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Non-tame Mice from Tame Failures of the Unique Branch Hypothesis

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Abstract. In this paper, we show that the failure of the unique branch hypothesis (UBH) for tame trees implies that in some homogenous generic extension of V there is a transitive model M containing $\mathrm{Ord} \cup \mathbb{R}$ such that $M \vDash \mathrm{AD}^+ + \Theta > \theta_0$. In particular, this implies the existence (in V) of a non-tame mouse. The results of this paper significantly extend J. R. Steel's earlier results for tame trees.

In this paper, we establish, using the core model induction, a lower bound for certain failures of the *Unique Branch Hypothesis* (UBH), which is the statement that every iteration tree that acts on V has at most one cofinal well-founded branch. For the rest of this paper, all trees considered are nonoverlapping, that is whenever E and F are extenders such that E is used before F along a branch of the tree, then $\mathrm{lh}(E) \leq \mathrm{crit}(F)$. The following is our main theorem. Tame trees[1] are defined in Definition 5.1. Roughly speaking, these are the trees in which the critical point of any branch embedding is above a strong cardinal that reflects strong cardinals.

Main Theorem *Suppose there is a proper class of strong cardinals and UBH fails for tame trees. Then in a set generic extension of V, there is a transitive inner model M such that* $\mathrm{Ord} \cup \mathbb{R} \subseteq M$ *and* $M \vDash \mathbf{AD}^+ + \theta_0 < \Theta$. *In particular, there is a non-tame mouse.*

The Unique Branch Hypothesis was first introduced by Martin and Steel in [3]. Towards showing UBH, Neeman [5] showed that a certain weakening of UBH called cUBH holds provided there are no non-bland mice.[2] However, in [17], Woodin showed that in the presence of supercompact cardinals UBH can fail for tame trees.[3] It is, however, still an important open problem whether UBH holds for trees that use extenders that are $2^{\aleph_0}$-closed in the models that they are chosen from. A positive resolution of this problem will lead to the resolution of the inner model problem for superstrong cardinals and beyond. It is worth remarking that the aforementioned form of UBH for tame trees will also lead to the resolution of the inner model problem for superstrong cardinals and beyond. Our work can be viewed as an attempt to prove UBH for tame trees by showing that its failure is strong consistency-wise.

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[1] The term "tame trees" is our ad-hoc terminology and has nothing to do with the well-established term "tame" used to define a certain first-order property of premice.

[2] We will not use this terminology.

[3] Woodin constructs alternating chains whose branches are well-founded. Extenders of such trees can be demanded to reflect the set of strong cardinals which reflect strong cardinals. Hence critical points of the branch embeddings can certainly be demanded to be above the first strong cardinal that reflects strong cardinals.

In this direction, Steel [16] showed that the failure of UBH for (nonoverlapping) trees implies that there is an inner model with infinitely many Woodin cardinals. If in addition UBH fails for some tree $\mathcal{T}$ such that $\delta(\mathcal{T})$ is in the image of two branch embeddings witnessing the failure of UBH for $\mathcal{T}$, then Steel obtained an inner model with a strong cardinal that is a limit of Woodin cardinals. For tame trees (which, as mentioned in the footnote, include a class of examples constructed by Woodin [17]), the Main Theorem considerably strengthens the aforementioned result of Steel and because the proof presented here is via the core model induction, we expect that it will yield much more. We believe that our proof, coupled with arguments from [8], will give the existence of a transitive inner model M such that $\mathrm{Ord} \cup \mathbb{R} \subseteq M$ and $M \vDash$ "$\mathbf{AD}_{\mathbb{R}} + \Theta$ is regular". However, we still do not know if an arbitrary failure of UBH implies the existence of a non-tame mouse. Various arguments presented in this paper resemble the arguments given in [7, 11], and some familiarity with those articles will be useful.

1 Preliminaries

In this paper, we will need to make use of the material presented in [7, Section 1], most of which, especially Section 1.1, carries over to the hybrid context by just changing the word "mouse" to "hybrid mouse". Because of this, we will only introduce a few main notions and will use [7, Section 1] as our main background material. In particular, we assume that the reader has already translated the material of [7, Section 1.1] into the language of hybrid mice.

1.1 Stacking Mice

Following the notation of [7, Section 1.3], we fix some uncountable cardinal λ and assume **ZF**. Notice that any function $f\colon H_\lambda \to H_\lambda$ can be naturally coded by a subset of $\mathcal{P}(\bigcup_{\kappa<\lambda} \mathcal{P}(\kappa))$. We then let $\mathrm{Code}^*_\lambda\colon H_\lambda^{H_\lambda} \to \mathcal{P}(\bigcup_{\kappa<\lambda} \mathcal{P}(\kappa))$ be one such coding. If $\lambda = \omega_1$, then we just write Code^*. Because for $\alpha \leq \lambda$, any (α, λ)-iteration strategy[4] for a hybrid premouse of size $<\lambda$ is in $H_\lambda^{H_\lambda}$, we have that any such strategy is in the domain of Code^*_λ.

Suppose $\Lambda \in \mathrm{dom}(\mathrm{Code}^*_\lambda)$ is a strategy with hull condensation and $\mu \leq \lambda$. Recall that we say F is a (μ, Λ)-mouse operator if for some $X \in H_\lambda$ and formula ϕ in the language of Λ-mice, if Y is such that $X \in Y$, then $F(Y)$ is the minimal μ-iterable Λ-mouse satisfying $\phi[Y]$.

We then let Code_λ be Code^*_λ restricted to $F \in \mathrm{dom}(\mathrm{Code}^*_\lambda)$, which is defined by the following recursion:

(a) for some $\alpha \leq \lambda$, F is a (α, λ)-iteration strategy with hull condensation,[5]

(b) for some $\alpha \leq \lambda$ and for some (α, λ)-iteration strategy $\Lambda \in \mathrm{dom}(\mathrm{Code}^*_\lambda)$ with hull condensation, F is a (λ, Λ)-mouse operator,

[4]This is an iteration strategy for stacks of fewer than α normal trees, each of which has length less than λ. Typically these are fine-structural n-maximal iteration trees (as defined in [4]), where n is the degree of soundness of the premouse we iterate. We will suppress this parameter thoughout our paper.

[5]In this case as well as in the cases below $\alpha = 0$ is allowed.

(c) for some $\alpha \le \lambda$, for some (α, λ)-iteration strategy $\Lambda \in \operatorname{dom}(\mathrm{Code}^*_\lambda)$ with hull condensation, for some (λ, Λ)-mouse operator $G \in \operatorname{dom}(\mathrm{Code}^*_\lambda)$ and for some $\beta \le \lambda$, F is a (β, Λ)-iteration strategy with hull condensation for some G-mouse $\mathcal{M} \in H_\lambda$.

When $\lambda = \omega_1$, we just write Code instead of Code_{ω_1}. Given an $F \in \operatorname{dom}(\mathrm{Code}_\lambda)$, we let $\mathcal{M}_F$ be, in the case F is an iteration strategy, the structure that F iterates, and in the case F is a mouse operator, the base of the cone on which F is defined.

Let $\mathcal{P} \in H_\lambda$ be a hybrid premouse, and for some $\alpha \le \lambda$ let Σ be (α, λ)-iteration strategy with hull condensation for $\mathcal{P}$. Suppose now that $\Gamma \subseteq \mathcal{P}(\cup_{\kappa<\lambda}\mathcal{P}(\kappa))$ is such that $\mathrm{Code}_\lambda(\Sigma) \in \Gamma$. Given a Σ-premouse $\mathcal{M}$, we say $\mathcal{M}$ is Γ-*iterable* if $|\mathcal{M}| < \lambda$ and $\mathcal{M}$ has a λ-iteration strategy (or (α, λ)-iteration strategy for some $\alpha \le \lambda$) Λ such that $\mathrm{Code}_\lambda(\Lambda) \in \Gamma$[6]. We let $\mathrm{Mice}^{\Gamma,\Sigma}$ be the set of Σ-premice that are Γ-iterable.

Definition 1.1 Given a Σ-premouse $\mathcal{M} \in H_\lambda$, we say $\mathcal{M}$ is countably α-iterable provided that, if $\pi \colon \mathcal{N} \to \mathcal{M}$ is a countable submodel of $\mathcal{M}$, then $\mathcal{N}$, as a Σ^π-mouse, is α-iterable. When $\alpha = \omega_1 + 1$, we just say that $\mathcal{M}$ is countably iterable. We say that $\mathcal{M}$ is countably Γ-iterable if, whenever π and $\mathcal{N}$ are as above, $\mathcal{N}$ is Γ-iterable.

Suppose $\mathcal{M}$ is a Σ-premouse. We then let $o(\mathcal{M}) = \mathrm{Ord} \cap \mathcal{M}$. We also let $\mathcal{M}||\xi$ be $\mathcal{M}$ cut off at ξ, *i.e.*, we keep the predicate indexed at ξ. We let $\mathcal{M}|\xi$ be $\mathcal{M}||\xi$ without the last predicate. We say ξ is a *cutpoint* of $\mathcal{M}$ if there is no extender E on $\mathcal{M}$ such that $\xi \in (\mathrm{cp}(E), lh(E)]$. We say ξ is a *strong cutpoint* if there is no E on $\mathcal{M}$ such that $\xi \in [\mathrm{cp}(E), lh(E)]$. We say $\eta < o(\mathcal{M})$ is *overlapped* in $\mathcal{M}$ if η is not a cutpoint of $\mathcal{M}$. Given $\eta < o(\mathcal{M})$ we let

$$\mathcal{O}^{\mathcal{M}}_\eta = \bigcup\{\mathcal{N} \triangleleft \mathcal{M} : \rho(\mathcal{N}) = \eta \text{ and } \eta \text{ is not overlapped in } \mathcal{N}\}.$$

Given a self-wellordered[7] $a \in H_\lambda$, we define the stacks over a as follows.

