\ &\leq C\Big(\sum_0^p (2k+1)^2\Big)^{1/2}\Big(\sum_0^p \|a_k\|_2^2\Big)^{1/2} \\ &\leq 2C\Big(\sum_0^p (k+1)^2\Big)^{1/2}\|a\|_2.\end{aligned}$$

But

$$\sum_0^p (k+1)^2 \leq \int_0^{p+1} (t+1)^2\, dt = (1/3)((p+2)^3 - 1).$$

Absorbing several factors into the constant, we obtain the desired inequality. ■

4 Hyperbolic Groups

In this section we show that our Main Theorem applies to word-hyperbolic groups. There are several equivalent definitions of what it means for a metric space to be hyperbolic [8]. We will find the following version well-suited to our purposes.

Definition 4.1 A metric space (X, ρ) is *hyperbolic* if there is a constant $\delta \geq 0$ such that for any four points $x, y, z, w \in X$ we have

$$\rho(x, y) + \rho(z, w) \leq \max\{\rho(x, z) + \rho(y, w), \rho(x, w) + \rho(y, z)\} + \delta.$$

If it is important to specify δ, we say that X is δ-hyperbolic.

Let G be a finitely generated discrete group, and let S be a finite generating subset for G, with $S = S^{-1}$. Let ℓ be the word-length function on G determined by S, and let ρ be the corresponding left-invariant metric on G defined by $\rho(x, y) = \ell(x^{-1}y)$. Then G is said to be hyperbolic if the metric space (G, ρ) is hyperbolic. It is not difficult to show [8] that this is independent of the choice of the finite generating set S.

For any discrete group G and any integer-valued length function ℓ on G we obtain a $*$-filtration $\{A_n\}$ of the convolution algebra $A = C_c(G)$ of complex-valued functions of finite support on G by setting

$$A_n = \{f \in A : f(x) = 0 \text{ if } \ell(x) > n\}.$$

The involution on A is defined, as usual, by $f^*(x) = (f(x^{-1}))^-$. We define a faithful trace, σ, on A by $\sigma(f) = f(e)$, where e denotes the identity element of G. The resulting GNS Hilbert space is $\ell^2(G)$, and the left regular representation of A on $\ell^2(G)$ is by bounded operators. The C^*-algebra generated by the left regular representation is the reduced C^*-algebra of G, $C_r^*(G)$. Thus we are in the setting of Section 1. (With a bit of care with the bookkeeping, all the above applies also to the convolution algebra of G twisted by a 2-cocycle, in the way that was explicitly carried out in [15]. Our results below also work for this case too.)

The Dirac operator corresponding to the filtration is just the operator M_ℓ of pointwise multiplication by ℓ on $\ell^2(G)$. We can then define the seminorm L on A by $L(f) = \|[D, f]\|$, where f on the right is viewed as the convolution operator on $\ell^2(G)$. We can then ask whether L is a Lip-norm. Our Main Theorem provides a possible tool for giving an affirmative answer to this question.

Definition 4.2 Let ℓ be an integer-valued length function on a group G. We say that (G, ℓ) satisfies a Haagerup-type condition if, for the filtration of $C_c(G) \subseteq C_r^*(G)$ defined above, with its canonical trace, the main condition of Theorem 1.2 is satisfied.

Proposition 4.3 *Let G be a word-hyperbolic group, and let ℓ be the word-length function for a finite generating subset of G. Then (G, ℓ) satisfies a Haagerup-type condition.*

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Proof A proof is essentially contained within Connes' proof of the Haagerup inequality for hyperbolic groups given in [5, p. 241]. But since some significant details are not included there, we give a complete proof here. The special case of this proposition for the free group on finitely many generators with its standard word-length function relative to the given generators is explicitly given by Haagerup as Lemma 1.3 in [9], with $C = 1$. (See also Lemma 1.1 of [7], where it is remarked right after the proof of Theorem 1.3 that it also works for the free group with countably many generators. But with an infinite number of generators the subspaces A_n of the filtration are infinite dimensional, and so the proof of our Main Theorem 1.2 only shows that the state space has finite diameter.)

For any integer $j \geq 0$ let $E_j = \{x \in G : \ell(x) = j\}$. We must find a constant, C, such that for any integers k, m, n, and any f supported on E_k we have $\|P_m f P_n\| \leq C\|f\|_2$. This means that for any ξ supported on E_n we must have

$$\Big(\sum_{x \in E_m} |(f * \xi)(x)|^2\Big)^{1/2} \leq C\|f\|_2\|\xi\|_2.$$

We examine $(f * \xi)(x)$. Let δ be a constant for which G, equipped with the metric from ℓ, is δ-hyperbolic as in Definition 4.1. Now

$$(f * \xi)(x) = \sum_{yz=x} f(y)\xi(z).$$

If $(f * \xi)(x) \neq 0$ there must be some $y, z \in G$ such that $x = yz$ with $\ell(y) = k$, $\ell(z) = n$, and so if $x \in E_m$ we must have $m \leq k + n$. But also $z = y^{-1}x$, so we must have $n \leq k + m$, and so $|m - n| \leq k$. In the same way we obtain $|n - k| \leq m$. Let $p = k + n - m$. If p is even set $q = p/2$, while if p is odd set $q = (p - 1)/2$. In either case set $\tilde{q} = p - q$, and notice that $q \leq \tilde{q} \leq q + 1$. Then $m = (k - q) + (n - \tilde{q})$, and from $|m - n| \leq k$ it is easy to check that $k - q \geq 0$, while from $|n - k| \leq m$ it is easy to check that $n - \tilde{q} \geq 0$. Consequently, for each $x \in E_m$ we can choose $\bar{x}, \tilde{x} \in G$ such that $x = \bar{x}\tilde{x}$ and $\ell(\bar{x}) = k - q$, while $\ell(\tilde{x}) = n - \tilde{q}$. This choice is usually not unique, but we fix it for the rest of the proof.

Suppose now that $x \in E_m$ and $x = yz$ for some $y \in E_k$ and $z \in E_n$. We apply Definition 4.1 to the four points $(e, x, \bar{x}, y)$ to obtain

$$\rho(e, x) + \rho(y, \bar{x}) \leq \max\{\rho(e, \bar{x}) + \rho(y, x),\ \rho(e, y) + \rho(x, \bar{x})\} + \delta.$$

But $\rho(e, \bar{x}) + \rho(y, x) = (k - q) + n$, while $\rho(e, y) + \rho(x, \bar{x}) = k + (n - \tilde{q})$. Consequently

$$\rho(y, \bar{x}) \leq k - q + n - m + \delta = \tilde{q} + \delta.$$

Thus $y = \bar{x}u$ for some u with $\ell(u) \leq \tilde{q} + \delta$. Then $z = y^{-1}x = u^{-1}\bar{x}^{-1}x = u^{-1}\tilde{x}$. Since this is true for all such x, y, we see that

$$(f * \xi)(x) = \sum\{f(\bar{x}u)\xi(u^{-1}\tilde{x}) : \ell(u) \leq \tilde{q} + \delta\}.$$

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We can apply the Cauchy–Schwarz inequality to this to get

$$|(f*\xi)(x)|^2 \le \Big(\sum\{|f(\bar{x}u)|^2 : \ell(u) \le \tilde{q}+\delta\}\Big)\Big(\sum\{|\xi(v\tilde{x})|^2 : \ell(v) \le \tilde{q}+\delta\Big).$$

For any $y \in E_k$ let us consider how many decompositions there are of the form $y = su$ such that $\ell(s) = k - q = \ell(\bar{x})$ and $\ell(u) \le \tilde{q} + \delta$. Let $y = tw$ be another such decomposition. We apply Definition 4.1 to the four points e, y, s, t to obtain

$$\rho(e,y) + \rho(s,t) \le \max\{\rho(e,s) + \rho(t,y),\ \rho(e,t) + \rho(s,y)\} + \delta.$$

But $\rho(e,s) + \rho(y,t) = k - q + \tilde{q} + \delta = \rho(e,t) + \rho(s,y)$. It follows that $k + \rho(s,t) \le \tilde{q} - q + k + 2\delta$, so that $\rho(s,t) \le 1 + 2\delta$. In the same way we find that for any two factorizations $z = vs = wt$ with $\ell(s) = \ell(t) = \ell(\tilde{x}) = n - \tilde{q}$ and $\ell(v), \ell(w) \le \tilde{q} + \delta$ we have $\rho(v,w) \le 2\delta$.

Let C be the number of elements of G in a ball of radius $1 + 2\delta$. Then the number of different s's which can enter as above into the factorization of y is no larger than C, and thus the number of u's is also no larger than C. Similarly, the number of v's which can enter as above into the factorization of z is no larger than C.

