2
Loop coordinates and the extended group of loops

2.1 Introduction

Continuing with the idea of describing gauge theories in terms of loops, we will now introduce a set of techniques that will aid us in the description of loops themselves. The idea is to represent loops with a set of objects that are more amenable to the development of analytical techniques. The advantages of this are many: whereas there is limited experience in dealing with functions of loops, there is a significant machinery to deal with analytic functions. They even present advantages for treatment with computer algebra.

Surprisingly, we will see that the end result goes quite beyond our expectations. The quantities we originally introduced to describe loops immediately reveal themselves as having great potential to replace loops altogether from the formulation and go beyond, allowing the development of a reformulation of gauge theories that is entirely new. This formulation introduces new perspectives with respect to the loop formulation that have not been fully developed yet, though we will see in later chapters some applications to gauge theories and gravitation.

The plan for the chapter is as follows: in section 2.2 we will start by introducing a set of tensorial objects that embody all the information that is needed from a loop to construct the holonomy and therefore to reconstruct any quantity of physical relevance for a gauge theory. In section 2.3 we will show how the group of loops is a subgroup of a Lie group with an associated Lie algebra, the extended loop group. The generators of this Lie group will turn out to be coordinates in the extended loop space, which we discuss in section 2.4. In section 2.5 we will study how the differential operators introduced in the previous chapter act on the loop coordinates. In particular we will study the action of the generator of diffeomorphisms. In section 2.6 we will discuss how to construct

https://doi.org/10.1017/9781009290203.004 Published online by Cambridge University Press

diffeomorphism invariant quantities in terms of loop coordinates and, in particular, knot invariants. In the conclusion we will discuss the differences and similarities between the group structures we have introduced and the usual Lie groups. The subject of this chapter has been discussed in detail in reference [20], the reader is referred to it for a more technical approach.

2.2 Multitangent fields as description of loops

As we discussed in the previous chapter, all the gauge invariant information present in a gauge field can be retrieved from the holonomy. Therefore the only information we really need to know from loops is that used in the definition of the holonomy,

$$H_A(\gamma) = P \, \exp\left(i \oint_\gamma A_a \, dy^a\right) . \tag{2.1}$$

We can write this definition more explicitly as

$$H_A(\gamma) = 1 + \sum_{n=1}^{\infty} i^n \int dx_1^3 \dots dx_n^3 A_{a_1}(x_1) \dots A_{a_n}(x_n) X^{a_1 \dots a_n}(x_1, \dots, x_n, \gamma), \tag{2.2}$$

where the loop dependent objects X are given by

$$\begin{aligned} & X^{a_1 \dots a_n}(x_1, \dots, x_n, \gamma) = \\ & \quad \oint_\gamma dy_n^{a_n} \int_o^{y_n} dy_{n-1}^{a_{n-1}} \dots \int_o^{y_2} dy_1^{a_1} \delta(x_n - y_n) \dots \delta(x_1 - y_1) = \\ & \quad \oint_\gamma dy_n^{a_n} \dots \oint_\gamma dy_1^{a_1} \delta(x_n - y_n) \dots \delta(x_1 - y_1) \Theta_\gamma(o, y_1, \dots, y_n) \end{aligned} \tag{2.3}$$

and $\Theta_\gamma(o, y_1, \dots, y_n)$ is a generalized Heaviside function that orders the points along the contour starting at the origin of the loop, i.e.,

$$\Theta_\gamma(o, y_1, \dots, y_n) = \begin{cases} 1 \text{ if } o < y_1 < y_2 < \dots y_n \text{ along the loop} \\ \qquad 0 \text{ otherwise.} \end{cases} \tag{2.4}$$

These relations define the X objects of "rank" n. We shall call them the multitangents of the loop γ.

By writing the holonomy in the non-standard form (2.2) we have been able to isolate all the loop dependent information in the multitangents of the loop. No more information from the loop is needed in order to compute the holonomy than that present in the multitangents of all orders.

In what follows, it will be convenient to introduce the notation

$$X^{\mu_1 \dots \mu_n}(\gamma) \equiv X^{a_1 \, x_1 \dots a_n \, x_n}(\gamma) \equiv X^{a_1 \dots a_n}(x_1, \dots, x_n, \gamma) \ , \tag{2.5}$$

with $\mu_i \equiv (a_i x_i)$, which is more suggestive of the role played by the x variables under diffeomorphisms. The X objects transform as multivector densities (they behave as a vector density at the point x_i on the index a_i) under the subgroup of coordinate transformations that leaves the base point o fixed. In other words if

$$x^a \longrightarrow x'^a = D^a(x) \tag{2.6}$$

then

$$X^{a_1 x'_1 \ldots a_n x'_n}(D\gamma) = \frac{\partial x'^{a_1}_1}{\partial x^{b_1}_1} \cdots \frac{\partial x'^{a_n}_n}{\partial x^{b_n}_n} \frac{1}{J(x_1)} \cdots \frac{1}{J(x_n)} X^{b_1 x_1 \ldots b_n x_n}(\gamma), \tag{2.7}$$

where J is the Jacobian of the transformation.

The Xs are not really "coordinates" in the sense that they are not independent. They are constrained by algebraic and differential relations.

The algebraic constraints stem from relations satisfied by the generalized Heaviside function,

$$\begin{aligned} &\Theta_\gamma(o, y_1, y_2, y_3) + \Theta_\gamma(o, y_2, y_1, y_3) + \Theta_\gamma(o, y_2, y_3, y_1) = \Theta_\gamma(o, y_2, y_3), \\ &\Theta_\gamma(o, y_1) = 1\,, \qquad\qquad \Theta_\gamma(o, y_1, y_2) + \Theta_\gamma(o, y_2, y_1) = 1, \end{aligned} \tag{2.8}$$

which imply the following kind of relations among the Xs,

$$\begin{aligned} X^{\mu_1\mu_2} + X^{\mu_2\mu_1} &= X^{\mu_1}\, X^{\mu_2}, \\ X^{\mu_1\mu_2\mu_3} + X^{\mu_2\mu_1\mu_3} + X^{\mu_2\mu_3\mu_1} &= X^{\mu_1}\, X^{\mu_2\mu_3}\,. \end{aligned} \tag{2.9}$$

And in general,

$$X^{\underline{\mu_1 \ldots \mu_k}\mu_{k+1}\ldots\mu_n} \equiv \sum_{P_k} X^{P_k(\mu_1 \mu_n)} = X^{\mu_1\ldots\mu_k}\, X^{\mu_{k+1}\ldots\mu_n}, \tag{2.10}$$

where the sum goes over all the permutations of the μ variables which preserve the ordering of the $\mu_1, \ldots, \mu_k$ and the $\mu_{k+1}, \ldots, \mu_n$ among themselves. We have introduced the notation of underlined indices to symbolize the permutation for future use.

The differential constraint ensures that the holonomy has the correct transformation properties under gauge transformations, and can be readily derived from equation (2.2). It is given by

$$\begin{aligned} &\frac{\partial}{\partial x^{a_i}_i} X^{a_1 x_1 \ldots a_i x_i \ldots a_n x_n} = \\ &\qquad \big(\delta(x_i - x_{i-1}) - \delta(x_i - x_{i+1})\big) X^{a_1 x_1 \ldots a_{i-1} x_{i-1}\, a_{i+1} x_{i+1} \ldots a_n x_n}. \end{aligned} \tag{2.11}$$

In this expression, both x_0 and x_{n+1} represent the base point of the loop.

An important property of the differential constraint is that *any* multitensor density $D^{a_1 x_1 \ldots a_n x_n}$ that satisfies it can be put into equation (2.2) and the resulting object is a *gauge covariant* quantity. When restricted to the multitangents of a loop, the resulting object is the holonomy. It is this property that exhibits the relevance of this formulation. In it, loops are only a particular case. One can, in general, deal with arbitrary multitensor densities and construct gauge invariant objects, for instance by taking the trace. The multitensor densities need not have the same distributional character as the multitangents associated with a loop. Their divergence structure is dictated by the differential constraint, which requires its solutions to be distributional. This will have important consequences later. We will call the space of all multitensors that satisfy the differential constraints $\mathcal{D}_o$.

