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RADICAL REGULARITY IN DIFFERENTIAL RINGS

HOWARD E. GORMAN

1. Introduction. In [**1**], we discussed completions of differentially finitely generated modules over a differential ring R. It was necessary that the topology of the module be induced by a differential ideal of R and it was natural that this ideal be contained in $J(R)$, the Jacobson radical of R. The ideal to be chosen, called $J_d(R)$, was the intersection of those ideals which are maximal among the differential ideals of R. The question as to when $J_d(R) \subseteq J(R)$ led to the definition of a class of rings called radically regular rings. These rings do satisfy the inclusion, and we showed in [**1**, Theorem 2] that R could always be "extended", via localization, to a radically regular ring in such a way as to preserve all its differential prime ideals.

In the present paper, we discuss the stability of radical regularity under quotient maps, localization, adjunction of a differential indeterminate, and integral extensions.

2. Preliminaries. All rings will be commutative with 1. By a *differential ring* we mean a ring R with distinguished derivation which we always denote by "′". For $r \in R$, the successive derivatives of r will be denoted by $r', r'', \ldots, r^{(n)}, \ldots$. A differential ring R is a *Ritt algebra* if it contains the rational numbers. For any ring R, Rad R will denote the *radical* of R, and Rad I will denote the *radical* of I for any ideal I of R.

If R is a differential ring, then the *differential polynomial ring* in the *differential indeterminate* x, written $R\{x\}$, is the polynomial ring over R in a countable number of variables $x = x^{(0)}, x^{(1)}, \ldots, x^{(n)}, \ldots$, with derivation defined by $x^{(n)\prime} = x^{(n+1)}$. We define the *weight* of the monomial

$$m = x^{(n_1)d_1} \ldots x^{(n_r)d_r}$$

to be

$$\omega(m) = \sum_{i=1}^{r} (n_i + 1)d_i$$

with n_i and d_i non-negative integers.

(∗) It is easily checked that $\omega(m^{(k)}) = \omega(m) + k$.

(This makes sense since the monomial terms in the derivative of a monomial all have the same weight.)

Let R be a differential ring. Then we call R a d-MP ring if the radical of a differential ideal is again a differential ring. This property is equivalent to each of the following by [**1**, Lemma 2].

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(a) Prime ideals minimal over differential ideals are differential ideals.

(b) If I is a differential ideal of R and S is a multiplicatively closed set of R disjoint from I, then ideals maximal among differential ideals which contain I and are disjoint from S are prime.

In particular, Ritt algebras are d-MP rings [**2**, Lemma 18].

If T is a subset of the differential ring R, then write $[T]/R$ and $\{T\}/R$ for the smallest differential ideal and radical differential ideal, respectively, which contain T. Call R radically regular if, for every $x \in R$, $\{x\}/R = R$ if and only if x is a unit of R. Note that for a d-MP ring, $\{x\}/R = R$ if and only if $[x]/R = R$, and that $\{T\}/R = \operatorname{Rad}([T]/R)$.

Denote the Jacobson radical of R by $J(R)$ and the intersection of all ideals maximal among differential ideals by $J_d(R)$. Then

(**) R radically regular implies that $J_d(R) \subseteq J(R)$ with equality if and only if $J(R)$ is a differential ideal (see [**1**, Lemma 10]).

If S is a multiplicatively closed set in the differential ring R with $0 \notin S$, then the derivation on R extends to R_S via the definition

$$\left(\frac{r}{s}\right)' = \frac{r's - rs'}{s^2} \quad \text{for all } r \in R \text{ and } s \in S$$

(see [**1**, Lemma 1]).

3. Quotients and localizations. The property of radical regularity is not stable under quotients. For, let K be a field containing the rationals, equipped with trivial derivation, and let $K[x]$ be the polynomial ring over K equipped with the derivation of calculus. Then $K[x]$ has no proper differential ideals and so is not radically regular. But, by Theorem 1 below, $K\{x\}$ is radically regular and $K\{x\}/[1 - x'] \cong K[x]$.

The situation with regard to the stability of radical regularity with respect to localizations is settled by the following result.

LEMMA 1. *Let R be a differential ring and P a prime ideal of R. Then R_P is radically regular if and only if P is a differential ideal.*

Proof. Clearly, R_P is radically regular if and only if PR_P is a differential ideal. But this is equivalent to P being a differential ideal.

