ON SOME FORMULAS ABOUT VOLUME AND SURFACE AREA

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We prove in this paper some integral formulas about volume and surface area which are the extensions of the classical formulas such as Guldin-Pappus's theorem about the solid of rotation and the surface of rotation and Holditch's theorem about the area of the domains bounded by the loci of three points on a segment that moves on the euclidean plane. The formulas we prove are so elementary that they may be found in some literature, but the proofs here given are very simple by the use of moving frames and I assume that they are of some interest.

1. The generalization of Guldin-Pappus's theorem

1.1 Let the volume of a solid M in the n-dimensional euclidean space be V. We cut the solid M by one-parametric continuously differentiable set of hyperplanes such that through each point of M one and only one of the hyperplanes passes and let v be an $n-1$ dimensional volume of the section. We assume moreover that the locus of the center of gravity of the section of M by each one of the hyperplanes is a curve with continuous tangents with respect to the parameter t of the hyperplanes, and let $d\sigma$ be an orthogonal component of an arc element of the locus of the center of gravity of the section to the normal direction of the section. Then we have

$$V = \int v d\sigma. \tag{1}$$

Proof. We can take a rectangular frame R' with an origin $\mathbf{A}'$ at the center of gravity of the section and n-th fundamental vector $\mathbf{e}'_n$ on the normal of the section on the side for which the parameter t of the locus of the center of gravity increases, in such a way that the set of these frames is continuously differentiable with respect to t. Let the origin of the fundamental rectangular frame R_0 be $\mathbf{A}^0$ and n fundamental vectors be $\mathbf{e}^0_1, \ldots, \mathbf{e}^0_n$.

We put $\mathbf{A}' = \mathbf{A}^0 + \sum_{i=1}^{n} p_i \mathbf{e}^0_i$, $\mathbf{e}'_i = \sum_{j=1}^{n} p_{ij} \mathbf{e}^0_j$. Let the frame got by translation from R' along the section be R and its origin and vectors be $\mathbf{A}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Then we have $\mathbf{A} = \mathbf{A}' + \sum_{i=1}^{n-1} x_i \mathbf{e}'_i$, $\mathbf{e}_i = \mathbf{e}'_i$. Now we put

Received May 14, 1953.

https://doi.org/10.1017/S0027763000017049 Published online by Cambridge University Press

$$R_0 = \begin{pmatrix} A^0 \\ e_1^0 \\ \cdot \\ \cdot \\ \cdot \\ e_n^0 \end{pmatrix}, \quad R' = \begin{pmatrix} A' \\ e_1' \\ \cdot \\ \cdot \\ \cdot \\ e_n' \end{pmatrix}, \quad R = \begin{pmatrix} A \\ e_1 \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{pmatrix}$$

and let the matrices of transformations between them be P, T. Then we have $R' = PR_0$, $R = TR'$ and hence

$$R = TPR_0 \tag{2}$$

where

$$P = \begin{pmatrix} 1 & p_1 \cdots p_n \\ 0 & \\ \cdot & \\ \cdot & (p_{ij}) \\ \cdot & \\ 0 & \end{pmatrix}, \quad T = \begin{pmatrix} 1 & x_1 & \cdots & x_{n-1} & 0 \\ 0 & 1 & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ 0 & & & & 1 \end{pmatrix}.$$

The coefficients ω_i, ω_{ij} of the infinitesimal relative displacement of the motion of R are obtained as the coefficients of the matrix

$$d(TP)(TP)^{-1} = dT \cdot T^{-1} + T(dPP^{-1})T^{-1}. \tag{3}$$

Let the infinitesimal relative displacement of R' be given by $dA' = \sum_{i=1}^{n} \pi_i e_i'$, $de_i' = \sum_{j=1}^{n} \pi_{ij} e_j'$. Then π_i, π_{ij} are the coefficients of dPP^{-1}. Calculating (3) we get

$$\begin{aligned} \omega_i &= dx_i + \pi_i + \sum_{j=1}^{n-1} x_j \pi_{ji} \qquad (i = 1, \ldots, n-1) \\ \omega_n &= \pi_n + \sum_{j=1}^{n-1} x_j \pi_{jn}. \end{aligned} \tag{4}$$