With this construction in hand, one could go further and forget loops and holonomies altogether. Since one can represent any gauge covariant object using the Ds, one could represent a gauge theory *entirely* in terms of Ds. This has not been done up to present for non-Abelian theories in a complete fashion (nor for gravity), but it can be easily worked out for an Abelian theory, as we will do in chapter 4.

When one allows arbitrary multitensors in (2.2) the convergence of the series is not guaranteed. There is no easy way to prescribe multitensors such that the series converges, so we will assume from now on that we work only with multitensors such that the series converges. Even this requirement is not enough to produce an object with a gauge invariant trace. The differential constraint (2.11) only ensures that if one performs a gauge transformation on the trace of the holonomy of a multitensor the resulting series has terms that cancel in pairs. For this to imply gauge invariance, it has to happen that [222]

$$\sum_{k=1}^{N} A_{\mu_1} \dots A_{\mu_k} [A, \Lambda]_{\mu_k} A_{\mu_{k+1}} \dots A_{\mu_n} X^{\mu_1 \dots \mu_n} \tag{2.12}$$

goes to zero as $N \to \infty$. Λ is the parameter of the gauge transformation and is therefore an arbitrary function. Notice that the vanishing of (2.12) is not guaranteed by the convergence of the holonomy alone. The question of selecting an appropriate set of multitensors in a precise way in order to ensure convergence of these expressions is at present not settled, see reference [21].

2.3 The extended group of loops

When we introduced the group of loops in the previous chapter, we noticed that no one-parameter subgroup existed (since one could only define integer powers of the generators) and therefore it did not form a Lie group. In this section we will introduce a Lie group, the "extended loop group". The group of loops will be a subgroup of it. This construction is of interest in itself, since it is clear that it is a great advantage to have at our disposal all the machinery of Lie groups to analyze loops. Among other results, by identifying the free parameters of the algebra associated with the extended loop group we will be able to solve automatically the homogeneous part of the differential and algebraic constraints (2.10), (2.11) of section 2.2. With some additional construction, we will have a definition for the portion of the multitensor density fields that is unconstrained, i.e., *that we can freely specify.* They can therefore genuinely be called "coordinates" and contain as a subspace the "loop coordinates" or coordinates on loop space. We will elaborate more on this concept in section 2.4. Now we will proceed to construct the extended loop group.

2.3.1 The special extended group of loops

Let us start by considering arbitrary* multitensor densities similar to those introduced in section 2.2 and define a quantity $\mathbf{E}$ by

$$\mathbf{E} = (E, E^{\mu_1}, \ldots, E^{\mu_1\ldots\mu_n}, \ldots) \equiv (E, \vec{E}), \tag{2.13}$$

where E is a real number and $E^{\mu_1,\ldots,\mu_n}$ (for any $n \neq 0$) is an arbitrary multivector density field. It can be readily checked that the set of these quantities has the structure of a vector space (denoted as $\mathcal{E}$) with the usual composition laws of addition and multiplication.

We will now introduce a product law in $\mathcal{E}$ in the following way: given two vectors $\mathbf{E}_1$ and $\mathbf{E}_2$, we define $\mathbf{E}_1 \times \mathbf{E}_2$ as the vector with components

$$\mathbf{E}_1 \times \mathbf{E}_2 = (E_1E_2, E_1\vec{E}_2 + \vec{E}_1E_2 + \vec{E}_1 \times \vec{E}_2), \tag{2.14}$$

where $\vec{E}_1 \times \vec{E}_2$ is given by

$$(\vec{E}_1 \times \vec{E}_2)^{\mu_1\ldots\mu_n} = \sum_{i=1}^{n-1} E_1^{\mu_1\ldots\mu_i}\, E_2^{\mu_{i+1}\ldots\mu_n} \quad . \tag{2.15}$$

For any value of n, the rank n component of the $\times$-product of elements of $\mathcal{E}$ can be expressed as

$$(\mathbf{E}_1 \times \mathbf{E}_2)^{\mu_1\ldots\mu_n} = \sum_{i=0}^{n} E_1^{\mu_1\ldots\mu_i}\, E_2^{\mu_{i+1}\ldots\mu_n} \tag{2.16}$$

with the convention

$$E^{\mu_1\ldots\mu_0} = E^{\mu_{n+1}\ldots\mu_n} = E \quad . \tag{2.17}$$

* In this chapter we will always discuss *real* multitensor fields. It is obvious that the formalism is unchanged if one allows complex fields. In some applications they seem to play an important role, as we will see in section 3.4.2 (see also [19]).

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The product law is associative and distributive with respect to the addition of vectors. It has a null element (the null vector) and an identity element, given by

$$\mathbf{I} = (1, 0, \ldots, 0, \ldots) \ . \tag{2.18}$$

An inverse element exists for all vectors with non-vanishing zeroth rank component. It is given by

$$\mathbf{E}^{-1} = E^{-1}\mathbf{I} + \sum_{i=1}^{\infty}(-1)^i E^{-i-1}(\mathbf{E} - E\mathbf{I})^i, \tag{2.19}$$

such that

$$\mathbf{E} \times \mathbf{E}^{-1} = \mathbf{E}^{-1} \times \mathbf{E} = \mathbf{I} \ . \tag{2.20}$$

When evaluating the components of $\mathbf{E}^{-1}$ it should be noticed that the sum involved in (2.19) is actually finite due to the fact that $(\mathbf{E} - E\mathbf{I})$ is a vector with its zeroth rank component equal to zero. Therefore,

$$[\,\vec{E} \times \overset{i}{\ldots} \times \vec{E}]^{\mu_1 \ldots \mu_n} = \left[\,(\mathbf{E} - E\mathbf{I})^i\,\right]^{\mu_1 \ldots \mu_n} = 0 \ \text{if} \ n < i \ . \tag{2.21}$$

The set of all vectors with non-vanishing zeroth rank component (notice the role of E^{-1} in equation (2.20)) forms a group with the $\times$-product as composition law.

The $\times$-product law has an interesting property when restricted to multitangents. In this case it just corresponds to the composition law of loops,

$$\mathbf{X}(\gamma_1) \times \mathbf{X}(\gamma_2) = \mathbf{X}(\gamma_1 \circ \gamma_2), \tag{2.22}$$

where $\mathbf{X}(\gamma) = (1, X^{\mu_1}(\gamma), \ldots, X^{\mu_1 \ldots \mu_n}(\gamma), \ldots)$. Therefore we see that the product law that gave rise to the group of loops is the same product law we are generalizing to the case of arbitrary multitensor fields. The $\times$-product law can also represent more general compositions than those of two loops sharing a common basepoint, such as the composition of an open path with a loop at its end, assuming a generalization of the definition of multitangents to open paths.

After all this construction, let us now make contact with the group of loops. First, let us restrict attention to multitensors (not necessarily associated with a loop) that satisfy the constraints (2.10), (2.11). Consider the set of vectors $\mathcal{X} \in \mathcal{E}$ that have their zeroth rank component equal to one, $\mathbf{X} = (1, \vec{X})$.

The set $\mathcal{X}$ is closed under the $\times$-product law. If $\mathbf{X}_1 \in \mathcal{X}$ and $\mathbf{X}_2 \in \mathcal{X}$, it is clear from the definition of the group product that $\mathbf{X}_1 \times \mathbf{X}_2$ satisfies the differential constraint. One can also demonstrate that $\mathbf{X}_1 \times \mathbf{X}_2$ satisfies the algebraic constraint. In a similar way one can show that the inverse $\mathbf{X}^{-1}$ given by (2.19) satisfies the constraints if $\mathbf{X}$ does. A detailed proof of these properties can be seen in the appendices of reference [20]. These results show that the group structure under the $\times$-composition law is preserved by the imposition of the algebraic and differential constraints. We call $\mathcal{X}$ the Special-extended Loop group (SeL group)†. Note that the zeroth rank component of $\mathbf{E}$ plays a role analogous to the determinant in a group of matrices. For this reason we introduce the name Special when selecting $E = 1$.