Definition 1.2 (i) $Lp^\Sigma(a) = \bigcup\{\mathcal{N} : \mathcal{N}$ is a countably iterable sound Σ-mouse over a such that $\rho(\mathcal{N}) = a\}$,

(ii) $\mathcal{K}^{\lambda,\Gamma,\Sigma}(a) = \bigcup\{\mathcal{N} : \mathcal{N}$ is a countably Γ-iterable sound Σ-mouse over a such that $\rho(\mathcal{N}) = a\}$,

(iii) $\mathcal{W}^{\lambda,\Gamma,\Sigma}(a) = \bigcup\{\mathcal{N} : \mathcal{N}$ is a Γ-iterable sound Σ-mouse over a such that $\rho(\mathcal{N}) = a\}$.

When $\Gamma = \mathcal{P}(\bigcup_{\kappa<\lambda}\mathcal{P}(\kappa))$, we omit it from our notation. We can define the sequences $\langle Lp^\Sigma_\xi(a) : \xi < \eta\rangle$, $\langle \mathcal{K}^{\lambda,\Gamma,\Sigma}_\xi(a) : \xi < \nu\rangle$, and $\langle \mathcal{W}^{\lambda,\Gamma,\Sigma}_\xi(a) : \xi < \mu\rangle$ as usual. The definition for operator Lp is as follows:

(a) $Lp^\Sigma_0(a) = Lp^\Sigma(a)$,
(b) for $\xi < \eta$, if $Lp^\Sigma_\xi(a) \in H_\lambda$ then $Lp^\Sigma_{\xi+1} = Lp^\Sigma(Lp^\Sigma_\xi(a))$,
(c) for limit $\xi < \eta$, $Lp^\Sigma_\xi = \bigcup_{\alpha<\xi} Lp^\Sigma_\alpha(a)$,
(d) η is least such that for all $\xi < \eta$, $Lp^\Sigma_\xi(a)$ is defined.

[6]Recall that iteration strategy for a Σ-mouse must respect Σ. In particular, all Λ-iterates of $\mathcal{M}$ are Σ-premice.

[7]*I.e.*, self well-ordered, a set a is called self well-ordered if $\mathrm{trc}(a \cup \{a\})$ is well-ordered in $L_1(a)$.

The other stacks are defined similarly.

1.2 (Γ, Σ)-suitable Premice

Again we fix an uncountable cardinal λ such that a large fragment of **ZF** holds in V_λ. We also fix $\Sigma \in \mathrm{dom}(\mathrm{Code}_\lambda)$ such that Σ is a (α, λ)-iteration strategy with hull condensation and $\Gamma \subseteq \mathcal{P}(\bigcup_{\kappa<\lambda} \mathcal{P}(\kappa))$ such that $\mathrm{Code}_\lambda(\Sigma) \in \Gamma$. We now start outlining how to import the material from [7, Subsection 1.3]. The most important notion we need from that subsection is that of a (Γ, Σ)-suitable premouse, which is defined as follows.

Definition 1.3 ((Γ, Σ)-suitable premouse) A Σ-premouse $\mathcal{P}$ is (Γ, Σ)-suitable if there is a unique cardinal δ such that

(i) $\mathcal{P} \vDash$ "δ is the unique Woodin cardinal",
(ii) $o(\mathcal{P}) = \sup_{n<\omega} (\delta^{+n})^{\mathcal{P}}$,
(iii) for every $\eta \neq \delta$, $\mathcal{W}^{\lambda,\Gamma,\Sigma}(\mathcal{P}|\eta) \vDash$ "η is not Woodin".
(iv) for any $\eta < o(\mathcal{P})$, $\mathcal{O}^{\mathcal{P}}_\eta = \mathcal{W}^{\lambda,\Gamma,\Sigma}(\mathcal{P}|\eta)$.

If $\Gamma = \mathcal{P}(\bigcup_{\alpha<\lambda} \mathcal{P}(a))$, then we use λ instead of Γ. In particular, we use λ-suitable to mean Γ-suitable. We will do the same with all the other notions, such as fullness preservation and short tree iterability, defined in this section

Suppose $\mathcal{P}$ is Γ-suitable. Then we let $\delta^{\mathcal{P}}$ be the δ of Definition 1.3. We then proceed as in [7, Section 1.3] to define (1) nice iteration tree, (2) (Γ, Σ)-short tree, (3) (Γ, Σ)-maximal tree, (4) (Γ, Σ)-correctly guided finite stack, and (5) the last model of a (Γ, Σ)-correctly guided finite stack, by using the $\mathcal{W}^{\lambda,\Gamma,\Sigma}$ operator instead of the $\mathcal{W}^\Gamma$ operator.

Definition 1.4 ($S(\Gamma, \Sigma)$ and $F(\Gamma, \Sigma)$) Let $S(\Gamma, \Sigma) = \{\mathcal{Q} : \mathcal{Q} \text{ is } (\Gamma, \Sigma)\text{-suitable}\}$. Also, we let $F(\Gamma, \Sigma)$ be the set of functions f such that $\mathrm{dom}(f) = S(\Gamma, \Sigma)$ and for each $\mathcal{P} \in S(\Gamma, \Sigma)$, $f(\mathcal{P}) \subseteq \mathcal{P}$ and $f(\mathcal{P})$ is amenable to $\mathcal{P}$, *i.e.*, for every $X \in \mathcal{P}$, $X \cap f(\mathcal{P}) \in \mathcal{P}$.

Given $\mathcal{P} \in S(\Gamma, \Sigma)$ and $f \in F(\Gamma, \Sigma)$, we let $f_n(\mathcal{P}) = f(\mathcal{P}) \cap \mathcal{P}|((\delta^{\mathcal{P}})^{+n})^{\mathcal{P}}$. Then $f(\mathcal{P}) = \bigcup_{n<\omega} f_n(\mathcal{P})$. We also let $\gamma^{\mathcal{P}}_f = \sup\big(\delta^{\mathcal{P}} \cap \mathrm{Hull}^{\mathcal{P}}_1\big(\{f_n(\mathcal{P}) : n < \omega\}\big)\big)$.

Notice that $\gamma^{\mathcal{P}}_f = \delta^{\mathcal{P}} \cap \mathrm{Hull}^{\mathcal{P}}_1\big(\gamma^{\mathcal{P}}_f \cup \{f_n(\mathcal{P}) : n < \omega\}\big)$. We then let

$$H^{\mathcal{P}}_f = \mathrm{Hull}^{\mathcal{P}}_1(\gamma^{\mathcal{P}}_f \cup \{f_n(\mathcal{P}) : n < \omega\}).$$

If $\mathcal{P} \in S(\Gamma, \Sigma)$, $f \in F(\Gamma, \Sigma)$ and $i\colon \mathcal{P} \to \mathcal{Q}$ is an embedding, then we let $i(f(\mathcal{P})) = \bigcup_{n<\omega} i(f_n(\mathcal{P}))$.

The following are the next block of definitions that routinely generalize into our context: (1) (f, Σ)-iterability, (2) $\vec{b} = \langle b_k : k < m\rangle$ witness (f, Σ)-iterability for $\vec{\mathcal{T}} = \langle \mathcal{T}_k, \mathcal{P}_k : k < m\rangle$, and (3) strong (f, Σ)-iterability. These definitions generalize by using $S(\Gamma, \Sigma)$ and $f \in F(\Gamma, \Sigma)$ instead of $S(\Gamma)$ and $F(\Gamma)$.

If $\mathcal{P}$ is strongly (f, Σ)-iterable and $\vec{\mathcal{T}}$ is a (Γ, Σ)-correctly guided finite stack on $\mathcal{P}$ with last model $\mathcal{R}$, then we let $\pi^{\Sigma}_{\mathcal{P},\mathcal{R},f}\colon H^{\mathcal{P}}_f \longrightarrow H^{\mathcal{R}}_f$ be the embedding given by any $\vec{b}$

that witnesses the (f,Σ)-iterability of $\vec{\mathcal{T}}$, *i.e.*, fixing $\vec{b}$ that witnesses f-iterability for $\vec{\mathcal{T}}$,

$$\pi^{\Sigma}_{\mathcal{P},\mathcal{R},f} = \pi_{\vec{\mathcal{T}},\vec{b}} \restriction H^{\mathcal{P}}_f.$$

Clearly, $\pi^{\Sigma}_{\mathcal{P},\mathcal{R},f}$ is independent of $\vec{\mathcal{T}}$ and $\vec{b}$. Here we keep Σ in our notation for $\pi^{\Sigma}_{\mathcal{P},\mathcal{R},f}$ because it depends on a (Γ,Σ)-correct iteration. It is conceivable that $\mathcal{R}$ might also be a (Γ,Λ)-correct iterate of $\mathcal{P}$ for another Λ, in which case $\pi^{\Sigma}_{\mathcal{P},\mathcal{R},f}$ might be different from $\pi^{\Lambda}_{\mathcal{P},\mathcal{R},f}$. However, the point is that these embeddings agree on $H^{\mathcal{P}}_f$. Also, we do not carry Γ in our notation, as it is usually understood from the context.

Given a finite sequence of functions $\vec{f} = \langle f_i : i < n\rangle \in F(\Gamma,\Sigma)$, we let $\bigoplus_{i<n} f_i \in F(\Gamma,\Sigma)$ be the function given by $(\bigoplus_{i<n} f_i)(\mathcal{P}) = \langle f_i(\mathcal{P}) : i < n\rangle$. We set $\bigoplus \vec{f} = \bigoplus_{i<n} f_i$.

We then let

$$\mathcal{I}_{\Gamma,F,\Sigma} = \left\{ (\mathcal{P},\vec{f}) : \mathcal{P} \in S(\Gamma,\Sigma), \quad \vec{f} \in F^{<\omega} \text{and } \mathcal{P} \text{ is strongly } \bigoplus \vec{f}\text{-iterable} \right\}.$$

Definition 1.5 Given $F \subseteq F(\Gamma,\Sigma)$, we say F is *closed* if for any $\vec{f} \subseteq F^{<\omega}$ there is $\mathcal{P}$ such that $(\mathcal{P},\oplus\vec{f}) \in \mathcal{I}_{\Gamma,F,\Sigma}$ and for any $\vec{g} \subseteq F^{<\omega}$, there is a (Γ,Σ)-correct iterate $\mathcal{Q}$ of $\mathcal{P}$ such that $(\mathcal{Q},\vec{f}\cup\vec{g}) \in \mathcal{I}_{\Gamma,F,\Sigma}$.