We denote by dV a volume element of the solid M. As π_i, π_{ij} are linear differential forms with a single variable t we get

$$dV = [\omega_1 \ldots \omega_n] = [dx_1 \ldots dx_{n-1}, \ \pi_n + \sum_{i=1}^{n-1} x_i \pi_{in}].$$

Since A is a center of the gravity of the section we have

$$\int x_i dx_1 \ldots dx_{n-1} = 0 \qquad (i = 1, \ldots, n-1)$$

Hence we get (1) when we remark $\pi_n = d\sigma$.

As a special case we consider a tube. We mean by 'tube' a solid bounded by orthogonal trajectories of one-parametric continuously differentiable set of hyperplanes and two of the hyperplanes with the assumption that any two of the hyperplanes do not intersect each other in the solid. It is well known that all orthogonal sections are congruent (Cf. [2] p. 117). Our definition of a tube is different from that of the paper [1]. The volume of a tube is given by

$$V = vl, \tag{5}$$

where v is an $n-1$ dimensional volume of the orthogonal section and l is an arc-length of the locus of the center of gravity of the section. This is a generalization of Guldin-Pappus's theorem.

1.2. As to surface area we can not obtain a formula analogous to (1), but the one analogous to (5) can be obtained. We cut an $n-1$ dimensional surface M by one-parametric set of hyperplanes which have the property stated in **1.1** and let a surface element of M be dS. Then we have

$$dS^2 = [\omega_1 \ldots \omega_{n-1}]^2 + [\omega_1 \ldots \omega_{n-2}\omega_n]^2 + \ldots + [\omega_2 \ldots \omega_n]^2. \tag{6}$$

In our case variables $x_1, \ldots, x_{n-1}$, t are not independent contrary to the case stated in **1.1**. $x_1, \ldots, x_{n-1}$ are functions of t and $n-2$ variables $u_1, \ldots, u_{n-2}$ which determine the position of a point on the hyperplane of the section. If we denote by attaching $_0$ to a differential form the one got by putting t into a constant, then we have

$$[dx_1 \ldots dx_{n-2}]_0^2 + \ldots + [dx_2 \ldots dx_{n-1}]_0^2 = C(u)^2[du_1 \ldots du_{n-2}]^2.$$

As $C(u)[du_1 \ldots du_{n-2}]$ is a surface element ds of a section of M by a hyperplane, we can put

$$[dx_2 \ldots dx_{n-1}]_0 = c_1 ds, \ldots, [dx_1 \ldots dx_{n-2}]_0 = c_{n-1} ds$$

where $\sum_{i=1}^{n-1} c_i^2 = 1$. Hence the right side of (6) with the exception of the first term is equal to

$$[[dx_1 \ldots dx_{n-2}]_0 \omega_n]^2 + \ldots + [[dx_2 \ldots dx_{n-1}]_0 \omega_n]^2$$

$$= [c_{n-1} ds, \omega_n]^2 + \ldots + [c_1 ds, \omega_n]^2 = [ds, \omega_n]^2.$$

Hence (6) can be written as

$$dS^2 = [\omega_1 \ldots \omega_{n-1}]^2 + [ds, \pi_n + \sum_{j=1}^{n-1} x_j \pi_{jn}]^2.$$

Now we assume M is an $n-1$ dimensional surface generated by orthogonal trajectories of one-parametric continuously differentiable set of hyperplanes and is cut by two of the hyperplanes. Then all the sections are congruent and if we take a suitable frame with origin at the center of gravity of the section,