We now claim that $\|f*\xi\|_2 \le C\|f\|_2\|\xi\|_2$. From our earlier calculations we know that

$$\begin{aligned}\|f*\xi\|_2^2 &= \sum_x |(f*\xi)(x)|^2 \\ &\le \sum_x \Big(\sum_{\ell(u)\le\tilde{q}+\delta} |f(\bar{x}u)|^2\Big)\Big(\sum_{\ell(v)\le\tilde{q}+\delta} |\xi(v\tilde{x})|^2\Big),\end{aligned}$$

while of course

$$(\|f\|_2\|\xi\|_2)^2 = \sum_{\substack{\ell(y)=k\\ \ell(z)=n}} |f(y)|^2|\xi(z)|^2.$$

Thus to obtain our desired inequality it suffices to show that for any pair (y,z) with $\ell(y) = k$ and $\ell(z) = n$, the number of x's for which there are a u and v with $\ell(u) \le \tilde{q} + \delta$ and $\ell(v) \le \tilde{q} + \delta$ such that $y = \bar{x}u$ and $z = v\tilde{x}$ is no greater than C^2. But suppose we have such x, u, v. Then $x = \bar{x}\tilde{x} = yu^{-1}v^{-1}z$. Given our earlier bound on the number of such u's and v's, it is now clear that the number of such x's is indeed bounded by C^2. ■

Corollary 4.4 *Let G be a word-hyperbolic group, and let ℓ be the word-length function for a finite generating subset of G. Then the metric on $S(C_r^*(G))$ coming from using ℓ as a Dirac operator gives $S(C_r^*(G))$ the weak-$*$ topology.*

5 Failure of the Haagerup-Type Condition

In this section we show that the Haagerup-type condition often fails for groups which contain a copy of $\mathbb{Z}^d$ for $d \ge 2$, or other amenable groups with suitable growth. We begin with the following observation.

Proposition 5.1 *Let ℓ be a length function on a group G, and let ℓ_H denote the restriction of ℓ to a subgroup H. If (G,ℓ) satisfies a Haagerup-type condition, then so does (H,ℓ_H).*

Proof Since G is the disjoint union of right cosets of H, the restriction to H of the left regular representation of G is a direct sum of copies of the left regular representation of H. Thus $C_r^*(H)$ is isometrically embedded in $C_r^*(G)$. The restriction to $C_r^*(H)$ of the canonical trace on $C_r^*(G)$ is the canonical trace on $C_r^*(H)$. The filtration of $C_r^*(H)$ for ℓ_H is just the intersection of $C_r^*(H)$ with the filtration of $C_r^*(G)$ for ℓ. The desired conclusion follows easily. ■

Proposition 5.2 *The group $\mathbb{Z}^2$ with the word-length function for its standard basis does not satisfy a Haagerup-type condition. Thus neither does $\mathbb{Z}^d$ for $d > 2$ with its standard word-length function.*

Proof For $\mathbb{Z}^2$ and the standard word-length function ℓ, given by $\ell((p,q)) = |p|+|q|$, we need to show that there is no constant C such that $\|P_m f P_n\| \leq C\|f\|_2$ for all m,k,n, where f is supported on E_k. Let $k > 0$ be fixed, choose $n > k$, and set $m = n+k$. Let f be the function which has value $(1/k)$ on the points $(p, k-p)$ of E_k for which $1 \leq p \leq k$, and value 0 elsewhere. In the evident way we will consider f to be a function just of p when convenient. Notice that $\|f\|_1 = 1$, so that $\|P_m f P_n\| \leq 1$, while $\|f\|_2 = 1/\sqrt{k}$. Similarly, let ξ be the function which has value $1/\sqrt{n}$ on the points $(q, n-q)$ of E_n for which $1 \leq q \leq n$, and value 0 elsewhere. We can consider ξ as a function just of q. Note that $\|\xi\|_2 = 1$. We estimate $\|P_m f P_n \xi\|$. We will evaluate only on the points $(r, m-r)$ of E_m for which $k \leq r \leq n$. Then with this restriction,

$$\begin{aligned}(P_m f P_n \xi)(r, m-r) &= \sum_{1\leq p\leq k} f(p)\xi(r-p)\\ &= k(1/k)(1/\sqrt{n}) = 1/\sqrt{n}.\end{aligned}$$

Thus $\|P_m f P_n \xi\|_2^2 \geq (n-k)/n$, so that $\|P_m f P_n\| \geq ((n-k)/n)^{1/2}$. Notice that this approaches 1 as $n \to +\infty$. But we could have chosen k as large as desired, so that $\|f\|_2 = 1/\sqrt{k}$ is as small as desired. Thus there is no constant C such that $\|P_n f P_m\| \leq C\|f\|_2$ for all m,k,n, where f is supported on E_k. ■

This, of course, raises the question of whether there is a way to give a unified proof of both the Corollary 4.4 for hyperbolic groups and the corresponding result in [15] for $\mathbb{Z}^d$, as well as the question of what happens for other groups. Perhaps the "bolic" groups of Kasparov and Skandalis [3, 11] provide a good class of groups for which one might hope to find a unified proof.

Suppose now that G is an amenable group, so that $C_r^*(G) = C^*(G)$. Then the trivial representation of G gives a representation of $C_r^*(G)$. By using the trivial representation we see that if $f \in C_c(G)$ and if $f \geq 0$ as a function, then $\|f\| = \|f\|_1$. For each integer p let $B_p = \{x \in G : \ell(x) \leq p\}$, and let χ_p denote the characteristic function of B_p. Suppose that G satisfies a Haagerup-type condition. Then according to Proposition 3.4 there is a constant, C', such that

$$\|\chi_p\|_1 = \|\chi_p\| \le C'(p+1)^{3/2}\|\chi_p\|_2.$$

Let $|B_p|$ denote the number of elements in B_p. Then it follows that $|B_p| \le C'(p+1)^{3/2}|B_p|^{1/2}$. From this we obtain:

Proposition 5.3 *Let G be an amenable group, and let ℓ be an integer-valued length-function on G. If (G,ℓ) satisfies a Haagerup-type condition, then there is a constant, C', such that for every p we have*

$$|B_p| \le C'(p+1)^3.$$

We now recall some well-known definitions and facts. (See [8, p. 12].) For an integer-valued length-function on G we say that its rate of growth is polynomial if there is an integer n and a constant C such that $|B_p| \le Cp^n$ for all large enough p. We call the smallest such n the "growth rate" of G for ℓ. If $|B_p|$ grows at a faster than polynomial rate, then we say that the growth rate of G for ℓ is ∞.

The idea of comparing the 2-norm with the 1-norm came from [10], where Jolissaint showed that an amenable group with the property (RD) is of polynomial growth.

Let S be a finite generating set for G, and let ℓ_S be the corresponding word-length function. For any length function ℓ on G set $M = \max\{\ell(s) : s \in S\}$. Then it is easily seen that $\ell \le M\ell_S$. Consequently the growth rate of G for ℓ is no smaller than that for ℓ_S. In particular, the growth rates of G for any two word-length functions coincide. This common growth rate is called the growth rate of a given finitely generated group. From the above observations and Proposition 5.3 we obtain:

Corollary 5.4 *If G is a finitely generated amenable group, and if G satisfies a Haagerup-type condition for some length function, then the growth rate of G is no greater than* 3.

Corollary 5.5 *Let G be any discrete group. If G contains a finitely generated amenable group whose growth rate is ≥ 4, then there does not exist a length function ℓ on G such that (G,ℓ) satisfies a Haagerup-type condition.*

Corollary 5.6 *If a group G contains either $\mathbb{Z}^4$ or the discrete Heisenberg group, then there does not exist a length function ℓ on G such that (G,ℓ) satisfies a Haagerup-type condition.*

Proof Both $\mathbb{Z}^4$ and the discrete Heisenberg group have a growth of 4. (See [8, Ch. 1, §18] for the proof of this for the Heisenberg group.) ■

Question 5.7 Suppose that a group G admits a finite generating set for whose word-length function ℓ the pair (G,ℓ) satisfies a Haagerup-type condition. Must the group then be hyperbolic?

6 Free-Product C^*-Algebras

In this section we show that Main Theorem 1.2 applies to certain reduced free-product C^*-algebras. Jolissaint [10] showed that the property (RD) is preserved under forming free products, but his proof apparently does not work in our situation. Thus, we need a finer classification of types of words, which unfortunately complicates the notation.

Let A^1 and A^2 be unital pre-C^*-algebras with filtrations $\{A^1_m\}$ and $\{A^2_m\}$ respectively. Let $A = A^1 * A^2$ be the algebraic free product, with its evident involution. We define a filtration (respecting the involution) on A by setting A_n to be the linear span of all products $A^{i_1}_{n_1} \cdots A^{i_\alpha}_{n_\alpha}$ with each $i_j = 1, 2$, with $i_j \neq i_{j+1}$ for $1 \leq j \leq \alpha - 1$, and with $\sum n_j \leq n$.