The group of loops is a subgroup of the SeL group since $\mathbf{X}(\gamma) \in \mathcal{X}$ and the composition law of the group of loops $\circ$ is mapped via (2.22) to the $\times$-product.

An important question at this point is: is the group SeL just a fancy rewriting of the group of loops, or is it actually a more general structure? We will show that SeL is actually larger than the group of loops by direct construction. Consider the group element $\mathbf{X}^m \equiv \mathbf{X}\times \overset{m}{\dots} \times\mathbf{X}$. Note that if $\mathbf{X}$ gives the multitangent field of certain loop γ, $\mathbf{X}^m$ would be the multitangent field of the loop γ swept itself m times. Applying the binomial expansion we get,

$$\mathbf{X}^m \equiv [\mathbf{I} + (\mathbf{X} - \mathbf{I})]^m = \mathbf{I} + \sum_{i=1}^{m} \binom{m}{i} (\mathbf{X} - \mathbf{I})^i \ . \tag{2.23}$$

The extension of (2.23) to real values of m is straightforward, being defined as

$$\mathbf{X}^\lambda = \mathbf{I} + \sum_{i=1}^{\infty} \binom{\lambda}{i} (\mathbf{X} - \mathbf{I})^i \tag{2.24}$$

with λ real. We usually call this object the analytic extension of $\mathbf{X}$. Note that for $\lambda = -1$ we recover the expression of the inverse of $\mathbf{X}$. Also in this case, due to (2.21) the analytic extension is well defined for all elements of $\mathcal{X}$. One can prove that if $\mathbf{X}$ is constrained by the differential and algebraic identities, its analytic extension also satisfies the constraints (again see the appendices of [20]). So, the analytic extension of any $\mathbf{X}$ is in $\mathcal{X}$. Moreover, we have

$$\mathbf{X}^\lambda \times \mathbf{X}^\mu = \mathbf{X}^{\lambda+\mu} \ . \tag{2.25}$$

We conclude that the set $\{\mathbf{X}^\lambda / \lambda \in \mathrm{R}$ and $\mathbf{X}$ a given element of $\mathcal{X}\}$ defines an Abelian one-parameter subgroup of the $\mathcal{X}$ group.

For non-integer values of λ, the λth power of a multitangent is not a multitangent. This fact explicitly shows that there exist in $\mathcal{X}$ other elements besides the loop coordinates.

† Tavares [43] has also considered this group. His "shuffle product" is associated with the algebraic constraint in our terminology.

Matrix representations of the SeL group can be generated through a natural extension of the holonomy. The extended holonomy associated with a non-Abelian connection $A_{ax} \equiv A_a(x)$ is defined as $H_A(\mathbf{X}) = \mathbf{A} \cdot \mathbf{X}$, where

$$\mathbf{A} \equiv (1, iA_{a_1 x_1}, \ldots, i^n A_{a_n x_n}, \ldots), \tag{2.26}$$

$$\mathbf{X} \equiv (1, X^{a_1 x_1}, \ldots, X^{a_1 x_1 \ldots a_n x_n}, \ldots), \tag{2.27}$$

and the dot acts like a generalized Einstein convention including contractions of the discrete indices a_i and integrals over the three-manifold in the continuous indices x_i. We have

$$H_A(\mathbf{X}_1)\, H_A(\mathbf{X}_2) = \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} i^j A_{\mu_1 \ldots \mu_k} A_{\mu_{k+1} \ldots \mu_j} X_1^{\mu_1 \ldots \mu_k} X_2^{\mu_{k+1} \ldots \mu_j}$$

$$= \sum_{j=0}^{\infty} i^j A_{\mu_1 \ldots \mu_j} \left(\sum_{k=0}^{j} X_1^{\mu_1 \ldots \mu_k} X_2^{\mu_{k+1} \ldots \mu_j} \right) = H_A(\mathbf{X}_1 \times \mathbf{X}_2), \tag{2.28}$$

where convention (2.17) has been applied over all the indices. The correspondence $\mathbf{X} \rightarrow H_A(\mathbf{X})$ gives a representation of the SeL group into a particular gauge group. In the case of the $\mathcal{X}$ group and the connections A belonging to the algebra of a unitary group, $H_A(\mathbf{X})$ is an element of the given unitary group. If one considers multitensors that do not satisfy the algebraic constraint, one still has a group and can construct a representation by considering As that belong to a unitary gauge algebra. However, the corresponding representation will give a holonomy that is not an element of the gauge group. It will, in general, be an element of the general linear group of the same dimension as the gauge group. This highlights the role of the algebraic constraint in this formalism. The differential constraint imposed on $\mathbf{X}$ ensures that $H_A(\mathbf{X})$ is a gauge covariant quantity provided that the expressions involved in the proofs converge (see chapter 12 for some subtleties on this issue).

We have shown that the analytic extension of any element of the SeL group defines a one-parameter subgroup. By studying its properties one can find the algebra associated with the SeL group.

2.3.2 Generators of the SeL group

Consider the one-parameter subgroup $\{\mathbf{X}^\lambda\}$ and suppose that we increase λ by an infinitesimal amount. We can write

$$\mathbf{X}^{\lambda + d\lambda} = \mathbf{X}^\lambda \times \mathbf{X}^{d\lambda} = \mathbf{X}^\lambda + \frac{d\mathbf{X}^\lambda}{d\lambda} d\lambda \ , \tag{2.29}$$

and taking $\lambda = 0$ we get

$$\mathbf{X}^{d\lambda} = \mathbf{I} + \mathbf{F}\, d\lambda, \tag{2.30}$$

where

$$\mathbf{F} \equiv \frac{d\mathbf{X}^\lambda}{d\lambda}|_{\lambda=0} = \left(0, \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \vec{X}^i\right) = (0, \vec{F}) \ . \tag{2.31}$$

Introducing (2.30) in (2.29) we obtain the following differential equation for the elements of $\{\mathbf{X}^\lambda\}$

$$\frac{d\mathbf{X}^\lambda}{d\lambda} = \mathbf{X}^\lambda \times \mathbf{F} = \mathbf{F} \times \mathbf{X}^\lambda \ . \tag{2.32}$$

This equation can be iteratively integrated to give

$$\mathbf{X}^\lambda = \mathbf{I} + \sum_{k=1}^{n} \frac{\lambda^k}{k!}\mathbf{F}^k + \mathbf{F}^{n+1} \times \int_0^\lambda d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \ldots \int_0^{\lambda_n} d\lambda_{n+1}\, \mathbf{X}^{\lambda_{n+1}} \ . \tag{2.33}$$

The process actually stops for any finite rank n component ($\mathbf{F}^{n+1} = \mathbf{F}\times \overset{n+1}{\cdots} \times\mathbf{F} = 0$ in this case). Therefore

$$\mathbf{X}^\lambda = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!}\mathbf{F}^k = \exp(\lambda\mathbf{F}) \ . \tag{2.34}$$

We conclude that the vector $\mathbf{F}$ given by (2.31) is the generator of the one-parameter subgroup $\{\mathbf{X}^\lambda\}$. It is evident that the generator satisfies the differential constraint. We shall now prove the following fundamental property: $\mathbf{F}$ satisfies the homogeneous algebraic constraint (i.e., the sum over permutations defined in equation (2.10) vanishes). In other words, the generator of the one-parameter subgroup $\{\mathbf{X}^\lambda\}$ is the algebraic free part of $\mathbf{X}$.