Now fix a closed $F \subseteq F(\Gamma,\Sigma)$ and let $\mathcal{F}_{\Gamma,F,\Sigma} = \left\{ H^{\mathcal{P}}_f : (\mathcal{P},f) \in \mathcal{I}_{\Gamma,F,\Sigma}\right\}$. We then define $\preceq_{\Gamma,F,\Sigma}$ on $\mathcal{I}_{\Gamma,F,\Sigma}$ by letting $(\mathcal{P},\vec{f}) \preceq_{\Gamma,F,\Sigma} (\mathcal{Q},\vec{g})$ if and only if $\mathcal{Q}$ is a (Γ,Σ)-correct iterate of $\mathcal{P}$ and $\vec{f} \subseteq \vec{g}$. Given $(\mathcal{P},\vec{f}) \preceq_{\Gamma,F,\Sigma} (\mathcal{Q},\vec{g})$, we have that

$$\pi^{\Sigma}_{\mathcal{P},\mathcal{Q},\vec{f}} \colon H^{\mathcal{P}}_{\oplus\vec{f}} \longrightarrow H^{\mathcal{Q}}_{\oplus\vec{f}}.$$

Notice that if F is closed then $\preceq_{\Gamma,F,\Sigma}$ is directed. Then let $\mathcal{M}_{\infty,\Gamma,F,\Sigma}$ be the direct limit of $(\mathcal{F}_{\Gamma,F,\Sigma},\preceq_{\Gamma,F,\Sigma})$ under $\pi^{\Sigma}_{\mathcal{P},\mathcal{Q},\vec{f}}$'s. Given $(\mathcal{P},\vec{f}) \in \mathcal{I}_{\Gamma,F,\Sigma}$, we let

$$\pi^{\Sigma}_{\mathcal{P},\vec{f},\infty} \colon H^{\mathcal{P}}_{\oplus\vec{f}} \longrightarrow \mathcal{M}_{\infty,\Gamma,F,\Sigma}$$

be the direct limit embedding. Using the proof of [7, Lemma 1.19], we get the following lemma.

Lemma 1.6 *$\mathcal{M}_{\infty,\Gamma,F,\Sigma}$ is well founded.*

Let F be as above and let $G \subseteq F$. The following list is then the next block of definitions that carry over to our context with no significant changes (see [7, Section 1.4]): (1) semi (F,G,Σ)-quasi iteration, (2) the embeddings of the (F,G,Σ)-quasi iteration (in this context, we will have Σ in the superscripts), (3) (F,G,Σ)-quasi iterations, (4) the last model of (F,G)-quasi iterations, (5) $\vec{f}$-guided strategies, (6) a Σ-quasi-self-justifying-system (Σ-qsjs), and (7) (ω,Γ,Σ)-suitable premice.

1.3 HOD_Σ under $\mathbf{AD}^+$

It turns out that for certain iteration strategies Σ, $V_\Theta^{\mathrm{HOD}_\Sigma}$ of many models of determinacy can be obtained as $\mathcal{M}_{\infty,\Gamma,F,\Sigma}$ for some Γ and F. For the rest of this section we assume $\mathbf{AD}^+$. Suppose Σ is an iteration strategy of some hod mouse $\mathcal{Q}$ and suppose Σ is $\mathcal{P}(\mathcal{P}(\omega))$-fullness preserving (see [8]) and has branch condensation (*i.e.*, $\lambda = \omega_1$ from the notation of Subsections 1.1 and 1.2). Assume further that $V = L(\mathcal{P}(\mathbb{R})) + MC(\Sigma)$[8] $+ \Theta = \theta_\Sigma$ and that $\mathcal{P}$ is below "θ is measurable", *i.e.*, below a measurable limit of Woodins. We let $\Gamma = \mathcal{P}(\mathcal{P}(\omega))$ and for the duration of this subsection, we drop Γ from our notation. Thus, a Σ-suitable premouse is a (Γ, Σ)-suitable premouse, etc.

Suppose $\mathcal{P}$ is Σ-suitable and $A \subseteq \mathbb{R}$ is OD_Σ. We say that $\mathcal{P}$ *weakly term captures* A if, letting $\delta = \delta^{\mathcal{P}}$ for each $n < \omega$, there is a term relation $\tau \in \mathcal{P}^{\mathrm{Coll}(\omega,(\delta^{+n})^{\mathcal{P}})}$ such that for comeager many $\mathcal{P}$-generics, $g \subseteq \mathrm{Coll}(\omega, (\delta^{+n})^{\mathcal{P}})$, we have $\tau_g = \mathcal{P}[g] \cap A$. We say $\mathcal{P}$ term captures A if the equality holds for all generics. The following lemma is essentially due to Woodin, and the proof for mice can be found in [9].

Lemma 1.7 *Suppose $\mathcal{P}$ is Σ-suitable and $A \subseteq \mathbb{R}$ is OD_Σ. Then $\mathcal{P}$ weakly term captures A. Moreover, there is a Σ-suitable $\mathcal{Q}$ that term captures A.*

Given a Σ-suitable $\mathcal{P}$ and an OD_Σ set of reals A, we let $\tau_{A,n}^{\mathcal{P}}$ be the standard name for a set of reals in $\mathcal{P}^{\mathrm{Coll}(\omega,(\delta^{+n})^{\mathcal{P}})}$ witnessing the fact that $\mathcal{P}$ weakly captures A. We then define $f_A \in F(\Gamma, \Sigma)$ by letting

$$f_A(\mathcal{P}) = \langle \tau_{A,n}^{\mathcal{P}} : n < \omega \rangle.$$

Let $F_{\Sigma,od} = \{ f_A : A \subseteq \mathbb{R} \wedge A \in OD_\Sigma \}$.

All the notions we have defined above using $f \in F(\Gamma, \Sigma)$ can be redefined for OD_Σ sets $A \subseteq \mathbb{R}$ using f_A as the relevant function. To save some ink, in what follows, we will say A-iterable instead of f_A-iterable and similarly for other notions. Also, we will use A in our subscripts instead of f_A.

The following lemma is one of the most fundamental lemmas used to compute HOD and is originally due to Woodin. Again, the proof can be found in [9].

Theorem 1.8 *For each $f \in F_{\Sigma,od}$, there is a $\mathcal{P} \in S(\Gamma, \Sigma)$ that is $(F_{\Sigma,od}, f)$-quasi iterable.*

Let $\mathcal{M}_\infty = \mathcal{M}_{\infty,F_{od},\Sigma}$.

Theorem 1.9 (Woodin [9]) *$\delta^{\mathcal{M}_\infty} = \Theta$, $\mathcal{M}_\infty \in \mathrm{HOD}_\Sigma$ and*

$$\mathcal{M}_\infty | \Theta = \left(V_\Theta^{\mathrm{HOD}_\Sigma}, \vec{E}^{\mathcal{M}_\infty|\Theta}, S^{\mathcal{M}_\infty}, \in \right),$$

where $S^{\mathcal{M}_\infty}$ is the predicate of $\mathcal{M}_\infty$ describing Σ.

Finally, if $a \in H_{\omega_1}$ is self-wellordered, then we could define $\mathcal{M}_\infty(a)$ by working with Σ-suitable premice over a. Everything we have said about Σ-suitable premice can also be said about Σ-suitable premice over a, and in particular, the equivalent of Theorem 1.9 can be proved using $\mathrm{HOD}_{(\Sigma,a)\cup\{a\}}$ instead of HOD_Σ and $\mathcal{M}_\infty(a)$ instead of $\mathcal{M}_\infty$.

[8] $MC(\Sigma)$ stands for the Mouse Capturing relative to Σ, which says that for $x, y \in \mathbb{R}$, x is $OD(\Sigma, y)$ if and only if x is in some Σ-mouse over y.

2 The Maximal Model

The core model induction is a method for constructing models of determinacy while working under various hypothesis. During the induction one climbs up through the Solovay hierarchy. This is a hierarchy of axioms that extends $\mathbf{AD}^+$ and roughly describes how complicated the Solovay sequence is. To pass the successive stages of the Solovay hierarchy, (*i.e.*, the stages where the length of the sequence is a successor), one defines a large enough model, called the *maximal model*, and shows that it satisfies $\mathbf{AD}^+$. The next step is to construct a hod pair beyond the maximal model. In this section our goal is to introduce the maximal model and prove some correctness results, such as Lemma 2.5. For more on the Solovay hierarchy, see [6].

We start by introducing universally Baire iteration strategies and mouse operators. We assume **ZFC**. Throughout this paper we fix a canonical method for sets in HC by reals. Given a real x that is a code of a set in HC, we let M_x be the structure coded by x and let $\pi_x\colon M_x \to N_x$ be the transitive collapse of M_x. We let WF be the set of reals that code sets in HC.

Definition 2.1 (uB operators) Suppose $\Lambda \in \mathrm{dom}(\mathrm{Code})$ and $\lambda \geq \omega_1$ is a cardinal. We say Λ is λ-uB if there exist $<\lambda$-complementing trees[9] (T, S) witnessing that $\mathrm{Code}(\Lambda)$ is $< \lambda$-uB in the following stronger sense: for all $x \in WF$ and $n, m \in x$,

$$(x, n, m) \in p[T] \iff \pi_x(m) \in \Lambda(\pi_x(n)).$$

If g is a $<\lambda$-generic then we let Λ^g be the canonical interpretation of Λ onto $V[g]$; *i.e.*, given $a, b \in HC^{V[g]}$, $\Lambda^g(a) = b$ if and only if, whenever $x \in WF^{V[g]}$ is such that $a \in N_x$ and $n \in x$ is such that $\pi_x(n) = a$, we have

$$b = \pi_x\big[\{m : (x, n, m) \in (p[T])^{V[g]}\}\big].$$

If Λ is λ-uB for all λ, then we say Λ is uB.