Let σ^1 and σ^2 be faithful tracial states on A^1 and A^2. We let $\sigma = \sigma^1 * \sigma^2$ be the corresponding faithful tracial state on A which is used to define [2, 16, 18] the reduced free-product C^*-algebra structure on A. Its defining properties are that its restrictions to A^1 and A^2 coincide with σ^1 and σ^2, and that $\sigma(a^{i_1}_1 \cdots a^{i_\alpha}_\alpha) = 0$ if $\sigma^{i_j}(a^{i_j}_j) = 0$ for all $j = 1, \ldots, \alpha$ and $i_j \neq i_{j+1}$ for $j = 1, \ldots, \alpha - 1$. The reduced C^*-norm on A (for σ_1 and σ_2) is then the operator norm for the GNS representation for σ on $L^2(A, \sigma)$.

Theorem 6.1 *If (A^1, σ^1) and (A^2, σ^2) both satisfy a Haagerup-type condition with constant C, then $(A^1 * A^2, \sigma^1 * \sigma^2)$ satisfies a Haagerup-type condition with constant $\sqrt{5}C$.*

We remark that there are many examples to which this theorem applies. In addition to the reduced group C^*-algebras of hyperbolic groups studied in the earlier sections of this paper, one can take any finite-dimensional C^*-algebras with any filtrations.

This theorem is related to Lemma 3.3 of [6], but in [6] the algebras A^1 and A^2 are not assumed to be filtered, and so our situation is substantially different from that considered there.

We now establish some notation which will be used in the proof. As in Section 1, we let $\{P^i_n\}$ be the family of mutually orthogonal projections corresponding to the filtration $\{A^i_n\}$, for $i = 1, 2$, and we let $\{P_n\}$ be the corresponding family on A for $\{A_n\}$. We let E^i_n denote the range of P^i_n, and similarly for E_n. Thus E_0 is the span of 1, while if $n \geq 1$ then E_n is the orthogonal sum of the spans of products $E^{i_1}_{n_1} \cdots E^{i_\alpha}_{n_\alpha}$ such that $n_j \geq 1$ for all j and $i_j \neq i_{j+1}$ for $j = 1, \ldots, \alpha - 1$ while $\sum n_j = n$. In order to reduce notational clutter we will often omit the superscripts when they can be inferred from the context. In particular, we will let $P^\perp_0$ denote the projection onto the orthogonal complement of 1 for all three algebras.

Much as in section 2 of [6] we choose for $i = 1, 2$ an orthonormal basis $\mathcal{B}^i_n$ for each E^i_n, with $\{1\}$ as the basis for E^i_0. But for convenience we also require that each basis element be self-adjoint. We can do this because σ_i is tracial. We let $\mathcal{B}^i = \bigcup_n \mathcal{B}^i_n$, so that $\mathcal{B}^i$ is a basis for A^i. We define ℓ on each $\mathcal{B}^i$ by $\ell(x) = n$ if $x \in \mathcal{B}^i_n$. For $x \in (\mathcal{B}^1 \cup \mathcal{B}^2)$ we define μ by $\mu(x) = i$ if $x \in \mathcal{B}^i$, and we define ν by $\nu(x) = i$ if $x \notin \mathcal{B}^i$. As in [6] we obtain from $\mathcal{B}^1$ and $\mathcal{B}^2$ an orthonormal basis $\mathcal{B}$ for A. An element of $\mathcal{B}$ will be either 1, or a product $\boldsymbol{x} = x_1 \cdots x_\alpha$ with $x_i \in (B^1 \cup B^2) \setminus \{1\}$ for each i while $\mu(x_i) \neq \mu(x_{i+1})$ for $i = 1, \dots, \alpha - 1$. We extend the definitions of μ and ν to $\mathcal{B} \setminus \{1\}$ by setting $\mu(\boldsymbol{x}) = \mu(x_1)$ and $\nu(\boldsymbol{x}) = \nu(x_1)$ for any $\boldsymbol{x} \neq 1$. Although $\mu(1)$ is undefined (because A is really the free product amalgamated over $\mathbb{C}1$), we will make the unusual convention that both $\mu(\boldsymbol{x}) = \mu(\boldsymbol{y})$ and $\mu(\boldsymbol{x}) \neq \mu(\boldsymbol{y})$ are simultaneously true if $\boldsymbol{y} = 1$. We set $\ell(\boldsymbol{x}) = \sum \ell(x_j)$, with $\ell(1) = 0$. We then set $\mathcal{B}_n = \{\boldsymbol{x} : \ell(\boldsymbol{x}) = n\}$, and note that $\mathcal{B}_n$ is an orthonormal basis for E_n. (But we note also that the elements of $\mathcal{B}_n$ need not be self-adjoint, though the involution carries $\mathcal{B}_n$ into itself.) We will often write an element a of E_n as $a = \sum_{\boldsymbol{x} \in \mathcal{B}_n} a(\boldsymbol{x})\boldsymbol{x}$.

Our objective is to show that for any $a \in E_k$ and any m, n we have $\|P_m a P_n\|_2 \leq \sqrt{5}C\|a\|_2$, where on the left side a is viewed as an operator on $L^2(A, \sigma)$. Thus we must show that if $\xi \in E_n$ then

$$\|P_m(a\xi)\|_2 \leq \sqrt{5}C\|a\|_2\|\xi\|_2.$$

So we now fix m, k, and n for the rest of the proof. We can assume that m, k and n are all ≥ 1, since the desired inequality is very easily verified if any one of them is 0. Somewhat as in Section 4, we set $q = (k + n - m)/2$, but now q need not be an integer. Some of the objects considered below will depend on m, k and n, but to avoid notational clutter we often will not indicate that dependence explicitly.

For any $a \in E_k$ we have $a = \sum_{\boldsymbol{y} \in \mathcal{B}_k} a(\boldsymbol{y})\boldsymbol{y}$. In the same way, for $\xi \in E_n$ we have $\xi = \sum_{\boldsymbol{z} \in \mathcal{B}_n} \xi(\boldsymbol{z})\boldsymbol{z}$. We find it notationally convenient to work with $a^*\xi$ instead of $a\xi$. Then

$$a^*\xi = \sum_{\boldsymbol{y}, \boldsymbol{z}} \bar{a}(\boldsymbol{y})\xi(\boldsymbol{z})\boldsymbol{y}^*\boldsymbol{z}.$$

Thus we need information about $P_m(\boldsymbol{y}^*\boldsymbol{z})$. So we need to see how $\boldsymbol{y}^*\boldsymbol{z}$ can be expressed in terms of "reduced words". Let $\boldsymbol{y} = y_1 \cdots y_\beta$ and $\boldsymbol{z} = z_1 \cdots z_\gamma$. If $\mu(\boldsymbol{y}) \neq \mu(\boldsymbol{z})$, then $\boldsymbol{y}^*\boldsymbol{z}$ is already a reduced word, and $\boldsymbol{y}^*\boldsymbol{z} \in \mathcal{B}_{k+n}$. Otherwise, if $\mu(\boldsymbol{y}) = \mu(\boldsymbol{z})$ then there is some integer $\delta \geq 1$ such that $y_i = z_i$ for $i < \delta$ while $y_\delta \neq z_\delta$ (with the latter including the possibility that y_δ or z_δ is not present, *i.e.*, $\beta < \delta$ or $\gamma < \delta$). If $\delta = 1$ then $y_1 \neq z_1$ so that $P_0(y_1 z_1) = 0$, and

$$\boldsymbol{y}^*\boldsymbol{z} = y_\beta \cdots y_2 P_0^\perp(y_1 z_1) z_2 \cdots z_\gamma,$$

which is a reduced word. If $\delta > 1$ then $P_0(y_i z_i) = 1$ for $i < \delta$, and so

$$\begin{aligned} \boldsymbol{y}^*\boldsymbol{z} &= y_\beta \cdots y_2 P_0(y_1 z_1) z_2 \cdots z_\gamma + y_\beta \cdots y_2 P_0^\perp(y_1 z_1) z_2 \cdots z_\gamma \\ &= y_\beta \cdots y_2 z_2 \cdots z_\gamma + y_\beta \cdots y_2 P_0^\perp(y_1 z_1) z_2 \cdots z_\gamma. \end{aligned}$$

Continuing in this way, we obtain, even for $\delta = 1$ or $\mu(\boldsymbol{y}) \neq \mu(\boldsymbol{z})$:

Lemma 6.2 *Let $\boldsymbol{y}, \boldsymbol{z} \in \mathcal{B}$ with $\boldsymbol{y} = y_1 \cdots y_\beta$ and $\boldsymbol{z} = z_1 \cdots z_\gamma$, and let $\delta \geq 1$ be the integer such that $y_i = z_i$ for all $i < \delta$ while $y_\delta \neq z_\delta$ (including the case $\beta = \delta - 1$ or $\gamma = \delta - 1$). Then*

$$\boldsymbol{y}^*\boldsymbol{z} = \sum_{i=1}^{\delta} y_\beta \cdots y_{i+1} P_0^\perp(y_i z_i) z_{i+1} \cdots z_\gamma,$$

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where

(1) *One should replace $P_0^\perp(y_i z_i)$ by 1 if $\mu(y) \neq \mu(z)$ so that $\boldsymbol{y}^*\boldsymbol{z} \in \mathcal{B}$.*
(2) *One should replace the summand for $i = \delta$ by 1 if y_δ and z_δ are both not present,* i.e., *if $\boldsymbol{y} = \boldsymbol{z}$.*
(3) *If $\beta = \delta - 1$ then no y_j's should appear on the left of the term for $i = \delta$, and similarly if $\gamma = \delta - 1$.*

Suppose now that for some $i \leq \delta$ we have

$$P_m(y_\beta \cdots y_{i+1} P_0^\perp(y_i z_i) z_{i+1} \cdots z_\gamma) \neq 0.$$

Then there must be an $r \in \mathcal{B}^{\mu(y_i)}$, $r \neq 1$, such that $\sigma(r y_i z_i) \neq 0$ and

$$\ell(y_\beta \cdots y_{i+1}) + \ell(r) + \ell(z_{i+1} \cdots z_\gamma) = m.$$

But because $\sigma(r y_i z_i) \neq 0$ we also have, by the properties of filtrations,

$$\ell(y_i) + \ell(z_i) \geq \ell(r) \geq |\ell(y_i) - \ell(z_i)|.$$

Thus

$$\ell(y_\beta \cdots y_i) + \ell(z_i \cdots z_\gamma) \geq \ell(y_\beta \cdots y_{i+1}) + |\ell(y_i) - \ell(z_i)| + \ell(z_{i+1} \cdots z_\gamma).$$

Let $\boldsymbol{w} = y_1 \cdots y_{i-1} = z_1 \cdots z_{i-1}$. It follows from above that

$$\min\{\ell(y_1 \cdots y_i), \ell(z_1 \cdots z_i)\} \geq (\ell(\boldsymbol{y}) + \ell(\boldsymbol{z}) - m)/2 \geq \ell(\boldsymbol{w}).$$

Recall that $q = (k + n - m)/2$. Since $\ell(\boldsymbol{y}) = k$ and $\ell(\boldsymbol{z}) = n$, we see that $\ell(\boldsymbol{w}) = \ell(y_1 \cdots y_{i-1}) \leq q$, while $\ell(y_1 \cdots y_i) \geq q$ so that

$$\ell(y_\beta \cdots y_{i+1}) \leq k - q.$$

Similarly

$$\ell(z_{i+1} \cdots z_\gamma) \leq n - q.$$

Notice that

$$(k - q) + (n - q) = m,$$

so that we cannot have simultaneously $\ell(y_\beta \cdots y_{i+1}) = k - q$ and $\ell(z_{i+1} \cdots z_\gamma) = n - q$. We summarize the above observations by:

Lemma 6.3 *Suppose that $\boldsymbol{y}$ and $\boldsymbol{z}$ are such that $\mu(\boldsymbol{y}) = \mu(\boldsymbol{z})$. let δ be as defined above. If for some $i \leq \delta$ we have*

$$P_m(y_\beta \cdots y_{i+1} P_0^\perp(y_i z_i) z_{i+1} \cdots z_\gamma) \neq 0,$$

then $\boldsymbol{y}$ and $\boldsymbol{z}$ are of the form $\boldsymbol{y} = \boldsymbol{w}^* u \hat{\boldsymbol{s}}$ and $\boldsymbol{z} = \boldsymbol{w}^* v \hat{\boldsymbol{t}}$ where $\ell(\boldsymbol{w}) \leq q$, $\ell(\hat{\boldsymbol{s}}) \leq k - q$, $\ell(\hat{\boldsymbol{t}}) \leq n - q$, and $u, v \in \mathcal{B}^1 \cup \mathcal{B}^2$ with $\mu(\boldsymbol{w}) \neq \mu(u) \neq \mu(\hat{\boldsymbol{s}})$ and $\mu(\boldsymbol{w}) \neq \mu(v) \neq \mu(\hat{\boldsymbol{t}})$. At least one of u, v is not 1, and if $u = 1$ then also $\hat{\boldsymbol{s}} = 1$, and similarly for v. Specifically, $\boldsymbol{w} = y_{i-1} \cdots y_1 = z_{i-1} \cdots z_1$ and $u = y_i$ and $v = z_i$, while $\hat{\boldsymbol{s}} = y_{i+1} \cdots y_\beta$ and $\hat{\boldsymbol{t}} = z_{i+1} \cdots z_\gamma$. If $\beta \leq i$ then $\hat{\boldsymbol{s}} = 1$, and similarly for $\gamma \leq 1$. Then

$$P_m(y_\beta \cdots y_{i+1} P_0^\perp(y_i z_i) z_{i+1} \cdots z_\gamma) = P_m(\hat{\boldsymbol{s}}^* P_0^\perp(uv)\hat{\boldsymbol{t}}).$$

Either $\ell(\hat{\boldsymbol{s}}) < k - q$ or $\ell(\hat{\boldsymbol{t}}) < n - q$ (or both).

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In order to be in a position to apply our assumption that (A^1, σ^1) and (A^2, σ^2) satisfy a Haagerup-type condition, we need to consider collectively all the $\boldsymbol{x}$'s which may occur in the support of a fixed term $y_\beta \cdots y_{i+1} P_0^\perp(y_i z_i) z_{i+1} \cdots z_\gamma$. For this purpose it is convenient to assume now that both $k - q \neq 0$ and $n - q \neq 0$. At the end of the proof we will give separately the argument for the remaining cases. We also need to divide the situation into two cases, depending on the structure of the $\boldsymbol{x}$'s. Let $\boldsymbol{x} = x_1 \cdots x_\alpha$. For the first case we assume that there is a j such that $\ell(x_1 \cdots x_j) < k - q$ while $\ell(x_1 \cdots x_{j+1}) > k - q$. (This will always happen if q is not an integer.) Thus we can express $\boldsymbol{x}$ as $\boldsymbol{x} = \boldsymbol{s}^* r \boldsymbol{t}$ where $\mu(\boldsymbol{s}) = \mu(\boldsymbol{t})$ and $\mu(r) \neq \mu(\boldsymbol{s})$, with $\ell(\boldsymbol{s}) < k - q$ and $\ell(\boldsymbol{t}) < n - q$. The second case will be that in which there is a j such that $\ell(x_1 \cdots x_j) = k - q$.

Notation 6.4 Assume that $k - q \neq 0$ and $n - q \neq 0$. For any pair $(\boldsymbol{s}, \boldsymbol{t})$ of elements of $\mathcal{B}$ such that $\ell(\boldsymbol{s}) < k - q$ and $\ell(\boldsymbol{t}) < n - q$ we set:

(a) If $\mu(\boldsymbol{s}) = \mu(\boldsymbol{t})$ (with $\boldsymbol{s} = 1$ and/or $\boldsymbol{t} = 1$ permitted — recall our convention about $\mu(1)$), then

$$\mathcal{B}_{\boldsymbol{s},\boldsymbol{t}} = \{\boldsymbol{x} \in \mathcal{B}_m : \boldsymbol{x} = \boldsymbol{s}^* r \boldsymbol{t}, r \in \mathcal{B}^1 \cup \mathcal{B}^2 \setminus \{1\}, \text{ and } \mu(\boldsymbol{s}) \neq \mu(r) \neq \mu(\boldsymbol{t})\}.$$

We let $E_{\boldsymbol{s},\boldsymbol{t}}$ denote the linear span of $\mathcal{B}_{\boldsymbol{s},\boldsymbol{t}}$, and we let $P_{\boldsymbol{s},\boldsymbol{t}}$ denote the projection onto $E_{\boldsymbol{s},\boldsymbol{t}}$.