$x_1, \ldots, x_{n-1}$ are functions of $n-2$ variables which are independent with t and moreover we have

$$\pi_i = 0, \qquad \pi_{ij} = 0 \qquad (i, j = 1, \ldots, n-1).$$

Hence we get by virtue of (4) $\omega_1 = dx_1, \ldots, \omega_{n-1} = dx_{n-1}$, hence $[\omega_1 \ldots \omega_{n-1}] = 0$. As the origin is at the center of gravity we have $\int x_i ds = 0$ $(i = 1, \ldots, n-1)$. Thus we obtain $dS = [ds, \omega_n]$. Integrating this we get

$$S = sl, \tag{7}$$

where S is a surface area of M, s is an $n-2$ dimensional area of the section and l is an arc-length of the locus of the center of gravity of the section. This is a generalization of Guldin-Pappus's theorem.

1.3 Next we treat the case of spherical space. Let the volume of a solid M in the n-dimensional spherical space with the radius 1 be V. We cut M by one-parametric continuously differentiable set of $n-1$ dimensional spheres of radius 1 which have the property stated at the beginning of **1.1**. Let R_0 be a fundamental rectangular frame and R' be a frame with the ends of first n fundamental vectors $e'_0, e'_1, \ldots, e'_{n-1}$ on the $n-1$ dimensional sphere of the section. We assume R' is defined uniquely for each section and let R be any frame with the ends of $e_0, e_1, \ldots, e_{n-1}$ on the section. Then putting

$$R_0 = \begin{pmatrix} e_0^0 \\ e_1^0 \\ \cdot \\ \cdot \\ \cdot \\ e_n^0 \end{pmatrix}, \quad R' = \begin{pmatrix} e'_0 \\ e'_1 \\ \cdot \\ \cdot \\ \cdot \\ e'_n \end{pmatrix}, \quad R = \begin{pmatrix} e_0 \\ e_1 \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{pmatrix}$$

we get $R' = PR_0$, $R = TR'$ and hence $R = TPR_0$, where T is a matrix such that

$$T = \begin{pmatrix} t_{00} \ldots\ldots\ldots t_{0n-1} & 0 \\ \ldots\ldots\ldots\ldots\ldots & \cdot \\ t_{n-10} \ldots\ldots t_{n-1,\,n-1} & 0 \\ 0 \ldots\ldots\ldots\ldots 0 & 1 \end{pmatrix} \qquad (t_{0i} = t_i).$$

The first row of dTT^{-1} is of the form $(0, \tau_1, \ldots, \tau_{n-1}, 0)$. Let the parameter of the sectional $n-1$ dimensional spheres be t. Then the coefficients of dPP^{-1}, namely π_{ij} $(i, j = 0, 1, \ldots, n)$ when we put $de'_i = \sum_{j=0}^{n} \pi_{ij} e'_j$, are linear differential forms with a single variable t. If we put $de_i = \sum_{j=0}^{n} \omega_{ij} e_j$ $(\omega_{0i} = \omega_i)$, ω_{ij}'s are the coefficients of $d(TP)(TP)^{-1}$ and we get as in **1.1**

$$\omega_i \equiv \tau_i \qquad (\text{mod. } dt,\ i = 1, \ldots, n-1)$$
$$\omega_n = \sum_{i=0}^{n-1} t_i \pi_{in}.$$

https://doi.org/10.1017/S0027763000017049 Published online by Cambridge University Press

Hence the volume element dV of the solid generated by the endpoint of e_0 is given by

$$dV = [\omega_1 \ldots \omega_{n-1}\omega_n] = [\tau_1 \ldots \tau_{n-1}, \sum_{i=0}^{n-1} t_i \pi_{in}].$$