Suppose now that λ is an uncountable cardinal, g is a less than λ-generic, $a \in (H_\lambda)^V[g]$ and $\Sigma \in \mathrm{dom}(\mathrm{Code})$ is λ-uB. Then we define $Lp^{\Sigma,g}(a)$, $\mathcal{W}^{\lambda,\Sigma,g}(a)$, and $\mathcal{K}^{\lambda,\Sigma,g}(a)$ in $V[g]$ according to Definition 1.2. The following connects the three stacks defined above.

Proposition 2.2 *For every $a \in H_\lambda^V$, $\mathcal{W}^{\lambda,\Sigma}(a) \trianglelefteq \mathcal{K}^{\lambda,\Sigma}(a) \trianglelefteq Lp^\Sigma(a)$. Moreover, for any $\eta < \lambda$ and V-generic $g \subseteq \mathrm{Coll}(\omega, \eta)$ or $g \subseteq \mathrm{Coll}(\omega, <\eta)$, $\mathcal{W}^{\lambda,\Sigma,g}(a) \trianglelefteq \mathcal{W}^{\lambda,\Sigma}(a)$, $\mathcal{K}^{\lambda,\Sigma,g}(a) \trianglelefteq \mathcal{K}^{\lambda,\Sigma}(a)$, and $Lp^{\Sigma,g}(a) \trianglelefteq Lp^\Sigma(a)$.*

We are now in a position to introduce the maximal model of $\mathbf{AD}^+$.

Definition 2.3 (Maximal model of $\mathbf{AD}^+$) Suppose $\Sigma \in \mathrm{Code}$ is λ-uB and $\mu < \lambda$ is a cardinal. Let $g \subseteq \mathrm{Coll}(\omega, <\mu)$[10] be generic. Then we let $\mathcal{S}^{\lambda,\Sigma}_{\mu,g} = L(\mathcal{K}^{\lambda,\Sigma,g}(\mathbb{R}^{V[g]}))$.

[9]This means that the trees project to complements in all $<\lambda$-generic extensions.
[10]In this paper, μ is typically an inaccessible cardinal.

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Thus far strategy mice have been discussed only in situations when the underlying set was self-wellordered. However, $\mathcal{S}^{\lambda,\Sigma}_{\mu,g}$ is a Σ-mouse over the set of reals. Such hybrid mice were defined in [8, Section 2.10]. We say that $\mathcal{S}^{\lambda,\Sigma}_{\mu,g}$ is the λ-Σ-*maximal model at* μ. Next we define hod pairs below a cardinal.

Definition 2.4 (Hod pair below λ) Suppose now that $(\mathcal{P}, \Sigma)$ is a hod pair[11] such that $\Sigma \in \text{dom(Code)}$ is λ^+-uB. We say that $(\mathcal{P}, \Sigma)$ is a hod pair below λ if Σ has branch condensation and whenever $g \subseteq \text{Coll}(\omega, \lambda)$ is V-generic, and Σ^g is ω_1-fullness preserving in $V[g]$.

The next lemma connects various degrees of iterability. Below, if $\xi \in \text{Ord}$ and N is a transitive model of **ZFC**, then we let $N_\xi = V^N_\xi$.

For the purposes of the next lemma, suppose $\mu < \lambda$ are such that μ is a strong cardinal and λ is inaccessible. Let $j \colon V \to M$ be an embedding witnessing that μ is λ^+-strong and let $g \subseteq \text{Coll}(\omega, {<}\mu)$ and $h \subseteq \text{Coll}(\omega, {<}j(\mu))$ be two generics such that $g = h \cap \text{Coll}(\omega, {<}\mu)$. Let $j^+ \colon V[g] \to M[h]$ be the lift of j. Let $W = V[g]$.

Lemma 2.5 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair below μ and $a \in V_\lambda[g]$ is self-wellordered. Then*

$$\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{W}^{\lambda,\Sigma,h\cap\text{Coll}(\omega,<\lambda)}(a) = \mathcal{K}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\mu,\Sigma,g}(a) = (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}.$$

Proof We first show that $\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\mu,\Sigma,g}(a)$. Work in W. Clearly, $\mathcal{W}^{\lambda,\Sigma,g}(a) \trianglelefteq \mathcal{K}^{\mu,\Sigma,g}(a)$. Then let $\mathcal{M} \trianglelefteq \mathcal{K}^{\mu,\Sigma,g}(a)$ be such that $\rho(\mathcal{M}) = a$. We want to see that $\mathcal{M} \trianglelefteq \mathcal{W}^{\lambda,\Sigma,g}(a)$. To see this, notice that, by a standard absoluteness argument, there is $\sigma \colon \mathcal{M} \to j^+(\mathcal{M})$ such that $\sigma \in M[h]$, $\sigma(\mathcal{P}) = \mathcal{P}$ and $M[h] \vDash j(\Sigma^g)^\sigma = j(\Sigma^g)$ (this follows from the fact that Σ has branch condensation). Hence, in $M[h]$, $\mathcal{M}$ is a $\omega_1 + 1$-iterable $j(\Sigma^g)$-mouse. Let in $M[h]$, $\Lambda \in M[h]$ be the unique $\omega_1 + 1$-iteration strategy of $\mathcal{M}$ (as a $j(\Sigma^g)$-mouse). It follows from the homogeneity of the collapse and the uniqueness of Λ that $\Lambda \restriction H^W_\lambda \in W$. Hence, $\mathcal{M} \trianglelefteq \mathcal{W}^{\lambda,\Sigma,g}(a)$.

To see that $\mathcal{W}^{\lambda,g}(a) = (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}$, first suppose that $\mathcal{M} \trianglelefteq \mathcal{W}^{\lambda,\Sigma,g}(a)$. Then in $M[h]$ we have $j(\mathcal{M}) \trianglelefteq \mathcal{W}^{j(\lambda),j(\Sigma),h}(j^+(a))$. Since in $M[h]$ $\mathcal{M}$ is embeddable into $j^+(\mathcal{M})$ via σ with the above properties, we get that in $M[h]$, $\mathcal{M} \trianglelefteq \mathcal{W}^{j(\lambda),j(\Sigma),h}(a)$. Next, suppose that $\mathcal{M} \trianglelefteq (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}$ is such that $\rho(\mathcal{M}) = a$. It follows from the homogeneity of the collapse and the uniqueness of the strategy of $\mathcal{M}$ that $\mathcal{M} \in V[g]$ and that $\mathcal{M} \trianglelefteq \mathcal{W}^{\lambda,\Sigma,g}(a)$.

We thus have that

$$\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\mu,\Sigma,g}(a) = \big(\mathcal{W}^{j(\lambda),j(\Sigma),h}(a)\big)^{M[h]}. \tag{2.1}$$

Finally notice that

$$\big(\mathcal{W}^{j(\lambda),j(\Sigma),h}(a)\big)^{M[h]} \trianglelefteq \mathcal{W}^{\lambda,\Sigma,h\cap\text{Coll}(\omega,<\lambda)}(a) \trianglelefteq \mathcal{W}^{\lambda,\Sigma,g}(a) \trianglelefteq \mathcal{K}^{\lambda,\Sigma,g}(a) \trianglelefteq \mathcal{K}^{\mu,\Sigma,g}(a). \tag{2.2}$$

Equations (2.1) and (2.2) now easily imply the claim. ■

[11] Hod pairs are in the sense of [8]. They all satisfy that there is no measurable limit of Woodins.

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The following is an easy corollary of Lemma 2.5.

Corollary 2.6 *Suppose that $\mu < \kappa < \lambda$ and $j\colon V \to M$ are such that μ and κ are strong cardinals, λ is inaccessible, j witness that μ is λ-strong, and $M \vDash$ "κ is strong cardinal". Let $(\mathcal{P}, \Sigma)$ be a hod pair below μ that is λ-uB. Let $g \subseteq \mathrm{Coll}(\omega, {<}\kappa)$ and $h \subseteq \mathrm{Coll}(\omega, {<}j(\mu))$ be generic such that $g = h \cap \mathrm{Coll}(\omega, {<}\kappa)$. Let*

$$j^+\colon V[g \cap \mathrm{Coll}(\omega, {<}\mu)] \longrightarrow M[h]$$

be the lift of j. Then, whenever $a \in V_\lambda[g]$,

$$\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\kappa,\Sigma,g}(a) = \mathcal{W}^{\lambda,\Sigma,h\cap\mathrm{Coll}(\omega,<\lambda)}(a) = \left(\mathcal{W}^{j(\lambda),j(\Sigma),h}(a)\right)^{M[h]}.$$

Proof Let $k = g \cap \mathrm{Coll}(\omega, {<}\mu)$. Notice that because $j(\Sigma)$ has a unique extension in $M[h]$, we have that $j^+(\Sigma^k) \restriction V_\lambda[g] = \Sigma^g$. Because κ is a strong cardinal in V, it follows from Lemma 2.5 that

$$\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\kappa,\Sigma,g}(a).$$

Because κ is a strong cardinal in M, it follows from Lemma 2.5 that

$$\mathcal{K}^{\kappa,\Sigma,g}(a) = \mathcal{W}^{j(\lambda),j(\Sigma),g}(a) = (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}.$$

Therefore, $\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\kappa,\Sigma,g}(a) = (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}$. ■

3 The Core Model Induction

The goal of this section is to develop some basic notions in order to state Theorem 3.3 which we will use as a black box. Our core model induction is a typical one: we have two uncountable cardinals $\kappa < \lambda$, *the core model induction operators* (cmi operators) defined on bounded subsets of κ can be extended to act on bounded subsets of λ, and for any such cmi operator F acting on bounded subsets of λ, the minimal F-closed mouse with one Woodin cardinal exists and is λ-iterable. Having these three conditions is enough to show, by using the *scales analysis* developed in [10, 13], that the -λ-*maximal model at* κ indeed satisfies $\mathbf{AD}^+$. The details of the proof of Theorem 3.3 have appeared, in a less general form, in [9, 11].