(b) If q is an integer and $\mu(\boldsymbol{s}) \neq \mu(\boldsymbol{t})$ (with $\boldsymbol{s} = 1$ and/or $\boldsymbol{t} = 1$ permitted), then

$$\mathcal{C}_{\boldsymbol{s},\boldsymbol{t}} = \Big\{\boldsymbol{x} \in \mathcal{B}_m : \boldsymbol{x} = \boldsymbol{s}^* r_1 r_2 \boldsymbol{t},\ r_1 \in \mathcal{B}^{\nu(r_2)}_{k-q-\ell(\boldsymbol{s})},\ r_2 \in \mathcal{B}^{\nu(r_1)}_{n-q-\ell(\boldsymbol{t})}, \\ \mu(r_1) \neq \mu(\boldsymbol{s}), \mu(r_2) \neq \mu(\boldsymbol{t})\Big\}.$$

(Note that $\ell(r_i) \geq 1$ for $i = 1, 2$ since $\ell(\boldsymbol{s}) < k - q$ and $\ell(\boldsymbol{t}) < n - q$.) We let $F_{\boldsymbol{s},\boldsymbol{t}}$ denote the linear span of $\mathcal{C}_{\boldsymbol{s},\boldsymbol{t}}$ and we let $Q_{\boldsymbol{s},\boldsymbol{t}}$ denote the projection onto $F_{\boldsymbol{s},\boldsymbol{t}}$.

Lemma 6.5 *$\mathcal{B}_m$ is the disjoint union of all the $\mathcal{B}_{\boldsymbol{s},\boldsymbol{t}}$'s and $\mathcal{C}_{\boldsymbol{s},\boldsymbol{t}}$'s.*

Proof It is evident that the $\mathcal{B}_{\boldsymbol{s},\boldsymbol{t}}$'s are disjoint among themselves, as are the $\mathcal{C}_{\boldsymbol{s},\boldsymbol{t}}$'s. If $\boldsymbol{x} \in \mathcal{B}_{\boldsymbol{s},\boldsymbol{t}}$ for some $(\boldsymbol{s}, \boldsymbol{t})$ then $\boldsymbol{x}$ is not of the form $\boldsymbol{uv}$ where $\boldsymbol{u} \in \mathcal{B}_{k-q}$ and $\boldsymbol{v} \in \mathcal{B}_{n-q}$, whereas all elements of any $\mathcal{C}_{\boldsymbol{s},\boldsymbol{t}}$ are of this form. Thus the $\mathcal{B}_{\boldsymbol{s},\boldsymbol{t}}$'s are disjoint from the $\mathcal{C}_{\boldsymbol{s},\boldsymbol{t}}$'s.

Let $\boldsymbol{x} \in \mathcal{B}_m$ with $\boldsymbol{x} = x_1 \cdots x_\alpha$. Recall our assumption that $m \geq 1$. If $\boldsymbol{x}$ satisfies the conditions for the first case discussed just before Notation 6.4, then $\boldsymbol{x} \in \mathcal{B}_{\boldsymbol{s},\boldsymbol{t}}$ for the choice of $\boldsymbol{s}, \boldsymbol{t}$ given there. Suppose instead that $\boldsymbol{x}$ does not satisfy the conditions of the first case. Then there is a j such that $\ell(x_1 \cdots x_j) = k - q$. (Thus q is an integer.) Since $k \neq q$, $\ell(x_j) \geq 1$. Thus we can write $x_1 \cdots x_j = \boldsymbol{s}^* r_1$ with $r_1 = x_j$, so $\ell(r_1) \geq 1$ and $\ell(\boldsymbol{s}) + \ell(r_1) = k - q$, and $r_1 \in \mathcal{B}^{\nu(\boldsymbol{s})}$ unless $\boldsymbol{s} = 1$. Since $(k - q) + (n - q) = m$, we will also have $\ell(x_{j+1} \cdots x_\alpha) = n - q \neq 0$, so that $x_{j+1} \cdots x_\alpha = r_2 \boldsymbol{t}$ with $\ell(r_2) \geq 1$, $\ell(r_2) + \ell(\boldsymbol{t}) = n - q$ and $r_2 \in \mathcal{B}^{\nu(r_1)}$, and $r_2 \in \mathcal{B}^{\nu(\boldsymbol{t})}$ unless $\boldsymbol{t} = 1$. Thus $\boldsymbol{x} \in \mathcal{C}_{\boldsymbol{s},\boldsymbol{t}}$ for this choice of $(\boldsymbol{s}, \boldsymbol{t})$. ∎

Corollary 6.6 *Assume that $k \neq q$ and $n \neq q$. Then*

$$P_m = \Big(\bigoplus_{\mu(\boldsymbol{s})=\mu(\boldsymbol{t})} P_{\boldsymbol{s},\boldsymbol{t}} \Big) \oplus \Big(\bigoplus_{\mu(\boldsymbol{s})\neq\mu(\boldsymbol{t})} Q_{\boldsymbol{s},\boldsymbol{t}} \Big),$$

where $\boldsymbol{s} = 1$ and $\boldsymbol{t} = 1$ are permitted.

As this corollary suggests, we will now examine $P_{\boldsymbol{s},\boldsymbol{t}}(a^*\xi)$ and $Q_{\boldsymbol{s},\boldsymbol{t}}(a^*\xi)$ in order to obtain the estimate we need for $P_m(a^*\xi)$.

Lemma 6.7 *Let $(\boldsymbol{s},\boldsymbol{t})$ be such that $\mu(\boldsymbol{s}) = \mu(\boldsymbol{t})$, with $\boldsymbol{s} = 1$ and $\boldsymbol{t} = 1$ permitted. Let $\boldsymbol{y} \in \mathcal{B}_k$ and $\boldsymbol{z} \in \mathcal{B}_n$ be given. If $P_{\boldsymbol{s},\boldsymbol{t}}(\boldsymbol{y}^*\boldsymbol{z}) \neq 0$, then $\boldsymbol{y}$ and $\boldsymbol{z}$ are of the form $\boldsymbol{y} = \boldsymbol{w}^* u\boldsymbol{s}$ and $\boldsymbol{z} = \boldsymbol{w}^* v\boldsymbol{t}$ where*

$u, v \in \mathcal{B}^1 \cup \mathcal{B}^2 \setminus \{1\}$ and $\mu(u) = \mu(v)$,
$\mu(\boldsymbol{s}) \neq \mu(u) \neq \mu(\boldsymbol{w})$ and $\mu(v) \neq \mu(\boldsymbol{t})$,
$\ell(\boldsymbol{w}) \leq q$, with $\boldsymbol{w} = 1$ permitted.

(Consequently $\ell(u) = k - \ell(\boldsymbol{s}) - \ell(\boldsymbol{w})$ and $\ell(v) = n - \ell(\boldsymbol{t}) - \ell(\boldsymbol{w})$.) Then

$$P_{\boldsymbol{s},\boldsymbol{t}}(\boldsymbol{y}^*\boldsymbol{z}) = \boldsymbol{s}^* P_{m(\boldsymbol{s},\boldsymbol{t})}(uv)\boldsymbol{t},$$

where $m(\boldsymbol{s},\boldsymbol{t}) = m - \ell(\boldsymbol{s}) - \ell(\boldsymbol{t})$.

Proof This follows from Lemma 6.3 when we set $\boldsymbol{s} = \hat{\boldsymbol{s}}$ and $\boldsymbol{t} = \hat{\boldsymbol{t}}$ there. ■

Lemma 6.8 *Let $(\boldsymbol{s},\boldsymbol{t})$ be such that $\mu(\boldsymbol{s}) \neq \mu(\boldsymbol{t})$, with $\boldsymbol{s} = 1$ and $\boldsymbol{t} = 1$ permitted. Let $\boldsymbol{y} \in \mathcal{B}_k$ and $\boldsymbol{z} \in \mathcal{B}_n$ be given. If $Q_{\boldsymbol{s},\boldsymbol{t}}(\boldsymbol{y}^*\boldsymbol{z}) \neq 0$, then $\boldsymbol{y}$ and $\boldsymbol{z}$ are in one and only one of the forms:*

(a) *$\boldsymbol{y} = \boldsymbol{w}^* u\boldsymbol{s}$ and $\boldsymbol{z} = \boldsymbol{w}^* v r_2\boldsymbol{t}$ where*

$u, v \in \mathcal{B}^1 \cup \mathcal{B}^2 \setminus \{1\}$ and $\mu(u) = \mu(v)$,
$\mu(\boldsymbol{s}) \neq \mu(u) \neq \mu(\boldsymbol{w})$, and $\mu(v) \neq \mu(r_2) \neq \mu(\boldsymbol{t})$,
$\ell(r_2\boldsymbol{t}) = n - q$ while $\ell(\boldsymbol{w}) \leq q$.

Then

$$Q_{\boldsymbol{s},\boldsymbol{t}}(\boldsymbol{y}^*\boldsymbol{z}) = \boldsymbol{s}^* P_{m(\boldsymbol{s},r_2\boldsymbol{t})}(uv)r_2\boldsymbol{t},$$

where $m(\boldsymbol{s}, r_2\boldsymbol{t}) = m - \ell(\boldsymbol{s}) - \ell(r_2\boldsymbol{t})$.