We know that

$$(\mathbf{X}^\lambda)^{\underline{\mu_1\cdots\mu_k}\mu_{k+1}\cdots\mu_n} = (\mathbf{X}^\lambda)^{\mu_1\cdots\mu_k}\, (\mathbf{X}^\lambda)^{\mu_{k+1}\cdots\mu_n} \ . \tag{2.35}$$

Differentiating with respect to λ and evaluating for $\lambda = 0$ we get

$$\begin{aligned}\frac{d}{d\lambda}\left(\mathbf{X}^\lambda\right)^{\underline{\mu_1\cdots\mu_k}\mu_{k+1}\cdots\mu_n}_{\lambda=0} &= \left(\frac{d\mathbf{X}^\lambda}{d\lambda}\right)^{\mu_1\cdots\mu_k}_{\lambda=0} I^{\mu_{k+1}\cdots\mu_n} \\ &\quad + I^{\mu_1\cdots\mu_k}\left(\frac{d\mathbf{X}^\lambda}{d\lambda}\right)^{\mu_{k+1}\cdots\mu_n}_{\lambda=0} \ .\end{aligned} \tag{2.36}$$

As $1 \leq k < n$, we conclude

$$F^{\underline{\mu_1\cdots\mu_k}\mu_{k+1}\cdots\mu_n} = 0 \ , \quad 1 \leq k < n \ . \tag{2.37}$$

Reciprocally, one can demonstrate that the exponential of any algebraically free quantity produces an object that satisfies the algebraic constraint. It is important to stress that these results allow us to obtain the general solution for the algebraic constraint (equation (2.34) with $\lambda = 1$

and its inverse (2.31) give the relationship between an object that satisfies the algebraic constraint and its algebraic-free part).

The set of all **F**s that satisfy the differential constraint and the homogeneous algebraic constraint forms a vector space $\mathcal{F}$. One can define a bilinear operation on $\mathcal{F}$ in the following way,

$$[\mathbf{F_1}, \mathbf{F_2}] = \mathbf{F_1} \times \mathbf{F_2} - \mathbf{F_2} \times \mathbf{F_1} \quad \text{for any } \mathbf{F_1}, \mathbf{F_2} \in \mathcal{F}. \tag{2.38}$$

This operation is closed on $\mathcal{F}$. The vector space $\mathcal{F}$ together with the bracket operation (2.38) defines the Lie algebra associated with the SeL group.

2.4 Loop coordinates

The quantities **X** that we introduced in section 2.2 are not freely specifiable. That is, in order to be able to construct a gauge covariant object via equation (2.1), the **X**s had to satisfy the differential and algebraic constraints (2.10), (2.11). That they are not freely specifiable is a natural thing, since they are elements of a group. That is why it was important to find the associated algebra, since its free parameters give us a chance to separate the part of the multitangents that we can freely specify. In the previous section we saw how to construct the set of objects $\mathcal{F}$. These objects had the advantage of being constrained not by the algebraic constraint, but by the homogeneous algebraic constraint. This latter constraint is very easily solvable, simply by requiring some symmetries on the **F**s, given by equation (2.37). In terms of the **F**s one immediately is able to compute a solution to both the differential and algebraic constraints making use of equation (2.34),

$$\mathbf{X} = \exp(\mathbf{F}). \tag{2.39}$$

However, the **F**s are far from freely prescribable since they are constrained by the differential constraint. The main intention of this section is to give a prescription for generating the **F**s (and through them the **X**s) from freely specifiable quantities. In order to do this we will need to introduce some technology to deal with transverse tensors. This technology will also be useful for dealing with knot invariants.

2.4.1 Transverse tensor calculus

First of all notice that the notion of transversality (divergence equal to zero) is well defined for vector densities, since their divergence can be computed without introducing an external metric. For instance, statements such as

$$\partial_{ax} E^{ax\,by} = 0 \tag{2.40}$$

are well defined for an object like E which is a vector density on the index a at the point x.

Let us introduce the notion of transverse and longitudinal projectors in the multivector density space. In order to do this, it is convenient to endow the space of transverse vector densities of rank one with a natural metric structure. Given two transverse fields V^{ax} and W^{ax} one can define their inner product [22],

$$\begin{aligned} g(V,W) &= \int d^3x \;\; V^a A_a^W, \\ \partial_a V^a &= \partial_a W^a = 0, \end{aligned} \tag{2.41}$$

where A_a^W is a "potential" defined in the following way. Construct a two-form $W_{ab} = \epsilon_{abc} W^c$. This two-form is curl-free, $\partial_{[c} W_{ab]}$, due to the transversality of W^a. Then one can define the one-form ("potential") A_a^W by $\partial_{[b} A_{a]}^W = W_{ab}$. This one-form is defined up to the addition of a gradient. This will force us to give *ad-hoc* prescriptions when dealing with expressions in terms of A_a^W. However, the inner product (2.41) is well defined in a prescription independent way since the addition of a gradient to A_a^W only contributes a total divergence term.

The inner product introduced by (2.41) gives rise to a covariant metric on the space of transverse vectors,

$$g(V,W) = g_{0\;axby} V^{ax} W^{by}, \tag{2.42}$$

which can be explicitly written, for instance, in the transverse (non-covariant) prescription,

$$\partial^a A_a^W = 0 \tag{2.43}$$

as

$$g_{o\;axby} \;\; = \;\; -\frac{1}{4\pi}\, \epsilon_{abc} \, \frac{x^c - y^c}{\mid x-y \mid^3} \quad . \tag{2.44}$$

Notice that due to the use of a non-covariant prescription the final object has both coordinate and background metric dependence. g_0 is a well known object in knot theory, where it plays the role of the kernel of the Gauss knot invariant, as we will see in section 2.6. It is the expression in a particular prescription of the covariant metric in the space of transverse vector densities defined by (2.41). Notice that in what follows we will not need to specify a background metric unless we want to give a specific prescription. In general, the covariant metric is defined up to gradients

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that change according to the prescription chosen,

$$g_{axby} = g_{0\,axby} + \rho_{ax\,y,b} + \rho_{by\,x,a}\ . \tag{2.45}$$

Transverse and longitudinal projectors may easily be written without the use of a background metric in terms of g and its inverse in the transverse space,

$$g^{axby} = \epsilon^{abc}\ \partial_c\,\delta(x-y)\ . \tag{2.46}$$

We define the quantities δ_T and δ_L (the transverse and longitudinal Dirac deltas) as

$$\delta_T{}^{ax}{}_{by} \equiv g^{ax\,cz} g_{cz\,by} \tag{2.47}$$

and

$$\delta_L{}^{ax}{}_{by} \equiv \delta^{ax}{}_{by} - \delta_T{}^{ax}{}_{by}, \tag{2.48}$$

where $\delta^{ax}{}_{by} = \delta^a{}_b\delta(x-y)$. It is straightforward to check that they have the desired projection properties,

$$\begin{aligned}
\delta_T{}^{\mu}{}_{\rho}\ \delta_T{}^{\rho}{}_{\nu} &= \delta_T{}^{\mu}{}_{\nu}\ ,\\
\delta_L{}^{\mu}{}_{\rho}\ \delta_L{}^{\rho}{}_{\nu} &= \delta_L{}^{\mu}{}_{\nu}\ ,\\
\delta_L{}^{\mu}{}_{\rho}\ \delta_T{}^{\rho}{}_{\nu} &= \delta_T{}^{\mu}{}_{\rho}\ \delta_L{}^{\rho}{}_{\nu} = 0\ .
\end{aligned}$$

By using the explicit form of the covariant metric one can prove that

$$\delta_L{}^{ax}{}_{by} = \phi^{ax}{}_{y,b}\ , \tag{2.49}$$

where

$$\frac{\partial}{\partial x^a}\ \phi^{ax}{}_{y} = -\delta(x-y)\ . \tag{2.50}$$

The ambiguity in the definition of the metric induces an ambiguity in the decomposition into transverse and longitudinal parts. Each function ϕ that satisfies (2.50) determines a particular prescription of the decomposition. It is important to note that the transverse density fields and in particular the contravariant metric (2.46) are prescription independent. In the particular case in which we choose the transverse metric to be g_0 we have

$$\phi_0{}^{ax}{}_{y} = \frac{1}{4\pi}\,\frac{\partial}{\partial x^a}\frac{1}{\mid x-y\mid}, \tag{2.51}$$

$$\delta_{0T}{}^{ax}{}_{by} = \delta^{ax}{}_{by} + \frac{\partial^a\partial_b}{4\pi}\frac{1}{\mid x-y\mid}\ . \tag{2.52}$$