The mouse operators that are constructed during core model induction have two additional properties: they transfer and relativize well. To make these notions precise, fix $\Sigma \in \mathrm{dom}(\mathrm{Code})$, which is λ-uB. Given a Σ-mouse operator $F \in \mathrm{dom}(\mathrm{Code}_\lambda)$, we say that:

(Relativizes well) *F relativizes well* if there is a formula $\phi(u, v, w)$ such that, whenever $X, Y \in \mathrm{dom}(F)$ and N are such that $X \in L_1(Y)$ and N is a transitive rudimentarily closed set such that $Y, F(Y) \in N$, we have that $F(X) \in N$ and $F(X)$ is the unique U such that $N \vDash \phi[U, X, F(Y)]$.

(Transfers well) *F transfers well* if, whenever $X, Y \in \mathrm{dom}(F)$ are such that X is generic over $L_1(Y)$, we have that $F(L_1(Y)[X])$ is obtained from $F(Y)$ via S-constructions (see [8, Section 2.11]) and in particular, $F(L_1(Y))[X] = F(L_1(Y)[X])$.

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We are now in a position to introduce the core model induction operators that we will need in this paper.

Definition 3.1 (Core model induction operator) Suppose $|\mathbb{R}| = \kappa$, $(\mathcal{P}, \Sigma)$ is a hod pair below κ^+. We say $F \in \mathrm{dom}(\mathrm{Code})$ is a Σ core model induction operator, or just Σ-cmi operator, if one of the following holds:

(i) For some $\alpha \in \mathrm{Ord}$, letting $M = \mathcal{S}_\omega^{\kappa^+,\Sigma}||\alpha$, $M \vDash \mathbf{AD}^+ + MC(\Sigma)$ and one of the following holds:

(a) F is a Σ-mouse operator that transfers and relativizes well.

(b) For some self-wellordered $b \in HC$ and some Σ-premouse $\mathcal{Q} \in HC^V$ over b, F is an (ω_1, ω_1)-iteration strategy for $\mathcal{Q}$ that is $(\mathcal{P}(\mathbb{R}))^M$-fullness preserving, has branch condensation, and is guided by some $\vec{A} = (A_i : i < \omega)$ such that $\vec{A} \in OD^M_{b,\Sigma,x}$ for some $x \in b$. Moreover, α ends either a weak or a strong gap in the sense of [10].

(c) For some $H \in \mathrm{dom}(\mathrm{Code})$, H satisfies a or b above and for some $n < \omega$, F is $x \to \mathcal{M}_n^{\#,H}(x)$ operator or for some $b \in HC$, F is the ω_1-iteration strategy of $\mathcal{M}_n^{\#,H}(b)$.

(ii) For some $\alpha \in \mathrm{Ord}$, $a \in HC$ and $\mathcal{M} \trianglelefteq \mathcal{W}^{\kappa^+,\Sigma}(a)$ such that $\rho(\mathcal{M}) = a$ letting Λ be $\mathcal{M}$'s unique strategy, the above conditions hold for F with $L^{\Lambda}_{\kappa^+}(\mathbb{R})$ used instead of $\mathcal{S}_\omega^{\kappa^+,\Sigma}$ and Λ used instead of Σ.

When $\Sigma = \varnothing$, we omit it from our notation. Often times, when doing core model induction, we have two uncountable cardinals $\kappa < \lambda$ and we need to show that cmi operators in $V^{\mathrm{Coll}(\omega,<\kappa)}$ can be extended to act on $V_\lambda^{\mathrm{Coll}(\omega,<\kappa)}$. This is a weaker notion than being λ-uB. We also need to know that for any cmi operator $F \in V^{\mathrm{Coll}(\omega,<\kappa)}$, $\mathcal{M}_1^{\#,F}$-exists. We make these statements more precise.

Definition 3.2 (Lifting cmi operators) Suppose $\kappa < \lambda$ are two cardinals such that κ is an inaccessible cardinal and suppose $(\mathcal{P}, \Sigma)$ is a hod pair below κ.

(i) $\mathrm{Lift}(\kappa, \lambda, \Sigma)$ is the statement that for every generic $g \subseteq \mathrm{Coll}(\omega, <\kappa)$, in $V[g]$, for every every Σ^g-cmi operator F there is an operator $F^* \in \mathrm{dom}(\mathrm{Code}_\lambda)$ such that $F = F^* \restriction HC$. In this case we say that F is λ-extendable. Such an F^* is necessarily unique, as can be easily shown by a Skolem hull argument.[12] If $\mathrm{Lift}(\kappa, \lambda, \Sigma)$ holds, $g \subseteq \mathrm{Coll}(\omega, <\kappa)$ is generic, and F is a Σ^g-cmi operator, then we let F^λ be its extended version.

(ii) We let $\mathrm{Proj}(\kappa, \lambda, \Sigma)$[13] be the conjunction of the statements "$\mathrm{Lift}(\kappa, \lambda, \Sigma)$" and "for every generic $g \subseteq \mathrm{Coll}(\omega, <\kappa)$, in $V[g]$, we have

(a) for every Σ^g-cmi operator F, $\mathcal{M}_1^{\#,F}$ exists and is λ-iterable;

(b) for every $a \in H_{\omega_1}$, $\mathcal{K}^{\omega_1,\Sigma,g}(a) = \mathcal{W}^{\lambda,\Sigma,g}(a)$".

[12] Suppose $H_0, H_1 \in \mathrm{dom}(\mathrm{Code}_\lambda^{V[g]})$ are two extensions of F. Working in $V[g]$, let $\pi \colon N \to H_{\lambda^+}[g]$ be elementary such that N is countable and $H_0, H_1 \in \mathrm{rng}(\pi)$. Let $(\overline{H}_0, \overline{H}_1) = \pi^{-1}(H_0, H_1)$. Then it follows from the definition of being a Σ-cmi operator that $\overline{H}_0 = H_0 \restriction N$ and $\overline{H_1} = H \restriction N$. However, since $H_0 \restriction N = F \restriction N = H_1 \restriction N$, we get that $N \vDash \overline{H}_0 = \overline{H}_1$, a contradiction.

[13] Proj stands for projective determinacy. The meaning is taken from clause (a).

Recall that under **AD**, if X is any set, then θ_X is the least ordinal that is not a surjective image of $\mathbb{R}$ via an OD_X function. The following is the core model induction theorem that we will use.

Theorem 3.3 *Suppose $\kappa < \lambda$ are two uncountable cardinals and suppose $(\mathcal{P}, \Sigma)$ is a hod pair below κ such that* $\mathrm{Proj}(\kappa, \lambda, \Sigma)$ *holds. Then for every generic $g \subseteq \mathrm{Coll}(\omega, {<}\kappa)$, $\mathcal{S}^{\lambda,\Sigma}_{\kappa,g} \models \mathbf{AD}^+ + \theta_\Sigma = \Theta$.*

We will not prove the theorem here, as the proof is very much like the proof of the core model induction theorems in [7, Theorems 2.4 and 2.6], [9, Chapter 7], and [11]. To prove the theorem we have to use the unpublished scales analysis for $\mathcal{S}^{\lambda,\Sigma}_{\kappa,g}$; see [10]. Also, the readers familiar with the scales analysis of $Lp(\mathbb{R})$ as developed by Steel [13, 14] should have no problem seeing how the general theory should be developed. However, there is one point worth going over.

Suppose we are doing core model induction to prove Theorem 3.3. Fix $g \subseteq \mathrm{Coll}(\omega, {<}\kappa)$. During this core model induction, we climb through the levels of $\mathcal{S}^{\lambda,\Sigma}_{\kappa,g}$, some of which project to $\mathbb{R}$ but do not satisfy that "$\Theta = \theta_{\Sigma^g}$". It is then the case that the scales analysis of [10] cannot help us in producing the next "new" set. However, such levels can never be problematic for proving that $\mathbf{AD}^+$ holds in $\mathcal{S}^{\lambda,\Sigma}_{\kappa,g}$. This follows from the next lemma.

Lemma 3.4 *Suppose in $V[g]$, $\mathcal{M} \trianglelefteq \mathcal{S}^{\lambda,\Sigma}_{\kappa,g}$ is such that $\rho(\mathcal{M}) = \mathbb{R}$ and $\mathcal{M} \models$ "$\Theta \neq \theta_{\Sigma^g}$". Then there is $\mathcal{N} \trianglelefteq \mathcal{S}^{\lambda,\Sigma}_{\kappa,g}$ such that $\mathcal{M} \trianglelefteq \mathcal{N}$, $\mathcal{N} \models$ "$\mathbf{AD}^+ + \Theta = \theta_{\Sigma^g}$".*

Proof Since $\mathcal{M} \models$ "$\Theta \neq \theta_{\Sigma^g}$" it follows that $\mathcal{P}(\mathbb{R})^{\mathcal{M}} \cap (Lp^{\Sigma^g}(\mathbb{R}))^{\mathcal{M}} \neq \mathcal{P}(\mathbb{R})^{\mathcal{M}}$. It then follows that there is some $\alpha < o(\mathcal{M})$ such that $\rho(\mathcal{M}|\alpha) = \mathbb{R}$ but $\mathcal{M}|\alpha \ntrianglelefteq (Lp^{\Sigma^g}(\mathbb{R}))^{\mathcal{M}}$. Let $\pi\colon \mathcal{N} \to \mathcal{M}|\alpha$ be such that $\mathcal{N}$ is countable and its iteration strategy is not in $\mathcal{M}$. Let $\Lambda \in V[g]$ be the λ-iteration strategy of $\mathcal{N}$. Then a core model induction through $L^\Lambda(\mathbb{R})$ shows that $L^\Lambda(\mathbb{R}) \models \mathbf{AD}^+$ (this is where we needed Definition 3.1(ii)). However, its not hard to see that $L^\Lambda(\mathbb{R}) \models$ "$\Theta = \theta_{\Sigma^g}$". It then follows from an unpublished result of the the first author and Steel that $L^\Lambda(\mathbb{R}) \models \mathcal{P}(\mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap Lp^{\Sigma^g}(\mathbb{R})$ (for the case $\Sigma^g = \varnothing$, see [12]). Then let $\mathcal{K} \trianglelefteq (Lp^{\Sigma^g}(\mathbb{R}))^{L^\Lambda(\mathbb{R})}$ be such that $\rho(\mathcal{K}) = \mathbb{R}$, $\mathcal{K} \models \Theta = \theta_{\Sigma^g}$ and $\Lambda \restriction HC^{V[g]} \in \mathcal{K}$ (there is such a $\mathcal{K}$ by an easy application of $\Sigma^2_1(\mathrm{Code}(\Sigma^g))$ reflection). Since countable submodels of $\mathcal{K}$ are λ-iterable (see clause (b) of $\mathrm{Proj}(\kappa, \lambda, \Sigma)$), we have that $\mathcal{K} \trianglelefteq \mathcal{S}^{\lambda,\Sigma}_{\kappa,g}$. Also we cannot have that $\mathcal{K} \triangleleft \mathcal{M}$, because otherwise $\mathcal{N}$ would have a strategy in $\mathcal{M}$. Therefore, $\mathcal{M} \trianglelefteq \mathcal{K}$. ∎