(b) *$\boldsymbol{y} = \boldsymbol{w}^* u r_1\boldsymbol{s}$ and $\boldsymbol{z} = \boldsymbol{w}^* v\boldsymbol{t}$ with similar restrictions as above, and $\ell(r_1\boldsymbol{s}) = k - q$ while $\ell(\boldsymbol{w}) \leq q$. Then*

$$Q_{\boldsymbol{s},\boldsymbol{t}}(\boldsymbol{y}^*\boldsymbol{z}) = \boldsymbol{s}^* r_1 P_{m(\boldsymbol{s}^* r_1,\boldsymbol{t})}(uv)\boldsymbol{t}.$$

Proof In the notation of Lemma 6.3 this is the case in which either $\ell(\hat{\boldsymbol{s}}) = k - q$ or $\ell(\hat{\boldsymbol{t}}) = n - q$ (but not both). If $\ell(\hat{\boldsymbol{t}}) = n - q$ with $\hat{\boldsymbol{t}} = z_{i+1} \cdots z_\gamma$, then we must have $r_2 = z_{i+1}$ and $\boldsymbol{t} = z_{i+2} \cdots z_\gamma$. We also have $\boldsymbol{s} = \hat{\boldsymbol{s}}$. This gives case (a). If, instead, $\ell(\hat{\boldsymbol{s}}) = n - q$, then we must have $r_1 = y_{i+1}$ and $\boldsymbol{s} = y_{i+2} \cdots z_\gamma$, while $\boldsymbol{t} = \hat{\boldsymbol{t}}$. This gives case (b). ■

Proof of Theorem 6.1 Suppose now that $a \in E_k$ and $\xi \in E_n$, with $a = \sum a(\boldsymbol{y})\boldsymbol{y}$ and $\xi = \sum \xi(\boldsymbol{z})\boldsymbol{z}$. Let $(\boldsymbol{s}, \boldsymbol{t})$ be such that $P_{\boldsymbol{s},\boldsymbol{t}}$ is defined. For any $\boldsymbol{s}'$ and $\boldsymbol{w} \in \mathcal{B}$ set $k(\boldsymbol{s}', \boldsymbol{w}) = k - \ell(\boldsymbol{s}') - \ell(\boldsymbol{w})$ and $n(\boldsymbol{s}', \boldsymbol{w}) = n - \ell(\boldsymbol{s}') - \ell(\boldsymbol{w})$. Then from Lemma 6.7 we have

$$P_{\boldsymbol{s},\boldsymbol{t}}(a^*\xi) = \sum_{\boldsymbol{w},u,v} \bar{a}(\boldsymbol{w}^* u\boldsymbol{s})\xi(\boldsymbol{w}^* v\boldsymbol{t})\boldsymbol{s}^* P_{m(\boldsymbol{s},\boldsymbol{t})}(uv)\boldsymbol{t}$$

where in the above sum

$u, v \in \mathcal{B}^1 \cup \mathcal{B}^2 \setminus \{1\}$ and $\mu(u) = \mu(v)$,
$\mu(\boldsymbol{s}) \neq \mu(u) \neq \mu(\boldsymbol{w})$ and $\mu(v) \neq \mu(\boldsymbol{t})$,
$\ell(\boldsymbol{w}) \leq q,\ \ell(u) = k(\boldsymbol{s}, \boldsymbol{w})$, and $\ell(v) = n(\boldsymbol{t}, \boldsymbol{w})$.

This sum can be rewritten as

$$\boldsymbol{s}^* \Big(\sum_{\substack{\ell(\boldsymbol{w}) \leq q \\ \mu(\boldsymbol{w}) = \mu(\boldsymbol{s})}} P_{m(\boldsymbol{s},\boldsymbol{t})}(\tilde{a}_{\boldsymbol{s},\boldsymbol{w}} \tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}}) \Big) \boldsymbol{t},$$

where we have set

$$\tilde{a}_{\boldsymbol{s},\boldsymbol{w}} = \sum_{u \in \mathcal{B}^{\nu(\boldsymbol{s})}_{k(\boldsymbol{s},\boldsymbol{w})}} \bar{a}(\boldsymbol{w}^* u\boldsymbol{s})u$$

$$\tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}} = \sum_{v \in \mathcal{B}^{\nu(\boldsymbol{t})}_{n(\boldsymbol{t},\boldsymbol{w})}} \xi(\boldsymbol{w}^* v\boldsymbol{t})v.$$

Note that for any $\boldsymbol{x} \in \mathcal{B}$ and $b \in A^{\nu(\boldsymbol{x})}$ we have $\|b\boldsymbol{x}\|_2 = \|b\|_2 = \|\boldsymbol{x}^* b\|_2$. Consequently

$$\begin{aligned}
\|P_{\boldsymbol{s},\boldsymbol{t}}(a^*\xi)\|_2^2 &\leq \Big(\sum_{\boldsymbol{w}} \|P^{\nu(\boldsymbol{s})}_{m(\boldsymbol{s},\boldsymbol{t})}(\tilde{a}_{\boldsymbol{s},\boldsymbol{w}}\, \tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}})\|_2 \Big)^2 \\
&\leq \Big(\sum_{\boldsymbol{w}} C \|\tilde{a}_{\boldsymbol{s},\boldsymbol{w}}\|_2 \|\tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}}\|_2 \Big)^2 \\
&\leq C^2 \Big(\sum_{\boldsymbol{w}} \|\tilde{a}_{\boldsymbol{s},\boldsymbol{w}}\|_2^2 \Big) \Big(\sum_{\boldsymbol{w}'} \|\tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}'}\|_2^2 \Big) \\
&= C^2 \Big(\sum_{\boldsymbol{w}} \sum_{u} |a(\boldsymbol{w}^* u\boldsymbol{s})|^2 \Big) \Big(\sum_{\boldsymbol{w}'} \sum_{v} |\xi(\boldsymbol{w}'^* v\boldsymbol{t}|^2 \Big).
\end{aligned}$$

The second inequality is the crucial place where we use the assumption that A^1 and A^2 satisfy a Haagerup-type condition with constant C. The third inequality comes from the Cauchy–Schwarz inequality.

We have seen that the $P_{\boldsymbol{s},\boldsymbol{t}}$'s form an orthogonal family of projections. Consequently, with the understanding that $\ell(\boldsymbol{s}) < k - q$, $\ell(\boldsymbol{t}) < n - q$, and $\mu(\boldsymbol{s}) = \mu(\boldsymbol{t})$, with $\boldsymbol{s} = 1$ and $\boldsymbol{t} = 1$ permitted, we obtain

$$\Big\| \sum_{\boldsymbol{s},\boldsymbol{t}} P_{\boldsymbol{s},\boldsymbol{t}}(a^*\xi) \Big\|_2^2 = \sum_{\boldsymbol{s},\boldsymbol{t}} \|P_{\boldsymbol{s},\boldsymbol{t}}(a^*\xi)\|_2^2$$

$$\leq C^2 \sum_{\boldsymbol{s},\boldsymbol{t}} \Big(\sum_{\boldsymbol{w}} \sum_{u} |a(\boldsymbol{w}^* u\boldsymbol{s})|^2 \Big) \Big(\sum_{\boldsymbol{w}'} \sum_{v} |\xi(\boldsymbol{w}'^* v\boldsymbol{t}|^2 \Big).$$

Now any given $\boldsymbol{y} \in \mathcal{B}_k$ has a unique expression as $\boldsymbol{y} = \boldsymbol{w}u\boldsymbol{s}$ for some $\boldsymbol{w}$ with $\ell(\boldsymbol{w}) \leq q$ and some $\boldsymbol{s}$ with $\ell(\boldsymbol{s}) < k - q$, and similarly for $\boldsymbol{z} \in \mathcal{B}_n$ as $\boldsymbol{z} = \boldsymbol{w}v\boldsymbol{t}$. It is easily seen from this that we obtain

$$\Big\| \sum_{\boldsymbol{s},\boldsymbol{t}} P_{\boldsymbol{s},\boldsymbol{t}}(a^*\xi) \Big\|_2^2 \leq C^2 \|a\|_2^2 \|\xi\|_2^2.$$

Notice that if q is not an integer, so that $P_m = \sum P_{\boldsymbol{s},\boldsymbol{t}}$, then this already gives the desired inequality, and the proof of the theorem is complete.