A transverse projector acting on the vector space $\mathcal{E}$ of multitensor densities can be immediately introduced through the matrix δ_T, defined in

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components as

$$\delta_T{}^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} \equiv \delta_{n,m}\ \delta_T{}^{\mu_1}{}_{\nu_1}\ldots\delta_T{}^{\mu_n}{}_{\nu_n}\,. \tag{2.53}$$

Given any multivector density $\mathbf{E}$ one can construct a multivector density $\mathbf{E}_T$ that is transverse or in other words that satisfies the homogeneous part of the differential constraint (2.11) by,

$$\mathbf{E}_T = \delta_T \cdot \mathbf{E}\,. \tag{2.54}$$

The set of all $\mathbf{E}_T$s forms a linear vector space $\mathcal{E}_T$. The definition of $\mathbf{E}_T$ is not unique, it depends on the prescription used in the definition of the projector.

Since δ_T a projector, relation (2.54) is obviously not invertible in general. However, it turns out that it can be inverted on a subspace of $\mathcal{E}$ given by $\mathcal{E}_D$, the multitensor densities that satisfy the differential constraint (2.11). In order to do this, let us start by evaluating

$$E_D^{\mu_1\ldots\mu_n} = \delta^{\mu_1}{}_{\nu_1}\ldots\delta^{\mu_n}{}_{\nu_n}E_D^{\nu_1\ldots\nu_n}, \tag{2.55}$$

making use of identity (2.48) and the differential constraint and recalling that the first rank component of $\mathbf{E}$ is transverse, we then get

$$\mathbf{E}_D = \sigma \cdot \mathbf{E}_T. \tag{2.56}$$

The soldering quantities σ only depend on the function ϕ which characterizes the choice of decomposition in transverse and longitudinal parts,

$$\sigma^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} = \begin{cases} \delta_T{}^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_n}\,, & \text{if } m=n \\ Q^{\mu_1\ldots\mu_n}_{\rho_1\ldots\rho_{n-1}}\ \sigma^{\rho_1\ldots\rho_{n-1}}{}_{\nu_1\ldots\nu_m}\,, & \text{if } m<n \\ 0\,, & \text{if } m>n \end{cases} \tag{2.57}$$

with

$$Q^{a_1x_1\ldots a_nx_n}_{c_1y_1\ldots c_{n-1}y_{n-1}} \equiv \sum_{j=1}^{n} \delta^{a_1x_1\ldots a_{j-1}x_{j-1}}_{c_1y_1\ldots c_{j-1}y_{j-1}} \left(\phi^{a_jx_j}_{y_j} - \phi^{a_jx_j}_{y_{j-1}}\right) \delta_T{}^{a_{j+1}x_{j+1}\ldots a_nx_n}_{c_jy_j\ldots c_{n-1}y_{n-1}}\,. \tag{2.58}$$

Again, this definition is not unique and will be prescription dependent. However, starting from a given $\mathbf{E}_D$ one can construct an $\mathbf{E}_T$ and then uniquely reconstruct the original $\mathbf{E}_D$ by applying σ.

A crucial property is that the quantities σ *satisfy the differential constraint in their upper indices*, as can be checked from their definition. That is, given an arbitrary transverse multitensor density $\mathbf{E}_T$, one can construct a solution of the differential constraint by applying equation (2.56).

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The quantities σ have definite transversality properties

$$\delta_T \cdot \sigma = \delta_T\,, \tag{2.59}$$

$$\sigma \cdot \delta_T = \sigma\,. \tag{2.60}$$

Notice that due to these properties we can relax the requirement to construct a solution to the differential constraint, i.e., given an *arbitrary* multitensor $\mathbf{E}$, the quantity $\sigma \cdot \mathbf{E}$ is a solution of the differential constraint.

Under a change of the prescription $\phi^{ax}_{1\,y} \to \phi^{ax}_{2\,y}$ we get a $\sigma[\phi_2]$ satisfying

$$\sigma[\phi_1] = \sigma[\phi_2] \cdot \sigma[\phi_1]. \tag{2.61}$$

The operations δ_T and σ define an isomorphism between vector spaces, $\mathcal{E}_D$ the space of multitensors that solve the differential constraint and $\mathcal{E}_T$ via,

$$\mathbf{E}_T = \delta_T \cdot \mathbf{E}_D, \tag{2.62}$$

$$\mathbf{E}_D = \sigma \cdot \mathbf{E}_T. \tag{2.63}$$

The vector product can be introduced in the vector space $\mathcal{E}_D$ and, due to the isomorphism, it is simply given by

$$\mathbf{E}_{D1} \times \mathbf{E}_{D2} = \sigma \;\cdot\; (\mathbf{E}_{T1} \times \mathbf{E}_{T2}). \tag{2.64}$$

This last property will have useful applications in section 2.6 where we construct diffeomorphism invariants.

We are now ready to combine this construction with the ideas of the last section to define the loop coordinates.

2.4.2 Freely specifiable loop coordinates

We saw in section 2.3.2 that one could generate a solution to the differential and algebraic constraints $\mathbf{X}$ by considering

$$\mathbf{X} = \exp(\mathbf{F}) \tag{2.65}$$

but for this to hold $\mathbf{F}$ had to satisfy the differential constraint and the homogeneous algebraic constraint.

Let us now consider an arbitrary transverse multitensor $\mathbf{E}_T$. Applying the results of the last subsection, we notice that the quantity $\sigma \cdot \mathbf{E}_T$ satisfies the differential constraint. Unfortunately, it does not satisfy the homogeneous algebraic constraint (if it did, we would be done, since it would be an element of $\mathcal{F}$).

We will remedy this situation now. We define a new matrix, given by

$$\Omega^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} \equiv \delta^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} + \sum_{k=1}^{n-1} \frac{(n-k)}{n}\,(-1)^k\,\delta^{\underline{\mu_1\ldots\mu_k}\mu_{k+1}\ldots\mu_n}{}_{\nu_1\ldots\nu_m}, \tag{2.66}$$

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where

$$\delta^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_m} \equiv \delta_{n,m} \; \delta^{\mu_1}{}_{\nu_1} \ldots \delta^{\mu_n}{}_{\nu_n} \, . \tag{2.67}$$

The matrix Ω has the following important property: it satisfies the homogeneous algebraic constraint in the upper indices. This fact immediately shows that Ω is a projector. Given an arbitrary vector $\mathbf{E}$, $\Omega \cdot \mathbf{E}$ is an algebraic-free object. In particular we have $\mathbf{F} = \Omega \cdot \mathbf{F}$.

Let us now introduce the following set of vectors

$$\mathcal{S}_{\nu_1 \ldots \nu_m} = (\, 0, \, \vec{\mathcal{S}}_{\nu_1 \ldots \nu_m}) \tag{2.68}$$

with

$$\mathcal{S} = \left(\sigma \cdot \Omega\right), \tag{2.69}$$

which written explicitly in components is

$$(\mathcal{S}_{\nu_1 \ldots \nu_m})^{\mu_1 \cdots \mu_n} = \sigma^{\mu_1 \cdots \mu_n}{}_{\alpha_1 \ldots \alpha_l} \Omega^{\alpha_1 \ldots \alpha_l}{}_{\nu_1 \ldots \nu_m} . \tag{2.70}$$

These vectors combine the action of σ, which converted an arbitrary multitensor into a solution of the differential constraint, and Ω, which projects into the space of solutions of the homogeneous algebraic constraint. That is, given an arbitrary multivector density $\mathbf{E}$, projecting it with $\mathcal{S}$ one obtains an element of $\mathcal{F}$. Simply by exponentiating this element, as we saw in section 2.3.2, we obtain a solution of the differential and algebraic constraint. That is, we just consider,

$$\mathbf{X} = \exp(\mathcal{S} \cdot \mathbf{E}), \tag{2.71}$$

and the $\mathbf{E}$s are unconstrained! Notice that expression (2.71) is the usual relation between elements of a Lie group ($\mathbf{X}$), a basis of generators $\mathcal{S}$ and their free parameters ($\mathbf{E}$).