We can now do core model induction through the levels of $\mathcal{S}^{\lambda,\Sigma}_{\kappa,g}$ as follows. If we have reached a gap satisfying "$\Theta = \theta_{\Sigma^g}$", then we can use the scales analysis of [10] to go beyond. If we have reached a level that satisfies "$\Theta \neq \theta_{\Sigma^g}$", then using Lemma 3.4 we can skip through it and go to the least level beyond it that satisfies "$\Theta = \theta_{\Sigma^g}$". We leave the rest of the details to the reader.

One final remark is that under the hypothesis of Theorem 3.3, whenever $\Lambda \in V[g]$ is an iteration strategy of some Σ-mouse $\mathcal{M}$ over some self-wellordered $a \in HC^{V[g]}$ with the property that $\rho(\mathcal{M}) = a$, then $L^\Lambda(\mathbb{R}^{V[g]}) \models \mathbf{AD}^+$ (which can be proved by a core model induction argument through $L^\Lambda(\mathbb{R}^{V[g]})$). It then follows that $\mathcal{S}^{\lambda,\Sigma}_{\kappa,g} \models$ "$\Theta = \theta_{\Sigma^g}$".

We end this section with the following useful fact on lifting strategies. Among other things it can be used to show clause (b) of $\mathrm{Proj}(\kappa, \lambda, \Sigma)$.

Lemma 3.5 (Lifting cmi operators through strongness embeddings) *Suppose $\kappa < \lambda$ are such that κ is a λ-strong cardinal. Then whenever $(\mathcal{P}, \Sigma)$ is a hod pair below κ,* $\mathrm{Lift}(\kappa, \lambda, \Sigma)$ *and clause* (b) *of* $\mathrm{Proj}(\kappa, \lambda, \Sigma)$ *hold.*

Proof Fix an embedding $j\colon V \to M$ witnessing that κ is λ-strong. We only show that $\mathrm{Lift}(\kappa, \lambda, \Sigma)$ holds, as the proof of clause (b) of $\mathrm{Proj}(\kappa, \lambda, \Sigma)$ is very similar. Let $g \subseteq \mathrm{Coll}(\omega, {<}\kappa)$ and $h \subseteq \mathrm{Coll}(\omega, {<}j(\kappa))$ be V-generic such that $g = h \cap \mathrm{Coll}(\omega, {<}\kappa)$. We can then extend j to $j^+\colon V[g] \to M[h]$. Working in $V[g]$, fix a Σ^g-cmi operator F. Let $F^\lambda = j^+(F) \restriction H_\lambda[g]$. Fix $X \in HC^{V[g]}$ such that $V[g] \vDash F \in OD_{\{X,\Sigma^g\}}$. It then follows that $M[h] \vDash j^+(F) \in OD_{\{X, j^+(\Sigma^g)\}}$. This in turn implies $F^\lambda \in V[g]$. ■

4 A Core Model Induction at a Strong Cardinal

In this section we present a useful application of Theorem 3.3 that we will later use to prove our main theorem. Recall that we say μ reflects the set of strong cardinals if for every λ there is an embedding $j\colon V \to M$ witnessing that μ is λ-strong and for any cardinal $\kappa \in [\mu, \lambda)$, $V \vDash$ "κ is strong" if and only if $M \vDash$ "κ is strong".

Theorem 4.1 *Suppose $\mu < \kappa < \lambda$ are such that λ is an inaccessible cardinal, μ and κ are strong such that μ reflects the set of strong cardinals, and whenever $(\mathcal{R}, \Psi)$ is a hod pair below κ such that $\lambda^{\mathcal{R}} = 0$, $\mathrm{Proj}(\kappa, \lambda, \Psi)$ holds. Suppose $m \subseteq \mathrm{Coll}(\omega, {<}\kappa)$ is generic. Then in $V[m]$, there is $A \subseteq \mathbb{R}$ such that $L(A, \mathbb{R}) \vDash \theta_0 < \Theta$.*

More specifically, let $g = m \cap \mathrm{Coll}(\omega, {<}\mu)$ and $\mathcal{P} = (\mathcal{M}_\infty)^{\mathcal{S}^\lambda_{\mu,g}}$. Then in $V[m]$, $\mathcal{P}$ has an (ω_1, ω_1)-iteration strategy Ψ such that Ψ is λ-fullness preserving. Moreover, there is a stack $\vec{\mathcal{T}} \in HC^{V[m]}$ on $\mathcal{P}$ according to Ψ with last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{T}}}$ exists and in $V[m]$ $(\mathcal{Q}, \Psi_{\mathcal{Q},\vec{\mathcal{T}}})$ is a hod pair below ω_1. Finally, in $V[m]$, Ψ is λ-extendible and $L(\Psi_{\mathcal{Q},\vec{\mathcal{T}}}, \mathbb{R}) \vDash \mathbf{AD}^+ + \theta_0 < \Theta$.

Clearly it is enough to prove the second part of the theorem, which we do with a sequence of lemmas. Fix $\mu < \kappa < \lambda$ as in Theorem 4.1. Fix a V-generics m, g as in the theorem and let $j\colon V \to M$ be an embedding witnessing that μ is λ^+-strong and such that κ is strong in M. Also fix a V-generic $h \subseteq \mathrm{Coll}(\omega, {<}j(\mu))$ such that $h \cap \mathrm{Coll}(\omega, {<}\kappa) = m$. It then follows that j lifts to $j^+\colon V[g] \to M[h]$. Notice that it follows from our hypothesis, Theorem 3.3, and Lemma 3.5 that $\mathcal{S}^\lambda_{\mu,g} \vDash \mathbf{AD}^+ + \Theta = \theta_0$.

Let $k = h \cap \mathrm{Coll}(\omega, {<}\lambda)$, $\mathcal{S} = \mathcal{S}^\lambda_{\mu,g}$, and $\Gamma^* = (F_{od})^{\mathcal{S}}$. The following is an immediate corollary of Lemma 2.5.

Corollary 4.2 *For any $a \in HC^{V[g]}$, $(Lp(a))^{\mathcal{S}} = \mathcal{W}^{\lambda,g}(a)$.*

We will use the next lemma along with [7, Lemma 1.29] to construct an iteration strategy for $\mathcal{P}$.

Lemma 4.3 *$j^+[\Gamma^*]$ is a qsjs for $j^+(S(\Gamma^*))$ as witnessed by $\mathcal{P}$.*

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Proof We first prove the following claim.

Claim Suppose $\mathcal{R} \in j(\mathcal{S})$ is such that there are $\pi\colon \mathcal{P} \to \mathcal{R}$ and $\sigma\colon \mathcal{R} \to j(\mathcal{P})$ such that $j \restriction \mathcal{P} = \sigma \circ \pi$. Then $\mathcal{R} \in S(j^+(\Gamma^*))$.

Proof of Claim First let $T \in j^+(\mathcal{S})$ be the tree projecting to the universal $(\Sigma^2_1)^{j^+(\mathcal{S})}$ set. We have that $L[T, \mathcal{P}] \vDash \mathcal{P} = H_{(\delta^{\mathcal{P}})^{+\omega}}$. Notice that $T \in V$. It then follows that we can lift $j \restriction \mathcal{P}$, π and σ to

$$j^*\colon L[T, \mathcal{P}] \longrightarrow L[j(T), j(\mathcal{P})], \quad \pi^*\colon L[T, \mathcal{P}] \longrightarrow L[\pi^*(T), \mathcal{R}],$$
$$\sigma^*\colon L[\pi^*(T), \mathcal{R}] \longrightarrow L[j(T), j(\mathcal{P})].$$

such that $j^* = \sigma^* \circ \pi^*$. The proof of [7, Lemma 2.21] now shows that $\mathcal{R} \in j^+(S(\Gamma^*))$. ■

To finish the proof, we need to show that for every $A \in \Gamma^*$, in $j^+(\mathcal{S})$:

(a) $\mathcal{P}$ is $(j^+[\Gamma^*], j(A))$-quasi iterable[14].

To see (a), fix $A \in \Gamma^*$ and fix $\mathcal{Q} \in S(\Gamma^*)$ such that in $\mathcal{S}$, $\mathcal{Q}$ is (Γ^*, A)-quasi iterable. Then $j^+(\mathcal{S}) \vDash$ "$\mathcal{Q}$ is $(j^+(\Gamma^*), j(A))$-quasi iterable". Since we have that $j^+(\mathcal{S}) \vDash$ "$\mathcal{P}$ is a $(j^+(\Gamma^*), j(A))$-quasi iterate of $\mathcal{Q}$", we have that $j^+(\mathcal{S}) \vDash$ "$\mathcal{P}$ is a $(j^+(\Gamma^*), j(A))$-quasi iterable". Repeating the argument for every A, we get that

(b) for every $A \in \Gamma^*$, $j^+(\mathcal{S}) \vDash$ "$\mathcal{P}$ is $(j^+(\Gamma^*), j(A))$-quasi iterable".