Suppose instead that q is an integer and that $\mu(\boldsymbol{s}) \neq \mu(\boldsymbol{t})$, so that $Q_{\boldsymbol{s},\boldsymbol{t}}$ is defined. Then from Lemma 6.8 we have

$$Q_{\boldsymbol{s},\boldsymbol{t}}(a^*\xi) = \sum_{\boldsymbol{w},u,v,r_2} \bar{a}(\boldsymbol{w}^* u\boldsymbol{s}) \xi(\boldsymbol{w}^* v r_2 \boldsymbol{t}) \boldsymbol{s}^* P_{m(\boldsymbol{s}, r_2\boldsymbol{t})}(uv) r_2 \boldsymbol{t}$$

$$+ \sum_{\boldsymbol{w},u,v,r_1} \bar{a}(\boldsymbol{w}^* u r_1 \boldsymbol{s}) \xi(\boldsymbol{w}^* v\boldsymbol{t}) \boldsymbol{s}^* r_1 P_{m(r_1\boldsymbol{s},\boldsymbol{t})}(uv)\boldsymbol{t},$$

where in both sums $\boldsymbol{w} \in \mathcal{B}$ with $\ell(\boldsymbol{w}) \leq q$ and $u, v, r_1, r_2 \in \mathcal{B}^1 \cup \mathcal{B}^2 \setminus \{1\}$ with $\mu(u) = \mu(v)$, while in the first sum

$$\mu(\boldsymbol{s}) \neq \mu(u) \neq \mu(\boldsymbol{w}) \quad \text{and} \quad \mu(v) \neq \mu(r_2) \neq \mu(\boldsymbol{t}),$$
$$\ell(u) = k(\boldsymbol{s}, \boldsymbol{w}), \quad \ell(v) = n(r_2\boldsymbol{t}, \boldsymbol{w}), \quad \text{and} \quad \ell(r_2\boldsymbol{t}) = n - q,$$

whereas in the second sum

$$\mu(\boldsymbol{s}) \neq \mu(r_1) \neq \mu(u) \quad \text{and} \quad \mu(\boldsymbol{w}) \neq \mu(v) \neq \mu(\boldsymbol{t}),$$
$$\ell(u) = k(r_1\boldsymbol{s}, \boldsymbol{w}), \quad \ell(v) = n(\boldsymbol{t}, \boldsymbol{w}) \quad \text{and} \quad \ell(r_1\boldsymbol{s}) = k - q.$$

For each $\boldsymbol{w}$ with $\ell(\boldsymbol{w}) \leq q$ let us define $\tilde{a}_{\boldsymbol{s},\boldsymbol{w}}$, etc. much as before by

$$\tilde{a}_{\boldsymbol{s},\boldsymbol{w}} = \sum_{u} a(\boldsymbol{w}^* u\boldsymbol{s}) u, \qquad \tilde{\xi}_{r_2\boldsymbol{t},\boldsymbol{w}} = \sum_{v} \xi(\boldsymbol{w}^* v r_2 \boldsymbol{t}) v,$$
$$\tilde{a}_{r_1\boldsymbol{s},\boldsymbol{w}} = \sum_{u} a(\boldsymbol{w}^* u r_1 \boldsymbol{s}) u, \qquad \tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}} = \sum_{v} \xi(\boldsymbol{w}^* v\boldsymbol{t}) v,$$

with the restrictions on u and v as above. Then in terms of this notation we have

$$Q_{\boldsymbol{s},\boldsymbol{t}}(a^*\xi) = \sum_{\boldsymbol{w},r_2} \boldsymbol{s}^* P_{m(\boldsymbol{s},r_2\boldsymbol{t})}(\tilde{a}_{\boldsymbol{s},\boldsymbol{w}} \tilde{\xi}_{r_2\boldsymbol{t},\boldsymbol{w}}) r_2 \boldsymbol{t} + \sum_{\boldsymbol{w},r_1} \boldsymbol{s}^* r_1 P_{m(r_1\boldsymbol{s},\boldsymbol{t})}(\tilde{a}_{r_1\boldsymbol{s},\boldsymbol{w}} \tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}}) \boldsymbol{t}.$$

Since the two summands above may not be orthogonal, but the terms within each sum over r_1 and r_2 are orthogonal, we obtain

$$
\begin{aligned}
\|Q_{\boldsymbol{s},\boldsymbol{t}}(a^*\xi)\|_2^2 &\le 2\Big(\Big\|\sum_{\boldsymbol{w},r_2}\boldsymbol{s}^*P_{m(\boldsymbol{s},r_2\boldsymbol{t})}(\tilde{a}_{\boldsymbol{s},\boldsymbol{w}}\tilde{\xi}_{r_2\boldsymbol{t},\boldsymbol{w}})r_2\boldsymbol{t}\Big\|_2^2 \\
&\qquad + \Big\|\sum_{\boldsymbol{w},r_1}\boldsymbol{s}^*r_1P_{m(r_1\boldsymbol{s},\boldsymbol{t})}(\tilde{a}_{r_1\boldsymbol{s},\boldsymbol{w}}\tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}})\boldsymbol{t}\Big\|_2^2\Big) \\
&\le 2\sum_{r_2}\Big(\sum_{\boldsymbol{w}}\|P_{m(\boldsymbol{s},r_2\boldsymbol{t})}(\tilde{a}_{\boldsymbol{s},\boldsymbol{w}}\tilde{\xi}_{r_2\boldsymbol{t},\boldsymbol{w}})\|_2\Big)^2 \\
&\qquad + 2\sum_{r_1}\Big(\sum_{\boldsymbol{w}}\|P_{m(r_1\boldsymbol{s},\boldsymbol{t})}(\tilde{a}_{r_1\boldsymbol{s},\boldsymbol{w}}\tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}})\|_2\Big)^2 \\
&\le 2\sum_{r_2}\Big(\sum_{\boldsymbol{w}}C\|\tilde{a}_{\boldsymbol{s},\boldsymbol{w}}\|_2\|\tilde{\xi}_{r_2\boldsymbol{t},\boldsymbol{w}}\|_2\Big)^2 \\
&\qquad + \ 2\sum_{r_1}\Big(\sum_{\boldsymbol{w}}C\|\tilde{a}_{r_1\boldsymbol{s},\boldsymbol{w}}\|_2\|\tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}}\|_2\Big)^2 \\
&\le 2C^2\Big(\sum_{\boldsymbol{w}}\|\tilde{a}_{\boldsymbol{s},\boldsymbol{w}}\|_2^2\Big)\Big(\sum_{\boldsymbol{w}',r_2}\|\tilde{\xi}_{r_2\boldsymbol{t},\boldsymbol{w}'}\|_2^2\Big) \\
&\qquad + \ 2C^2\Big(\sum_{\boldsymbol{w},r_1}\|\tilde{a}_{r_1\boldsymbol{s},\boldsymbol{w}}\|_2^2\Big)\Big(\sum_{\boldsymbol{w}'}\|\tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}'}\|_2^2\Big).
\end{aligned}
$$

We have seen that the $Q_{\boldsymbol{s},\boldsymbol{t}}$'s form an orthogonal family of projections. Consequently, with the understanding that $\ell(\boldsymbol{s}) < k - q$, $\ell(\boldsymbol{t}) < n - q$ and $\mu(\boldsymbol{s}) \neq \mu(\boldsymbol{t})$, with $\boldsymbol{s} = 1$ and/or $\boldsymbol{t} = 1$ permitted, we obtain

$$
\begin{aligned}
\Big\|\sum_{\boldsymbol{s},\boldsymbol{t}}Q_{\boldsymbol{s},\boldsymbol{t}}(a^*\xi)\Big\|_2^2 &= \sum_{\boldsymbol{s},\boldsymbol{t}}\|Q_{\boldsymbol{s},\boldsymbol{t}}(a^*\xi)\|_2^2 \\
&\le 2C^2\sum_{\boldsymbol{s},\boldsymbol{t}}\Big(\Big(\sum_{\boldsymbol{w}}\|\tilde{a}_{\boldsymbol{s},\boldsymbol{w}}\|_2^2\Big)\Big(\sum_{\boldsymbol{w}',r_2}\|\tilde{\xi}_{r_2\boldsymbol{t},\boldsymbol{w}'}\|_2^2\Big) \\
&\qquad + \Big(\sum_{\boldsymbol{w},r_1}\|\tilde{a}_{r_1\boldsymbol{s},\boldsymbol{w}}\|_2^2\Big)\Big(\sum_{\boldsymbol{w}'}\|\tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}'}\|_2^2\Big)\Big).
\end{aligned}
$$

Now again any given $\boldsymbol{y} \in \mathcal{B}_k$ has a unique expression as $\boldsymbol{y} = \boldsymbol{w}u\boldsymbol{s}$ for some $\boldsymbol{w}$ with $\ell(\boldsymbol{w}) \le q$ and some $\boldsymbol{s}$ with $\ell(\boldsymbol{s}) < k - q$; furthermore, if $\boldsymbol{y}$ can be expressed as $\boldsymbol{y} = \boldsymbol{w}ur_1\boldsymbol{s}$ with $\ell(r_1) = k - 1 - \ell(\boldsymbol{s})$ and $\ell(u) + \ell(\boldsymbol{w}) = q$, then this expression too is unique. A similar statement holds for any $\boldsymbol{z} \in \mathcal{B}_n$ as $\boldsymbol{z} = \boldsymbol{w}v\boldsymbol{t}$ or $\boldsymbol{z} = \boldsymbol{w}vr_2\boldsymbol{t}$. In the same way as for the $P_{\boldsymbol{s},\boldsymbol{t}}$'s it is then easily seen that

$$
\Big\|\sum_{\boldsymbol{s},\boldsymbol{t}}Q_{\boldsymbol{s},\boldsymbol{t}}(a^*\xi)\Big\|_2^2 \le 4C^2\|a\|_2^2\|\xi\|_2^2.
$$

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Since P_m is the orthogonal sum of the $P_{\boldsymbol{s},\boldsymbol{t}}$'s and the $Q_{\boldsymbol{s},\boldsymbol{t}}$'s, it follows that

$$\|P_m(a^*\xi)\|_2 \le \sqrt{5}C\|a\|_2\|\xi\|_2$$

as desired.