Lemma 1 shows that localization is of limited use in discussing radical regularity.

4. Differential polynomial rings. For d-MP rings, radical regularity is stable under adjunction of a differential indeterminate.

LEMMA 2. *Let R be a differential ring, and let r be a unit of R. Then for any $x \in R$, $[x] = [rx]$.*

Proof. The proof follows immediately from the fact that $x \in [rx]$ and $rx \in [x]$.

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THEOREM 1. *Let R be a differential ring and let x be a differential indeterminate. Suppose that $R\{x\}$ is a* d-MP *ring. Then R is radically regular if and only if $R\{x\}$ is.*

Proof. We note first that if $R\{x\}$ is a d-MP ring, then so is R; for if P is a prime ideal of R minimal over the differential ideal I of R, then $PR\{x\}$ is a minimal prime over $IR\{x\}$.

Necessity. Let $r \in R$ and $[r]/R = R$. Then $[r]/R\{x\} = R\{x\}$, and so there exists $f \in R\{x\}$ such that $rf = 1$. Let s be the constant term of f so that $f = s + g$ with $g \in R\{x\}$. Then $1 = r(s + g) = rs + rg$, from which it follows that $1 = rs$ and $rg = 0$; i.e., r is a unit of R.

Sufficiency. Let $f \in R\{x\}$ be such that $[f]/R\{x\} = R\{x\}$. Then

$$1 = \sum_{i=0}^{k} g_i f^{(i)} \tag{1}$$

with $g_i \in R\{x\}$ for each i. If f has no constant term, then (1) is impossible. Let $a \neq 0$ be the constant term of f, and let b_i be the constant term of g_i for each i. Then (1) implies that

$$1 = \sum_{i=0}^{k} b_i a^{(i)} \tag{2}$$

so that $[a]/R = R$. Hence a is a unit of R. By Lemma 2, $[f] = a^{-1}[f] = [a^{-1}f]$. Hence we may assume that $a = 1$, the advantage being that $1' = 0$. Now (2) becomes $b_0 = 1$, and now $f^{(k)}$ has no constant term if $k > 0$.

Suppose that $f \neq 1$. (We assume $a = 1$.) Let m be the sum of those terms of f which have minimal, non-zero weight. We note that, for $k > 0$, $f^{(k)}$ has no terms of this weight. (See (*) in § 2.) Since $b_0 = 1$, m appears in (1) and is not cancelled by any other terms in (1) because they all have larger weights. But this contradicts (1). Hence $f = 1$ and the proof is complete.

5. Integral extensions. Let R and S ($R \subseteq S$) be differential rings. When we say that S is an integral extension of R, we shall assume that the derivation of R is the restriction to R of the derivation of S.

The next lemma shows that for differential d-MP rings, the "lying over" and "going up" theorems for integral extensions behave well with respect to derivations.

LEMMA 3. *Let R and S ($R \subseteq S$) be differential* d-MP *rings, and let S be integral over R. Let P be a prime ideal of R, and let $Q \subseteq S$ lie over P. Then P is a differential ideal if and only if Q is.*

Proof. If Q is a differential ideal, then so is $Q \cap R = P$.

Conversely, let P be a differential ideal of R. Then PS is a differential ideal of S, and $PS \cap (S - Q) = \emptyset$. Let Q_1 be an ideal of S maximal among differential ideals of S which contain PS and exclude $S - Q$. Then Q_1 is prime

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because S is a d-MP ring, and $Q_1 \subseteq Q$ by construction. Further, Q_1 lies over P since

$$P = Q \cap R \supseteq Q_1 \cap R \supseteq PS \cap R = P.$$

By the lying over theorem, $Q = Q_1$; i.e., Q is a differential ideal.

Before stating Theorem 2, we prove the following lemma.