It follows from (b) that to finish the proof of (a) it is enough to show that

(c) for every $A \in \Gamma^*$, in $j^+(\mathcal{S})$, every $(j^+(\Gamma^*), j^+(A))$-quasi iteration is also a $(j^+[\Gamma^*], j^+(A))$-quasi iteration.

To prove (c), it is enough to show that whenever $\mathcal{Q}$ is a $j^+(\Gamma^*)$-quasi iterate of $\mathcal{P}$ then $\delta^{\mathcal{Q}} = \bigcup_{B \in j^+[\Gamma^*]} H^{\mathcal{Q}}_{\tau^{\mathcal{Q}}_B}$. Fix $\mathcal{Q}$ which is a $j^+(\Gamma^*)$-quasi iterate of $\mathcal{P}$. Let $\pi = \bigcup_{B \in j^+[\Gamma^*]} \pi_{\mathcal{P},\mathcal{Q},B}$. Let $\mathcal{W}$ be the transitive collapse of $\bigcup_{B \in j^+[\Gamma^*]} H^{\mathcal{Q}}_{\tau^{\mathcal{Q}}_B}$. Let $\sigma\colon \mathcal{W} \to \mathcal{Q}$ be the uncollapse map and $\tau = \bigcup_{B \in j^+(\Gamma^*)} \pi_{\mathcal{Q},\infty,B}$. Because $\mathcal{P} = \bigcup_{B \in j^+[\Gamma^*]} H^{\mathcal{P}}_B$, π is total. It then follows that

$$j \restriction \mathcal{P} = \tau \circ (\sigma^{-1} \circ \pi).$$

The claim then implies that $\mathcal{W} \in S(j^+(\Gamma^*))$. This finishes the proof of (a). A similar proof gives the following:

(d) whenever $\mathcal{Q}$ is a $j^+[\Gamma^*]$-quasi iterate of $\mathcal{P}$ and $\epsilon\colon \mathcal{R} \to_{\Sigma_1} \mathcal{Q}$ is such that for every $A \in \Gamma^*$, $\tau^{\mathcal{Q}}_{j^+(A)} \in \operatorname{rng}(\epsilon)$ then $\mathcal{R} \in j^+(\mathcal{S}(\Gamma^*))$.

The key point again is that the embedding π defined above is total. This finishes the proof of the lemma. ■

We can now use [7, Lemma 1.29] to get a strategy $\Sigma^* = \Sigma^{j^+[\Gamma^*]}$. In our current situation, there is one important difference with [7]. Here Σ^* may not act on all trees

[14] Technically we should write $(j^+[\Gamma^*], \{j(A)\})$, but we abuse notation here.

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that are in $M[h]$ as $j^+[\Gamma^*]$ is not in $M[h]$. However, it acts on all stacks that are in $V_\lambda[k]$. This is simply because

$$F = \{B \cap \mathbb{R}^{V[k]} : B \in j^+[\Gamma^*]\} \in V[k].$$

Also, $\Sigma = \Sigma^* \restriction V_\lambda[m]$ and $\Psi = \Sigma \restriction V_\kappa[m]$.

Lemma 4.4 *In $V[m]$, Σ is a (λ, λ)-iteration strategy that is λ-fullness preserving and is guided by F.*

Proof It is enough to show that Σ is λ-fullness preserving as we have already established the remaining clauses. That Σ is λ-fullness preserving follows easily from Corollary 2.6. ■

Next, we show that there is a stack $\vec{\mathcal{T}}$ on $\mathcal{P}$ according to Ψ with last model $\mathcal{Q} \in HC^{V[m]}$ such that $\pi^{\vec{\mathcal{T}}}$ exists and $\Psi_{\mathcal{Q},\vec{\mathcal{T}}}$ has branch condensation. We follow the proof of branch condensation that first appeared in [2] and also in [9, Chapter 7] (see especially the proofs of [9, Lemmas 7.9.6 and 7.9.7]). Below we summarize what we need in order to carry out the proof.

Recall that if Λ is a (possibly partial) iteration strategy for a λ-suitable premouse $\mathcal{R}$, then we say Λ has *weak-condensation* on its domain whenever $\mathcal{R}^*$ is a Λ-iterate of $\mathcal{R}$ such that the iteration embedding $i\colon \mathcal{R} \to \mathcal{R}^*$ exists and $\mathcal{R}^{**}$ is such that there are $\pi\colon \mathcal{R} \to \mathcal{R}^{**}$ and $\sigma\colon \mathcal{R}^{**} \to \mathcal{R}^*$ with the property that $i = \sigma \circ \pi$, it follows that $\mathcal{R}^{**}$ is λ-suitable.

Suppose (R, J) is a pair such that R is a transitive set such that for some ν that is a cardinal in R, $R \models$ "$V = H_{\nu^+} + J$ is a precipitous ideal on ω_1". We say (R, J) captures Ψ if in $V[m]$,

(a) (R, J) is countable and an iterable pair via taking generic ultrapowers by J and its images;
(b) $\mathcal{P} \in HC^R$, $\Psi \restriction HC^R \in R$, and letting $\Psi^R = \Psi \restriction HC^R$, $R \models$ "no tail of Ψ^R has branch condensation";
(c) whenever $\xi < \omega_1$ and $(R_\alpha, J_\alpha, G_\alpha$[15]$, \pi_{\alpha,\beta} : \alpha < \beta \le \xi)$ is some iteration of (R, J) of length $\xi + 1$, we have that $\pi_{0,\xi}(\Psi^R)$ has weak-condensation and fullness preservation on its domain.

The main lemma towards showing that some tail of Ψ has branch condensation follows.

Lemma 4.5 *In $V[m]$, there is no (R, J) that captures Ψ.*

We do not give the proof of the lemma as it can be found in [2] and in [9, Chapter 7]. We then derive a contradiction by showing the following lemma.

Lemma 4.6 *Suppose no tail of Ψ has branch condensation. Then in $V[m]$, there is a pair (R, J) that captures Ψ.*

[15] $G_\alpha \subseteq (\mathcal{P}(\omega_1)/J_\alpha)^{R_\alpha}$ is a generic over R_α.

Proof It follows from Theorem 3.3 and Lemma 3.5 that

$$\mathcal{S}^{\lambda}_{\kappa,m} \vDash \mathbf{AD}^{+} + \theta_0 = \Theta.$$

Then let $\mathcal{Q} = \mathcal{M}_{\infty}^{\mathcal{S}^{\lambda}_{\kappa,m}}$. It easily follows from the fact that $j^{+}(\mathcal{S}) \vDash$ "$\mathcal{Q}$ is a F_{od}-quasi iterate of $\mathcal{P}$", from Lemma 4.3 and Lemma 4.4 that $\mathcal{Q} = \mathcal{M}_{\infty}(\mathcal{P}, \Psi)$. It then follows that $V \vDash |\mathcal{Q}| < \kappa^{+}$.

To finish let $\pi\colon \mathcal{P} \to \mathcal{Q}$ be the iteration map according to Ψ. We also let T be the tree of the universal $(\Sigma^2_1)^{\mathcal{S}^{\lambda}_{\kappa,m}}$-set, $\nu = ((2^{\kappa})^{+})^{V}$ and μ be a κ-complete normal measure on κ. Working in $V[m]$, let $\sigma\colon R \to (H_{\nu^{+}})^{V}[m]$ be such that R is countable and $\{\Psi, \mathcal{Q}, \pi, T, \mu\} \in \operatorname{rng}(\sigma)$. Let $n \in \omega$ be such that T_n projects onto

$$\big\{ (x, \mathcal{M}) : x \in \mathbb{R}^{V[m]} \wedge \mathcal{M} \trianglelefteq \mathcal{W}^{\lambda,m}(x) \wedge \rho(\mathcal{M}) = x \big\}.$$

Also let $r \in \omega$ be such that T_r projects to the set of (x, y, z) such that x codes a self-wellordered X, y codes an $\mathcal{M} \triangleleft \mathcal{W}^{\lambda,m}(X)$ such that $\rho(\mathcal{M}) = X$ and z is a tree on $\mathcal{M}$ according to the unique iteration strategy of $\mathcal{M}$.

Then let

$$\{\overline{\Psi}, \overline{\mathcal{Q}}, \overline{\pi}, \overline{T}, \overline{\mu}\} = \sigma^{-1}\big(\{\Psi, \mathcal{Q}, \pi, T, \mu\}\big),$$

$\overline{R} = \sigma^{-1}((H_{\nu^{+}})^{V})$ and $\overline{m} = \sigma^{-1}(m)$. We then have that $R = \overline{R}[\overline{m}]$. Then let $J \in R$ be the precipitous ideal on ω_1 induced by $\overline{\mu}$ (see [1, Theorem 22.33]).

Suppose now that no tail of Ψ has branch condensation. It then follows by elementarity of σ that $R \vDash$ "no tail of $\sigma^{-1}(\Psi)$ has branch condensation". Since we already know that in $V[m]$, (R, J) is countable and iterable, to finish, it remains to show that the (R, J) captures Ψ.

Let $\Psi^{R} = \Psi \restriction HC^{R} = \sigma^{-1}(\Psi)$, $\mathcal{Q}^{R} = \sigma^{-1}(\mathcal{Q})$, and $\pi^{R} = \sigma^{-1}(\pi)$. Notice that by the construction of Ψ we have that whenever $\mathcal{R}$ is a Ψ-iterate of $\mathcal{P}$ via $\vec{\mathcal{T}}$ such that the iteration embedding $\pi^{\vec{\mathcal{T}}}$-exists, $\mathcal{M}_{\infty}(\mathcal{R}, \Psi_{\mathcal{R},\vec{\mathcal{T}}}) = \mathcal{Q}$ and letting $\pi_{\mathcal{R},\mathcal{Q}}$ be the iteration map, $\pi = \pi_{\mathcal{R},\mathcal{Q}} \circ \pi^{\vec{\mathcal{T}}}$. We then have that

(1) $R \vDash$"if $\mathcal{R}$ is a Ψ^{R}-iterate of $\mathcal{P}$ via $\vec{\mathcal{T}}$ such that the iteration embedding $\pi^{\vec{\mathcal{T}}}$-exists, then $\mathcal{M}_{\infty}(\mathcal{R}, \Psi^{R}) = \mathcal{Q}^{R}$, and letting $\pi_{\mathcal{R},\mathcal{Q}^{R}}$ be the iteration map, $\pi^{R} = \pi_{\mathcal{R},\mathcal{Q}^{R}} \circ \pi^{\vec{\mathcal{T}}}$".