Since $dv = [\tau_1 \ldots \tau_{n-1}]$ is a volume element of a section we can write

$$dV = \sum_{i=0}^{n-1} [t_i dv, \pi_{in}]. \tag{8}$$

Now we consider the integrals $T_i = \int t_i dv$ $(i = 0, 1, \ldots, n-1)$ over the section. $(T_0, T_1, \ldots, T_{n-1})$ are components of a vector defined by the section and its length

$$w = (\sum_{i=0}^{n-1} T_i^2)^{1/2} \tag{9}$$

is a geometric quantity attached to the section which is different from the volume. As $(T_0, T_1, \ldots, T_{n-1})$ are components of a vector we can put $T_1 = \ldots = T_{n-1} = 0$ by taking R' suitably. We call a center of gravity of the section the endpoint of e_0'. Then we get $w = \int t_0 dv$, $\int t_i dv = 0$ $(i = 1, 2, \ldots, n-1)$. Hence we get by virtue of (8)

$$V = \int w d\sigma \tag{10}$$

where $d\sigma = \pi_{0n}$ is an orthogonal component of an arc-element of the locus of the center of gravity of the section to the direction of the normal of the section. It is notable that w in (10) is not the volume of the section. A tube can be defined in our space as in **1.1** (cf. [2] p. 117), and a formula $V = wl$ analogous to (5) can be obtained.

In the special case $n = 2$ a solid M is a domain on the sphere in the euclidean space of dimension 3, and we cut M by great circles which do not cut in the domain M each other. It can easily be verified that the center of gravity of the arc of section is its middle point and w is a length of a chord corresponding to the arc. $d\sigma$ is an orthogonal component of an arc element of the locus of the middle points of the arcs of section to the normal direction of the arc. If one end of each arc is fixed at O, we have for a spherical area S a well known formula $S = \int 1/2\ l^2 d\varphi$ where l is a length of a chord and $d\varphi$ is an infinitesimal angle between two consecutive planes containing the chords and the diameter of the sphere through O.

1.4 We can obtain a formula analogous to (7) in the spherical case. Let M be an $n-1$ dimensional surface in the spherical space of dimension n with the radius 1 generated by orthogonal trajectories of one-parametric set of the

https://doi.org/10.1017/S0027763000017049 Published online by Cambridge University Press

$n-1$ dimensional sphere of radius 1 and cut by two of the spheres. Let the equation of the $n-1$ dimensional sphere be $\sum_{i=0}^{n-1} t_i^2 = 1$ and the surface element of the section be dS. We put $U_i = \int t_i ds$ $(i = 0, 1, \ldots, n-1)$ and $u = (\sum_{i=0}^{n-1} U_i^2)^{1/2}$ and call a center of gravity the point with $(U_0 u^{-1}, U_1 u^{-1}, \ldots, U_{n-1} u^{-1})$ as its coordinates. Then the surface area S of M is given by

$$S = ul \tag{11}$$

where l is a length of the locus of a center of gravity. The proof is analogous to that of **1.2.**

2. Generalized Holditch's theorem

2.1 Let a straight line move in the n-dimensional euclidean space and let local parameters of this motion be $u_1, \ldots, u_{n-1}$. We take on this line a point $\mathbf{A}$ and a unit vector $\mathbf{e}_1$ and let R be a rectangular frame with origin at $\mathbf{A}$ and one of its fundamental vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ on $\mathbf{e}_1$. We take on each line a fixed point $\mathbf{A}'$ and let R' be a frame with $\mathbf{A}'$ as its origin and $\mathbf{e}'_i = \mathbf{e}_i$ as its vectors. Then $\mathbf{A} = \mathbf{A}' + t\mathbf{e}_1$, $\mathbf{e}_i = \mathbf{e}'_i$, and putting $R = TR'$, $R' = PR_0$, where R_0 is a fundamental frame, we have $R = TPR_0$. Hence the coefficients ω_i, ω_{ij} of the infinitesimal relative displacement of R are given by calculating $d(TP)(TP)^{-1}$. Writing the coefficients of dPP^{-1} as π_i, π_{ij} we get as in **1.1**

$$\omega_1 = \pi_1 + dt, \quad \omega_i = \pi_i + t\pi_{1i} \qquad (i = 2, \ldots, n).$$