Expression (2.71) does not really depend on the portion of the $\mathbf{E}$s that does not satisfy the homogeneous algebraic and differential constraints since the contraction with the $\mathcal{S}$s is independent of that portion. Therefore, one will usually concentrate on the set of transverse vectors $\mathbf{Y}$ that satisfy the homogeneous algebraic constraint, and we will call this set $\mathcal{Y}$,

$$\mathcal{Y}: \qquad \mathbf{Y} = \delta_T \cdot \mathbf{Y} \qquad \text{and} \qquad Y^{\underline{\mu_1 \ldots \mu_k} \mu_{k+1} \cdots \mu_n} = 0 \;\; , \;\; 1 \leq k < n \; . \tag{2.72}$$

The situation is totally analogous, for instance, to that of the Lorentz group. In that case the generators are antisymmetric matrices and therefore one usually works with free parameters that are antisymmetric matrices in spite of the fact that any kind of matrix would do. It is just that one can only code relevant information in its antisymmetric part. Similarly here, any arbitrary multitensor $\mathbf{E}$ would work as a free parameter, but only information coded in the portion that satisfies the homogeneous constraints will be relevant for constructing the **X**s via equation (2.71),

$$\mathbf{X} = \exp(\mathcal{S} \cdot \mathbf{Y}). \tag{2.73}$$

The elements of $\mathcal{Y}$ are immediately related to those of $\mathcal{F}$ by

$$\mathbf{Y} = \delta_T \cdot \mathbf{F}. \tag{2.74}$$

When referring to multitangents rather than arbitrary multitensors we can therefore call the objects **Y** "loop coordinates" or coordinates in loop space. Abusing the terminology a bit we will also refer to them in this way when we talk about arbitrary multitensor densities not necessarily associated with loops.

Since they are solutions to the homogeneous algebraic and differential constraint, the $\mathcal{S}$s are elements of $\mathcal{F}$ and therefore they form a basis for the algebra as we suggested above. Details of their construction, the proof that they satisfy the algebra and the determination of the structure constants of the SeL can be seen in reference [20].

2.5 Action of the differential operators

In the previous chapter we introduced a series of differential operators that represented the infinitesimal generators of the group of loops. The loop coordinates provide us with an explicit representation in terms of which we can explore the action of the differential operators. We will not discuss in detail the action of all the differential operators, since as we saw, they are related to each other. We will only concentrate on the action of the loop derivative and of the contact derivative. The former can be used as the starting point to compute any other derivative. The latter is related to diffeomorphism invariance and therefore deserves a detailed treatment.

Let us therefore start by computing the action of the loop derivative on a multitangent field. By the definition of the loop derivative (1.17),

$$(1 + \tfrac{1}{2}\sigma^{ab}\Delta_{ab}(\pi_o^z))X^{a_1x_1\ldots a_nx_n}(\gamma) \equiv X^{a_1x_1\ldots a_nx_n}(\pi_o^z \circ \delta u\delta v\delta\bar{u}\delta\bar{v} \circ \pi_z^o \circ \gamma), \tag{2.75}$$

and recalling the relation between the $\times$-product and the composition law (2.22), we can write

$$\begin{aligned} X^{a_1x_1\ldots a_nx_n}&(\pi_o^z \circ \delta u\delta v\delta\bar{u}\delta\bar{v} \circ \pi_z^o \circ \gamma) = \\ &(\mathbf{X}(\pi_o^z) \times \mathbf{X}_z(\delta u\delta v\delta\bar{u}\delta\bar{v}) \times \mathbf{X}(\pi_z^o) \times \mathbf{X}(\gamma))^{a_1x_1\ldots a_nx_n}. \end{aligned} \tag{2.76}$$

Notice that $\mathbf{X}_z(\delta u\delta v\delta\bar{u}\delta\bar{v})$ is a multitangent basepointed at z, which is in line with the fact that it is composed with an open path that ends at z.

We therefore need to evaluate $\mathbf{X}(\delta u \delta v \delta \bar{u} \delta \bar{v})$ applying the definition of the multitangents (2.3). We can do this order by order. We will only make explicit the calculation of the first order,

$$\mathbf{X}_z(\delta u \delta v \delta \bar{u} \delta \bar{v})^{a_1 x_1} = \epsilon_1 u^{a_1} \delta(x_1 - z) + \epsilon_2 v^{a_1} \delta(x_1 + \epsilon_1 u - z) \\ -\epsilon_1 u^{a_1} \delta(x_1 + \epsilon_1 u + \epsilon_2 v - z) - \epsilon_2 v^{a_1} \delta(x_1 + \epsilon_2 v - z), \quad (2.77)$$

we now expand the Dirac deltas

$$\delta(x_1 + \epsilon_1 u - z) = \delta(x_1 - z) + \epsilon_1 u^b \partial_b \delta(x_1 - z), \quad (2.78)$$

and noticing that all linear terms cancel, we collect terms of order $\epsilon_1 \epsilon_2$ to get

$$\mathbf{X}_z(\delta u \delta v \delta \bar{u} \delta \bar{v})^{a_1 x_1} = \tfrac{1}{2} \sigma^{ab} \delta_{ab}^{a_1 c} \delta_{,c}(x_1 - z). \quad (2.79)$$

In this last expression $\sigma^{ab} = 2\epsilon_1 \epsilon_2 u^{[a} v^{b]}$ as usual and we have introduced the antisymmetrized Kronecker delta $\delta_{ab}^{cd} = \frac{1}{2}(\delta_a^c \delta_b^d - \delta_b^c \delta_a^d)$ and the notation $\delta_{,c}(x - z) = \partial_c \delta(x - z)$.

With this in mind, similar calculations follow for higher order multitangents. The results are

$$\Delta_{ab}(\pi_o^z) X^{a_1 x_1}(\gamma) = \delta_{ab}^{a_1 c} \delta_{,c}(x_1 - z), \quad (2.80)$$

$$\Delta_{ab}(\pi_o^z) X^{a_1 x_1 a_2 x_2}(\gamma) = \delta_{ab}^{a_1 a_2} \delta(x_1 - z) \delta(x_2 - z) \\ +\delta_{ab}^{a_2 c} \delta_{,c}(x_2 - z) X^{a_1 x_1}(\pi_o^z) + \delta_{ab}^{a_1 c} \delta_{,c}(x_1 - z) X^{a_2 x_2}(\pi_z^o \circ \gamma), \quad (2.81)$$

and, in general,

$$\Delta_{ab}(\pi_o^z) X^{a_1 x_1 \ldots a_n x_n}(\gamma) = \\ \delta_{ab}^{a_1 c} \delta_{,c}(x_1 - z) X^{a_2 x_2 \ldots a_n x_n}(\pi_z^o \circ \gamma) \\ +\delta_{ab}^{a_n c} \delta_{,c}(x_n - z) X^{a_1 x_1 \ldots a_{n-1} x_{n-1}}(\pi_o^z) \\ +\delta_{ab}^{a_1 a_2} \delta(x_1 - z) \delta(x_2 - z) X^{a_3 x_3 \ldots a_n x_n}(\pi_z^o \circ \gamma) \\ +\delta_{ab}^{a_{n-1} a_n} \delta(x_{n-1} - z) \delta(x_n - z) X^{a_1 x_1 \ldots a_{n-2} x_{n-2}}(\pi_o^z) \\ + \sum_{j=1}^{n-2} \delta_{ab}^{a_{j+1} c} \delta_{,c}(x_{j+1} - z) X^{a_1 x_1 \ldots a_j x_j}(\pi_o^z) X^{a_{j+2} x_{j+2} \ldots a_n x_n}(\pi_z^o \circ \gamma) \\ + \sum_{j=1}^{n-3} \delta_{ab}^{a_{j+1} a_{j+2}} \delta(x_{j+1} - z) \delta(x_{j+2} - z) \\ \times X^{a_1 x_1 \ldots a_j x_j}(\pi_o^z) X^{a_{j+3} x_{j+3} \ldots a_n x_n}(\pi_z^o \circ \gamma). \quad (2.82)$$

In terms of these expressions for the loop derivative one can reconstruct the action of any other differential operator. We will consider as an example the expressions for the contact derivative.