LEMMA 4. *Let R be a differential ring and let M be any maximal ideal of R. Let $\langle N_\lambda \rangle_{\lambda \in \Lambda}$ be the set of proper differential ideals of R such that $M \cap N_\lambda \neq 0$. Then R is radically regular if and only if, for each maximal ideal M, $M \subseteq \bigcup_\lambda N_\lambda$.*

Proof. Let $x \in R$. Then x is a non-unit of R if and only if $x \in M$ for some maximal ideal M of R. If $M \subseteq \bigcup_\lambda N_\lambda$, then $x \in N_{\lambda_0}$ for some λ_0 so that $[x]/R \subseteq N_{\lambda_0}$. Conversely, if $M \not\subseteq \bigcup_\lambda N_\lambda$, choose $x \in M - \bigcup_\lambda N_\lambda$, $x \neq 0$. Then x is not in any proper differential ideal of R so that $[x]/R = R$.

THEOREM 2. *Let R and S ($R \subseteq S$) be differential* d-MP *rings and let S be an integral extension of R. Then*

(i) *If S is radically regular, then so is R, and, in this case, $J(S) = J_d(S)$ implies $J(R) = J_d(R)$;*

(ii) *Let R be radically regular. Then S is radically regular if either:*

(a) *R has only a finite number of maximal ideals, or*

(b) *S is a finitely generated free R-module.*

Proof. (i) Suppose that S is radically regular. Let M be a maximal ideal of R and let N be a maximal ideal of S lying over M. Then, by Lemma 5, $N \subseteq \bigcup_\lambda N_\lambda$, a union of differential ideals of S. Hence $M = N \cap R \subseteq (\bigcup_\lambda N_\lambda) \cap R = \bigcup_\lambda (N_\lambda \cap R)$, and so M is also contained in a union of differential ideals of R; i.e., R is radically regular by Lemma 4. If $J(S) = J_d(S)$, then $J(S) \cap R = J(R)$ is a differential ideal. Hence $J(R) = J_d(R)$ by (*) in § 2.

(ii) (a) Suppose that R is radically regular and that R has only a finite number of maximal ideals. Then by [**1**, Theorem 3], each of them is differential. By Lemma 3, every maximal ideal of S is differential so that S is radically regular by Lemma 5; it follows also that $J_d(S) = J(S)$.

(ii) (b) Let $a_1, \ldots, a_n$ be a free basis of S over R and let x be a non-unit of S. Suppose that

$$xa_i = \sum_{j=1}^{n} r_{ij} a_j$$

for each j with $r_{ij} \in R$. Let $\det x = \det(r_{ij}) = r$. Then r is a non-unit of R, so that r lies in a proper, prime differential ideal P of R. Let $\hat{R} = R_P$ and $\hat{S} = S_{R-P}$. Then $P\hat{S} + x\hat{S}$ is a proper ideal of $\hat{S}$. For if $1 = (s/u)x + (t/v)p$, with $u, v \in R - P$, $s, t \in S$, and $p \in P$, then $uv \equiv svx \pmod{PS}$ and $u^n v^n = \det(uv) \equiv v^n \det s \det x \pmod{P}$. But $u^n v^n \notin P$ and $\det x \in P$, a contradiction. Now $(P\hat{S} + x\hat{S}) \cap \hat{R} = P\hat{R}$ since $P\hat{R}$ is the maximal ideal of $\hat{R}$. Hence there is a maximal ideal M of $\hat{S}$ containing P and x and lying over $P\hat{R}$. By Lemma 3, M is a differential ideal. Hence x lies in a proper, differential ideal of S, and our proof is complete.

Acknowledgement. I would like to thank the referee for supplying a correct proof of (ii) (b).

6. A result on rings without differential ideals. In Proposition 1 below we give conditions under which a d-MP ring without differential ideals is a field.

LEMMA 5. *Let R be a* d-MP *ring with no proper, non-zero, differential ideals. Then R is a domain.*

Proof. Since R is a d-MP ring, 0 is a prime ideal. Hence R is a domain.

PROPOSITION 1. *Let R be a* d-MP *ring which is either radically regular or consists entirely of units and zero divisors. Then R is a field if and only if it contains no non-zero, differential ideals.*

Proof. It is enough to prove the "if" part. Assume that R contains no non-zero differential ideals. Then, for every non-zero $x \in R$, $[x] = R$. Further, by Lemma 5, R is a domain; thus, if every element of R is a unit or a zero divisor, then R is a field. If R is radically regular, then this last statement implies that every non-zero x is a unit of R; i.e., R is a field, and our proof is complete.

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Stanford University,
Stanford, California