Hence a volume element dV of the solid generated by $\mathbf{A}$ is given by

$$\begin{aligned} dV &= [\omega_1 \ldots \omega_n] = [\pi_1 + dt, \pi_2 + t\pi_{12}, \ldots, \pi_n + t\pi_{1n}] \\ &= [dt\, \pi_2 \ldots \pi_n] + \sum [t dt\, \pi_{12}\pi_3 \ldots \pi_n] + \ldots + [t^{n-1} dt\, \pi_{12} \ldots \pi_{1n}]. \end{aligned}$$

Integrating this for a domain which is a direct product of a segment $t_0 \leqq t \leqq t_0 + a$ and an n-dimensional closed orientable domain D with the local parameters $u_1, \ldots, u_{n-1}$ in the space of all straight lines as its elements, we get

$$V = aC_1 + a^2 C_2 + \ldots + a^n C_n \tag{12}$$

where we have put

$$C_1 = \int [\pi_2 \ldots \pi_n], \quad C_2 = 1/2 \int \sum [\pi_{12}\pi_3 \ldots \pi_n], \ldots, C_n = 1/n \int [\pi_{12} \ldots \pi_{1n}],$$

and V is an algebraic volume of the part swept by the segment $t_0 \leqq t \leqq t_0 + a$. Here by an algebraic volume we mean one which is a sum of the volume counted with the sign according to the orientation of the euclidean space and the orientation of the parameter space with point $(t, u_1, \ldots, u_{n-1})$. Now we assume that for our motion the other end of a unit vector $\mathbf{e}_1$ with one end at a fixed point covers a domain on a sphere in such a way that its degree of mapping is k.

We take n points $P_i = \mathbf{A}' + a_i \mathbf{e}_1$ $(i = 1, \ldots, n)$ on this line and assume that $\mathbf{A}'$, P_i describe n-dimensional closed surfaces which enclose solids of volume V_0, V_i respectively. Now we make one more assumption, which I hope to be proved, though difficult for me. The assumption is that the difference of V_0 and V_i is equal to the algebraic volume of the domain swept by the segment $A'P_i$. Then we get by (12)

$$V_i - V_0 = a_i C_1 + \ldots + a_i^{n-1} C_{n-1} + a_i^n C_n.$$

Eliminating $C_1, \ldots, C_{n-1}$ we get

$$\begin{vmatrix} V_1 - V_0 - a_1^n C_n & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ V_n - V_0 - a_n^n C_n & a_n & a_n^2 & \ldots & a_n^{n-1} \end{vmatrix} = 0.$$

By calculation we get

$$(13) \qquad \sum_{i=0}^{n} V_i / (d_{0i} \ldots d_{i-1i} d_{i+1i} \ldots d_{ni}) = (-1)^n C_n,$$

where d_{ij} is an oriented distance from P_i to P_j and $C_n = kI_n$, I_n being a volume of a unit sphere in the n-dimensional space. This is a generalization of Holditch's theorem. If V_i's $(i = 1, \ldots, n)$ are all zero, we have $V_0 = C_n d_{10} d_{20} \ldots d_{n0}$. A volume of an ellipsoid is an example of this formula.

2.2 An analogous formula for the spherical case can be obtained, but it takes different forms for n even and for n odd. Moreover since C_n is not of the form kI_n in this case we eliminate C_n, too. Thus in the case $n = 2$ we get

$$(14) \qquad S_0/(d_{01}d_{02}d_{03}) + S_1/(d_{10}d_{12}d_{13}) + S_2/(d_{20}d_{21}d_{23}) + S_3/(d_{30}d_{31}d_{32}) = 0,$$

where S_i is an area bounded by a curve described by a point P_i on a moving great circle, and d_{ij} is an oriented length of a chord from P_i to P_j, namely $2r \sin(\alpha_{ij}/2)$, r being a radius of the circle and α_{ij} being an oriented angle corresponding to the arc P_iP_j.