The expression of the action of the contact derivative on a multitangent

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is,

$$\mathcal{C}_a(z)X^{a_1x_1\ldots a_nx_n}(\gamma) \equiv \oint_\gamma dy^b\delta(z-y)\Delta_{ab}(\gamma_o^y)X^{a_1x_1\ldots a_nx_n}(\gamma) =$$
$$\sum_{j=1}^{n}\delta_{ab}^{a_jc}\delta_{,c}(x_j-z)X^{a_1x_1\ldots a_{j-1}x_{j-1}bza_{j+1}x_{j+1}\ldots a_nx_n}(\gamma) +$$
$$\sum_{j=1}^{n-1}\delta_{ab}^{a_ja_{j+1}}\delta(x_j-z)\delta(x_{j+1}-z)X^{a_1x_1\ldots a_{j-1}x_{j-1}bza_{j+2}x_{j+2}\ldots a_nx_n}(\gamma). \tag{2.83}$$

This expression can be written as a linear transformation of the Xs. This is just an expression of the fact that a "passive" diffeomorphism where one deforms the loop is the same as an "active" diffeomorphism where one maintains the loop fixed but changes coordinates. Let us take a minute to explore this result in detail. We rewrite the expression for the contact derivative as

$$\mathcal{C}_a(z)X^{a_1x_1\ldots a_nx_n}(\gamma) = \sum_{j=1}^{n} A_{az}{}^{a_jx_j}{}_{by}X^{a_1x_1\ldots a_{j-1}x_{j-1}bya_{j+1}x_{j+1}\ldots a_nx_n}(\gamma)$$
$$+\sum_{j=1}^{n-1} B_{az}{}^{a_jx_ja_{j+1}x_{j+1}}{}_{by}X^{a_1x_1\ldots a_{j-1}x_{j-1}bya_{j+2}x_{j+2}\ldots a_nx_n}, \tag{2.84}$$

with

$$A_{az}{}^{a_1x_1}{}_{by} = \delta_{ab}^{a_1c}\delta_{,c}(x_1-z)\delta(y-z), \tag{2.85}$$
$$B_{az}{}^{a_1x_1a_2x_2}{}_{by} = \delta_{ab}^{a_1a_2}\delta(x_1-z)\delta(x_2-z)\delta(y-z), \tag{2.86}$$

where we have used a generalized Einstein convention on the index y.

Sometimes it will be useful to compute the action of differential operators on cyclic multitangents, for instance, if one wants to evaluate the contact derivative of a Wilson loop, which only depends on the cyclic portion of the multitangents,

$$X_c^{\mu_1\ldots\mu_n} = \frac{1}{n}\left(X^{\mu_1\ldots\mu_n} + X^{\mu_2\ldots\mu_n\mu_1} + \ldots X^{\mu_n\ldots\mu_1}\right). \tag{2.87}$$

It is given by

$$\mathcal{C}_a(z)X_c^{a_1x_1\ldots a_nx_n}(\gamma) = \sum_{j=1}^{n} C_{az}{}^{a_jx_j}{}_{by}X_c^{a_1x_1\ldots a_{j-1}x_{j-1}bya_{j+1}x_{j+1}\ldots a_nx_n}(\gamma), \tag{2.88}$$

where

$$C_{az}{}^{a_1x_1}{}_{by} = \delta_{ab}^{a_1c}\delta_{,c}(x_1-z)\delta(y-z) - \delta_a^{a_1}\delta(x_1-z)\delta_{,b}(z-y). \tag{2.89}$$

Equation (2.84) can also be rearranged in terms of the linear transformation matrix C making use of the differential constraint, which was also used to derive (2.88).

These expressions allow us to write the expression for the transformation law of the multitangents under an infinitesimal coordinate transformation $x^a \longrightarrow x'^a = D^a(x) \equiv x^a + N^a(x)$ simply by computing

$$\left(1 + \int d^3x N^a(x)\mathcal{C}_a(x)\right) X^{\mu_1\ldots\mu_n} = \Lambda_D{}^{\mu_1}_{\nu_1} \cdots \Lambda_D{}^{\mu_n}_{\nu_n} X^{\nu_1\ldots\nu_n}, \tag{2.90}$$

with the coordinate transformation matrices given by

$$\Lambda_D{}^{ay}{}_{bx} = \frac{1}{J(x)} \frac{\partial D^a(x)}{\partial x^b}\, \delta(x - D^{-1}(y)) = \frac{\partial D^a(x)}{\partial x^b}\, \delta(D(x) - y)), \tag{2.91}$$

where $J(x)$ is the Jacobian of the coordinate transformation.

2.6 Diffeomorphism invariants and knots

Any vector $\mathbf{F}$ belonging to the SeL algebra behaves as a multivector density under a diffeomorphism that leaves the basepoint fixed. In matrix form the transformation law corresponding to a coordinate transformation $x^a \longrightarrow x'^a = D^a(x)$ is

$$\mathbf{F}' = \Lambda_D \cdot \mathbf{F} \tag{2.92}$$

where

$$\Lambda_D{}^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} \equiv \delta_{n,m}\ \Lambda_D{}^{\mu_1}{}_{\nu_1} \cdots \Lambda_D{}^{\mu_n}{}_{\nu_n}. \tag{2.93}$$

From here it is immediate just by inspecting equation (2.62) to derive the transformation law for the transverse algebraic-free vectors $\mathbf{Y}$,

$$\mathbf{Y}' = \delta_T \cdot \mathbf{F}' = \mathcal{L}_D \cdot \mathbf{Y}, \tag{2.94}$$

where

$$\mathcal{L}_D \equiv \delta_T \cdot \Lambda_D \cdot \boldsymbol{\sigma}. \tag{2.95}$$

The diffeomorphism transformation given by (2.92) is just a particular example of a more general family of transformations: the automorphisms of the algebra. Other automorphisms can be considered, for instance, the conjugation $\mathbf{F}' = \mathbf{X} \times \mathbf{F} \times \mathbf{X}^{-1}$.

The isomorphism between the vector spaces $\mathcal{E}_D$ and $\mathcal{E}_T$ makes $\mathcal{L}_D$ a representation of the diffeomorphism group. This representation emerges as the push-forward of the natural action of diffeomorphisms on the space of solutions of the differential constraint through the isomorphism of that space with the space of transverse vectors $\mathcal{E}_T$.

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The presence of the non-diagonal matrix σ in $\mathcal{L}_D$ makes this representation highly non-trivial. This is an important result, due to the possibility of introducing objects that transform under the adjoint representation of the diffeomorphism group. In fact, the isomorphism guarantees the following property of the σs

$$\sigma = \Lambda_D \cdot \sigma \cdot \mathcal{L}_{D^{-1}}. \tag{2.96}$$

This relationship clearly shows the role played by the σs as the soldering quantities between the fundamental representation Λ_D and the adjoint representation $\mathcal{L}_D$. It is straightforward to see that the subspaces $\mathcal{F}$ and $\mathcal{Y}$ are invariant under diffeomorphisms.