Finally, we must treat the cases in which $k-q=0$ or $n-q=0$. If $k-q=0$ then $m+k=n$. We follow the pattern of proof of the previous cases, and so allow ourselves less detailed notation and discussion. For any $\boldsymbol{t}\in\mathcal{B}$ with $\ell(\boldsymbol{t})<m$ set

$$\mathcal{B}_{(\boldsymbol{t})} = \{\boldsymbol{x}\in\mathcal{B}_m : \boldsymbol{x} = r\boldsymbol{t} \text{ with } r\in\mathcal{B}^1\cup\mathcal{B}^2\setminus\{1\},\ \mu(r)\ne\mu(\boldsymbol{t})\}.$$

We permit $\boldsymbol{t}=1$. It is easily seen that the $\mathcal{B}_{(\boldsymbol{t})}$'s are disjoint and that their union is $\mathcal{B}_m$. We let $E_{(\boldsymbol{t})}$ denote the linear span of $\mathcal{B}_{(\boldsymbol{t})}$, and we let $P_{(\boldsymbol{t})}$ denote the projection onto $E_{(\boldsymbol{t})}$.

Lemma 6.9 *Let $\boldsymbol{y}\in\mathcal{B}_k$ and $\boldsymbol{z}\in\mathcal{B}_n$. If $P_{(\boldsymbol{t})}(\boldsymbol{y}^*\boldsymbol{z})\ne 0$ then $\boldsymbol{y}$ and $\boldsymbol{z}$ are of the form $\boldsymbol{y}=\boldsymbol{w}^*u$ and $\boldsymbol{z}=\boldsymbol{w}^*v\boldsymbol{t}$ where $u,v\in\mathcal{B}^1\cup\mathcal{B}^2$, with $\ell(\boldsymbol{w})\le k$ and $\mu(u)\ne\mu(\boldsymbol{w})\ne\mu(v)\ne\mu(\boldsymbol{t})$ and $v\ne 1$. (But we may have $u=1$.) Then $P_{(\boldsymbol{t})}(\boldsymbol{y}^*\boldsymbol{z}) = P_{m(\boldsymbol{t})}(uv)\boldsymbol{t}$ where $m(\boldsymbol{t}) = m-\ell(\boldsymbol{t})$.*

Proof According to Lemma 6.3 we can express $\boldsymbol{y}$ and $\boldsymbol{z}$ as $\boldsymbol{y}=\boldsymbol{w}^*u\hat{\boldsymbol{s}}$ and $\boldsymbol{z}=\boldsymbol{w}^*v\hat{\boldsymbol{t}}$ where among the conditions we have $\ell(\hat{\boldsymbol{s}})\le k-q=0$. Thus $\hat{\boldsymbol{s}}=1$. So $\boldsymbol{y}=\boldsymbol{w}^*u$ with $\mu(\boldsymbol{w})\ne\mu(u)$. We will also have $\ell(\boldsymbol{w})\le q=k$ and $\ell(\hat{\boldsymbol{t}})\le n-q=m$. Suppose that $v=1$. Then $\ell(\boldsymbol{w})+\ell(\hat{\boldsymbol{t}})=\ell(\boldsymbol{z})=n=k+m$, and so $\ell(\boldsymbol{w})=k$, $\ell(\hat{\boldsymbol{t}})=m$ and $u=1$, which contradicts Lemma 6.3. Thus $v\ne 1$. We can set $\boldsymbol{t}=\hat{\boldsymbol{t}}$. Then from Lemma 6.3 we have $P_{(\boldsymbol{t})}(\boldsymbol{y}^*\boldsymbol{z}) = P_{m(\boldsymbol{t})}(uv)\boldsymbol{t}$. ∎

Suppose now that $a\in E_k$ and $\xi\in E_n$. Then, much as in the previous cases, we have

$$P_{(\boldsymbol{t})}(a^*\xi) = \sum_{\boldsymbol{w},u,v}\bar{a}(\boldsymbol{w}^*u)\xi(\boldsymbol{w}^*v\boldsymbol{t})P_{m(\boldsymbol{t})}(uv)\boldsymbol{t},$$

where the conditions on $\boldsymbol{w}$, u, v are as above. We set

$$\tilde{a}_{\boldsymbol{w}} = \sum_u \bar{a}(\boldsymbol{w}^*u)u,\quad \tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}} = \sum_v \xi(\boldsymbol{w}^*v\boldsymbol{t})v\boldsymbol{t}.$$

Thus

$$\begin{aligned}\|P_{(\boldsymbol{t})}(a^*\xi)\|_2^2 &\le \Big(\sum_{\boldsymbol{w}}\|P_{m(\boldsymbol{t})}(\tilde{a}_{\boldsymbol{w}}\tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}})\|_2\Big)^2 \le \Big(\sum_{\boldsymbol{w}} C\|\tilde{a}_{\boldsymbol{w}}\|_2\|\tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}}\|_2\Big)^2\\ &\le C^2\Big(\sum_{\boldsymbol{w}}\|\tilde{a}_{\boldsymbol{w}}\|_2^2\Big)\Big(\sum_{\boldsymbol{w}'}\|\tilde{\xi}_{\boldsymbol{t},\boldsymbol{w}'}\|_2^2\Big)\\ &= C^2\Big(\sum_{\boldsymbol{w},u}|\bar{a}(\boldsymbol{w}^*u|^2\Big)\Big(\sum_{\boldsymbol{w}',v}|\xi(\boldsymbol{w}'^*v\boldsymbol{t}|^2\Big).\end{aligned}$$

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Consequently

$$\begin{aligned}|P_m(a^*\xi)|_2^2 &= \sum_{\boldsymbol{t}} \|P_{(\boldsymbol{t})}(a^*\xi)\|^2 \\ &\leq C^2 \sum_{\boldsymbol{t}} \Big(\sum_{\boldsymbol{w},u} |\bar{a}(\boldsymbol{w}^* u|^2\Big)\Big(\sum_{\boldsymbol{w}',v} |\xi(\boldsymbol{w}'^* v\boldsymbol{t}|^2\Big).\end{aligned}$$

Now because $k + m = n$ it is easily seen that any given $\boldsymbol{z} \in \mathcal{B}_n$ has a unique expression as $\boldsymbol{z} = \boldsymbol{w}v\boldsymbol{t}$ where $\ell(\boldsymbol{w}) \leq k$, $\ell(\boldsymbol{t}) < m$, and $v \in \mathcal{B}^1 \cup \mathcal{B}^2 \setminus \{1\}$. However a $\boldsymbol{y} \in \mathcal{B}_k$ will have two expressions as $\boldsymbol{w}u$ with $\ell(\boldsymbol{w}) \leq k$ and $u \in \mathcal{B}^1 \cup \mathcal{B}^2$ (and $\mu(\boldsymbol{w}) \neq \mu(u)$), one of which will be $\boldsymbol{y} = \boldsymbol{w}$. It follows that

$$\|P_m(a^*\xi)\|_2^2 \leq 2C^2 \|a\|_2^2 \|\xi\|_2^2,$$

which implies the desired inequality.

Finally, we must deal with the case in which $n - q = 0$. But this case follows from essentially the mirror image of the above argument, in which now for $\ell(\boldsymbol{s}) < m$ the elements of $\mathcal{B}_{(\boldsymbol{s})}$ have form $\boldsymbol{x} = \boldsymbol{s}r$, and later we find that $(\boldsymbol{y}, \boldsymbol{z})$ must have the form $\boldsymbol{y} = \boldsymbol{w}^* u\boldsymbol{s}$ and $\boldsymbol{z} = \boldsymbol{w}^* v$. ∎

Question 6.10 What happens for amalgamated free products of C^*-algebras? What happens if σ_1 and σ_2 are not tracial?

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