3. Some formulas about the area

3.1 Let a triangle $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$ move on a euclidean plane and $\mathbf{A}_0$ be a point which is relatively fixed to this triangle. Let this motion be represented by a rectangular frame $R = PR_0$, R_0 being a fundamental frame. Let the coordinates of A_i $(i = 0, 1, 2, 3)$ with respect to a frame R be (a_i, b_i). If we represent the infinitesimal relative displacement of R by $d\mathbf{A} = \pi_1 \mathbf{e}_1 + \pi_2 \mathbf{e}_2$, $d\mathbf{e}_1 = \pi_{12} \mathbf{e}_2$, $d\mathbf{e}_2$

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$= -\pi_{12}\mathbf{e}_1$, then the coefficients of the relative displacement of R_i with $\mathbf{A}_i$ as its origin and $\mathbf{e}_1$, $\mathbf{e}_2$ as its vectors are given by

$$\omega_1 = \pi_1 - b_i\pi_{12}, \quad \omega_2 = \pi_2 + a_i\pi_{12}, \quad \omega_{12} = \pi_{12}.$$

If the motion of $\mathbf{A}_0\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$ is two-parametric, the areal element of the domains swept by $\mathbf{A}_i$ is given by

$$dS = [\omega_1\omega_2] = [\pi_1\pi_2] + a_i[\pi_1\pi_{12}] + b_i[\pi_2\pi_{12}].$$

We denote by S_i an algebraic area swept by A_i. Then we get

$$S_i = \int[\pi_1\pi_2] + a_i\int[\pi_1\pi_{12}] + b_i\int[\pi_2\pi_{12}].$$

Hence eliminating $\int[\pi_1\pi_2]$, $\int[\pi_1\pi_{12}]$, $\int[\pi_2\pi_{12}]$ we get

$$\begin{vmatrix} S_0 & 1 & a_0 & b_0 \\ S_1 & 1 & a_1 & b_1 \\ S_2 & 1 & a_2 & b_2 \\ S_3 & 1 & a_3 & b_3 \end{vmatrix} = 0.$$

This can be written in the form

$$\sum_{i=0}^{3} S_i\sigma_i = 0 \tag{15}$$

where σ_i is an oriented area of the triangle with the vertices except A_i.

3.2 We take a curve on a euclidean plane and denote by s an arc length from a fixed point on it to an arbitrary point on it and by k a curvature. From each point of the curve we draw a segment of a length l and let θ be an angle which the segment makes with the tangent of the curve at the point. Then for an algebraic area swept by the segment we get by calculation

$$dS = \int l \sin\theta\, ds - \frac{1}{2}\int l^2(kds + d\theta). \tag{16}$$

We can apply this to the area S bounded by a curve C and a roulette. Here we mean by a roulette a curve described by a point on a closed curve C_0 which rolls on C without slipping and we take into consideration an arc of the curve corresponding to one circulation of C_0, two ends being on C. Let S_0 be an area bounded by C_0 and k, k_0 be curvatures of C, C_0 at the point of contact which we represent as functions of the common arc length s, while l is a distance from a point on the roulette to a corresponding point of contact. Then by calculation we get from (16) under a certain assumption

$$S = S_0 + \frac{1}{2}\int l^2(k_0 - k)ds. \tag{17}$$

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Here we assume that the area S_0 of the domain bounded by C_0 is swept positively by a chord drawn from the point on C_0 at which C_0 touches C at the initial state to a point on C_0 of contact with C at each instant. Especially if C_0 is a circle of radius r, we get

$$S = 3\pi r^2 - \alpha r^2 + \int r^2 \cos 2\theta \cdot k ds,$$

where the curvature $1/r$ of the circle C_0 is counted positively and the curvature k of C is of the sign corresponding to that of C_0, while $\alpha = \int k\, ds$ is an angle between the two tangents of C at the both ends.

About the area on the unit sphere we can obtain a formula analogous to (16). We take notations as in (16) and let θ be an angle in which the center of the sphere commands an arc of a great circle drawn from each point on the curve. Then we get

$$S = \int \sin\alpha \sin\theta\, ds - \int (1 - \mathrm{con}\, \alpha)(k\, ds + d\theta). \tag{18}$$

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