Our task is to construct quantities invariant under automorphisms. To illustrate the procedure to follow, let us consider what is usually done to construct invariants of a group, say $SU(2)$. One takes elements of the group $\omega_i\sigma^i$, where σ^i are the usual Pauli matrices and ω_i free parameters, and computes their trace

$$\mathrm{Tr}(\omega_i\sigma^i\omega_j\sigma^j) = \mathrm{Tr}(\sigma^i\sigma^j)\omega_i\omega_j = G^{ij}\omega_i\omega_j. \tag{2.97}$$

The result is obviously an invariant and it has the form of a metric G^{ij} (in this particular case equal to δ^{ij}), which is invariant under the action of the automorphisms of the group, contracted with the free parameters of the group. Analogously one can take traces of higher order products of elements and one would end up with invariants of the form $G^{i_1\cdots i_n}\omega_{i_1}\ldots\omega_{i_n}$. We will generically call the Gs "invariant metrics".

We will now follow a similar procedure to find invariants under automorphisms of the SeL group. Since we showed that diffeomorphisms are just a particular case of automorphisms, the result will be diffeomorphism invariant. Consider a covector in the space $\mathcal{Y}$, $\mathbf{g} = (0, g_{\mu_1\mu_2}, \cdots, g_{\mu_1\ldots\mu_n}, \cdots)$ with the following properties:

$$\mathbf{g} = \mathbf{g} \cdot \mathcal{L}_D, \tag{2.98}$$

$$g_{\mu_1\ldots\mu_n} = g_{(\mu_1\ldots\mu_n)_{cyclic}}. \tag{2.99}$$

With it, we can define a multilinear form from $\mathcal{Y} \times \cdots \times \mathcal{Y}$ into the complex numbers,

$$I_n = \mathbf{g} \cdot (\mathbf{Y}_1 \times \cdots \times \mathbf{Y}_n) \tag{2.100}$$

that is invariant with respect to all automorphisms described above. The invariance property (2.98) ensures that (2.100) is invariant under diffeomorphisms, (2.99) ensures invariance under conjugation. Why do we require the extra cyclicity property (2.99)? The reader should remember that all the multitangent formalism is basepointed, i.e., there is a preferred point in the manifold as was obvious, for instance, when writing the differential constraint (2.11). The diffeomorphisms under which the constructed quantity would end up being invariant would be those that leave the basepoint fixed. This is not what one is usually interested in, not even in the case of knot invariants, when the multitensors really are multitangents to loops. The cyclicity property ensures that the quantities constructed do not depend on any basepoint.

Unfortunately, we do not have a general technique for constructing the invariant tensors g. Taking traces as in the $SU(2)$ example does not work since we want objects not only invariant under conjugacy but also under other automorphisms, specifically the ones that represent diffeomorphisms and the traces are not invariant under these transformations. Some invariant tensors g are known and we will discuss them in some detail later.

This formalism appears to be a very powerful technique for constructing invariants associated with three-manifolds. Its implications have not been worked out in detail yet, so we will end the generic discussion here. However, it is quite clear that this construction can immediately be particularized to the case in which one is not dealing with arbitrary multitensor fields, but with multitangents associated with loops. The resulting invariants would be *knot invariants*. There is an abundant literature on the subject and therefore we will find it worthwhile to explore the implications of our formalism in some detail for this case in order to make contact with well known results.

Therefore, we will now consider the quantities

$$I_n(\gamma) = \mathbf{g} \cdot (\mathbf{Y}(\gamma) \times \cdots \times \mathbf{Y}(\gamma)), \tag{2.101}$$

and it is evident by construction that $I_n(\gamma) = I_n(\gamma')$ if γ and γ' are related by a diffeomorphism.

Let us consider some particular examples of these quantities. Take $n = 2$. In this case, the invariant metric has only one non-vanishing component,

$$g_{G\,\mu_1\ldots\mu_n} = \delta_{n,2}\, g_{\mu_1\mu_2}, \tag{2.102}$$

where $g_{\mu_1\mu_2}$ is the metric on the space of order one multitangents, already introduced in (2.45). It leads to the following invariant:

$$I_G(\gamma) = \mathbf{g}_G \cdot (\mathbf{Y}(\gamma)\, \times\, \mathbf{Y}(\gamma)) = g_{\mu_1\mu_2} Y^{\mu_1}(\gamma) Y^{\mu_2}(\gamma)\,. \tag{2.103}$$

For a first order multitangent $Y^\mu(\gamma) = X^\mu(\gamma)$; replacing the definition of the Xs (2.3) and of g (2.44) and performing the integrals over the three-manifold explicitly we get

$$GL = -\frac{1}{4\pi} \oint_\gamma ds \oint_\gamma dt \dot{\gamma}^a(s) \dot{\gamma}^b(t) \epsilon_{abc} \frac{(\gamma(s)^c - \gamma(t)^c)}{|\gamma(s) - \gamma(t)|^3}. \tag{2.104}$$

The reader may recognize in this expression the Gauss linking number.

https://doi.org/10.1017/9781009290203.004 Published online by Cambridge University Press

Since we computed it for only one curve, it is a "self-linking number", a quantity which is in general ill-defined and to which we will return in chapter 10.

Although there is not a systematic procedure for constructing the invariant metrics, an infinite family of them can be constructed applying results from Chern–Simons theories, a class of topological field theories that has recently attracted great attention [45]. Using these techniques other invariant metrics have been computed in explicit fashion [187, 47], but we will postpone their discussion until chapter 10 when we discuss Chern–Simons theory in some detail.

The metrics are prescription dependent objects, as can be readily seen from equation (2.98). The knot invariants, however, should be prescription independent. In order to see this let us fix some prescription for $\mathbf{g}$, $\mathbf{g}_1 = \mathbf{g}_1 \cdot \delta_{T1}$. Then

$$\mathbf{g}_1 \cdot \vec{Y}_1 = \mathbf{g}_1 \cdot \delta_{T1} \cdot \vec{F} = \mathbf{g}_1 \cdot \vec{F}. \tag{2.105}$$

But $\vec{F} = \sigma_2 \cdot \vec{Y}_2$, then

$$\mathbf{g}_1 \cdot \vec{Y}_1 = \mathbf{g}_1 \cdot \sigma_2 \cdot \vec{Y}_2 = \mathbf{g}_2 \cdot \vec{Y}_2, \tag{2.106}$$

where

$$\mathbf{g}_2 = \mathbf{g}_1 \cdot \sigma_2 \tag{2.107}$$

is the invariant tensor in the prescription 2. Using the algebraic-free coordinates we have

$$\mathbf{g}_1 \cdot \vec{F} = \mathbf{g}_1 \cdot \vec{Y}_1 = \mathbf{g}_2 \cdot \vec{Y}_2 = \mathbf{g}_2 \cdot \vec{F}. \tag{2.108}$$

If one is considering a specific representation of the group of loops in terms of a gauge group, as we will start to do in the next chapter, functionals of a loop and of multiloops will be related by a series of identities called the Mandelstam identities. With these identities one can build and relate invariants of links of more than one component. We will return to this subject in chapter 10.

2.7 Conclusions

In this chapter we introduced a series of analytic techniques for describing loops. We exhibited the important role of multitensor densities as representations of loops. In fact we noticed that multitensor density fields can play a more fundamental role than loops in physics altogether. We showed how to represent the group of loops and how to extend it to form a Lie group in terms of multitensor fields. We found, by constructing the associated Lie algebra and its free parameters, a set of freely prescribable multitensors that can be used as fundamental objects to describe loops or to build a more general framework. We showed how the diffeomorphisms are represented in terms of these objects and how to use them to construct invariants of three-manifolds and of knots. All these techniques will play a fundamental role in chapters 10 and 11 in the applications to quantum gravity. They will be especially useful for revealing the relations between quantum gravity and topological field theories and will possibly become the calculational bridge between the beautiful notions of knot theory and the Einstein equations. Of all the mathematical technology that we will introduce in chapters 1-3, the extended loop calculus is the most recently discovered and its implications are least explored. A great degree of improvement in the understanding of these issues is likely to appear in the years to come.