To show that (R, J) captures Ψ, let $(R_{\alpha}, J_{\alpha}, G_{\alpha}, \pi_{\alpha,\beta} : \alpha < \beta \leq \xi)$ be some iteration of (R, J) of length $\xi + 1$. Let $\vec{\mathcal{T}} \in HC^{R_{\xi}}$ be according to $\pi_{0,\xi}(\Psi^{R})$ with last model $\mathcal{R}$ such that $\pi^{\vec{\mathcal{T}}}$-exists. We need to show that $\mathcal{S}^{\lambda}_{\kappa,m} \vDash$ "$\mathcal{R}$ is Σ^2_1-suitable". By (1), we have that there is $p\colon \mathcal{R} \to \pi_{0,\xi}(\mathcal{Q})$ such that $\pi_{0,\xi}(\pi^{R}) = p \circ \pi^{\vec{\mathcal{T}}}$.

It follows from the construction of J that $\pi_{0,\xi} \restriction \overline{R}$ is actually an iteration of $\overline{R}$ via $\overline{\mu}$ and there is $q\colon \pi_{0,\xi}(\overline{R}) \to (H_{\nu^{+}})^{V}$ such that $\sigma \restriction \overline{R} = q \circ (\pi_{0,\xi} \restriction \overline{R})$. We then have that

$$\pi = \big(q \restriction (\pi_{0,\xi} \restriction \mathcal{Q}^{R})\big) \circ p \circ \pi^{\vec{\mathcal{T}}},$$

implying that, by weak condensation of Ψ, that $\mathcal{S}^{\lambda}_{\kappa,m} \vDash$ "$\mathcal{R}$ is Σ^2_1-suitable". The proof that $\pi_{0,\xi}(\Psi^{R})$ has weak branch condensation is very similar, and we omit it.

It remains to show that iterations according to $\pi_{0,\xi}(\Psi^R)$ are correctly guided. We do this only for normal trees, as the general case is only notationally more complicated. To show this, we first consider the case of trees that do not have fatal drops. Notice that if $\mathcal{T} \in HC^{V[m]}$ is a correctly guided tree[16] that is according to Ψ and letting $b = \Psi(\mathcal{T})$, then $\mathcal{Q}(b, \mathcal{T})$-exists whenever $x, y \in \mathbb{R}^{V[m]}$ are such that x codes $\mathcal{M}(\mathcal{T})$ and y codes $\mathcal{Q}(b, \mathcal{T})$ then $(x, y) \in p[T_n]$. We then have that

(2) if $\mathcal{T} \in HC^R$ is according to Ψ^R, is correctly guided, and, letting $b = \Psi^R(\mathcal{T})$, $\mathcal{Q}(b, \mathcal{T})$-exists, then whenever $x, y \in \mathbb{R}^R$ are such that x codes $\mathcal{M}(\mathcal{T})$ and y codes $\mathcal{Q}(b, \mathcal{T})$, $(x, y) \in p[\overline{T}_n]$.

Now let $\mathcal{T} \in HC^{R_\xi}$ be according to $\pi_{0,\xi}(\Psi^R)$ and such that it is correctly guided and if $b = \pi_{0,\xi}(\Psi^R)(\mathcal{T})$, then $\mathcal{Q}(b, \mathcal{T})$-exists. Let $x, y \in \mathbb{R}^{R_\xi}$ be such that x codes $\mathcal{M}(\mathcal{T})$ and y codes $\mathcal{Q}(b, \mathcal{T})$. By (2) we have that $(x, y) \in p[\pi_{0,\xi}(\overline{T}_n)]$. Keeping the above notation, we have that $(x, y) \in p[l \circ \pi_{0,\xi}(\overline{T}_n)] = p[T_n]$ implying that $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{W}^{\lambda,m}(\mathcal{M}(\mathcal{T}))$.

Lastly, we need to take care of trees with fatal drops. Notice that if $\mathcal{T} \in HC^{V[m]}$ is a tree that has a fatal drop at (α, η) then letting $\mathcal{U}$ be the tail of $\mathcal{T}$ after stage α on $\mathcal{O}_\eta^{\mathcal{M}_\alpha^\mathcal{T}}$ and letting $\mathcal{M} \trianglelefteq \mathcal{O}_\eta^{\mathcal{M}_\alpha^\mathcal{T}}$ be the least such that $\rho(\mathcal{M}) = \eta$ and $\mathcal{U}$ is a tree on $\mathcal{M}$ above η then whenever $x, y, z \in \mathbb{R}^{V[m]}$ are such that x codes $\mathcal{M}_\alpha^\mathcal{T}|\eta$, y codes $\mathcal{M}$ and z codes $\mathcal{U}$, we have $(x, y, z) \in p[T_r]$. It then follows that

(3) if $\mathcal{T} \in HC^R$ is a tree that has a fatal drop at (α, η) then letting $\mathcal{U}$ be the tail of $\mathcal{T}$ after stage α on $\mathcal{O}_\eta^{\mathcal{M}_\alpha^\mathcal{T}}$ and letting $\mathcal{M} \trianglelefteq \mathcal{O}_\eta^{\mathcal{M}_\alpha^\mathcal{T}}$ be the least such that $\rho(\mathcal{M}) = \eta$ and $\mathcal{U}$ is a tree on $\mathcal{M}$ above η, whenever $x, y, z \in \mathbb{R}^R$ are such that x codes $\mathcal{M}_\alpha^\mathcal{T}|\eta$, y codes $\mathcal{M}$ and z codes $\mathcal{U}$, we have $(x, y, z) \in p[\overline{T}_r]$.

The rest of the proof is just like the proof of the case when $\mathcal{T}$ does not have a fatal drop except that we now use (3) instead of (2). ■

Using Lemma 4.6 we can fix $\vec{\mathcal{T}} \in HC^{V[m]}$ on $\mathcal{P}$ according to Ψ with last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{T}}}$-exists and $\Psi_{\mathcal{Q},\vec{\mathcal{T}}}$ has branch condensation. To finish the proof of Theorem 4.1 we need to show that in $V[m]$, $(\mathcal{Q}, \Psi_{\mathcal{Q},\vec{\mathcal{T}}})$ is a hod pair below ω_1. It would then follow from Lemma 3.3 that in $V[m]$, $L(\Psi_{\mathcal{Q},\vec{\mathcal{T}}}, \mathbb{R}) \vDash \mathbf{AD}^+$. Because in $V[m]$, $\Psi_{\mathcal{Q},\vec{\mathcal{T}}}$ is ω_1-fullness preserving, it follows that $L(\Psi_{\mathcal{Q},\vec{\mathcal{T}}}, \mathbb{R}) \vDash \mathbf{AD}^+ + \Theta > \theta_0$. The following lemma then finishes the proof of Theorem 4.1. Let $\Lambda = \Psi_{\mathcal{Q},\vec{\mathcal{T}}}$.

Lemma 4.7 *$V[m] \vDash (\mathcal{Q}, \Lambda)$ is a hod pair below ω_1.*

Proof Let $\nu < \mu$ be such that letting $l = m \cap \mathrm{Coll}(\omega, \nu)$, $\mathcal{Q} \in HC^{V[l]}$. We claim that

(1) in $V[l]$ there are κ-complementing trees T, S such that in $V[m]$,

$$(p[T])^{V[m]} = \{(x, n, m) : x \in \mathbb{R}, n, m \in x \text{ and } \pi_x(m) \in \Lambda(\pi_x(n))\}.$$

We start our proof of (1) with the following claim.

Claim The fragment of $\Lambda \restriction HC^{V[l]}$ that acts on normal trees is κ-uB in $V[l]$.

[16] Recall that correctly guided trees do not have fatal drops, see the paragraph before [7, Definition 1.11].

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Proof of claim Given $\eta \in (\nu, \kappa)$ we let $l_\eta = m \cap \mathrm{Coll}(\omega, {<}\eta)$. Also, let $\mathcal{P}_\eta \in V[l]$ be the result of generically comparing all $\mathcal{R} \in HC^{V[l_\eta]}$ such that $V[l_\eta] \vDash$ "$\Vdash_{\mathrm{Coll}(\omega,<\kappa)} \mathcal{R}$ is λ-suitable and λ-short tree iterable". Also, let $\mathcal{Q}_\eta$ be the Λ-iterate of $\mathcal{Q}$ obtained by making $H_\eta[l_\eta]$ generically generic for $\mathbb{B}^{\mathcal{Q}_\eta}_{\delta^{\mathcal{Q}_\eta}}$. Let $\pi_\eta \colon \mathcal{Q} \to \mathcal{P}_\eta$ and $\sigma_\eta \colon \mathcal{Q} \to \mathcal{Q}_\eta$ be the iteration embeddings. In $V[g]$, let

$$U = \{(x, y) \in \mathbb{R}^2 : \mathcal{S} \vDash \text{“}x \text{ codes } a \text{ and } y \text{ codes a sound } a\text{-mouse projecting to } a\text{”}\},$$

$$Z = \{(x, y, z) \in \mathbb{R}^3 : \mathcal{S} \vDash \text{“}x \text{ codes } a,\ y \text{ codes a sound } a\text{-mouse } \mathcal{M} \text{ projecting to } a, \text{ and } z \text{ codes a normal tree according to the unique strategy of } \mathcal{M}\text{”}\}.$$

Then we have that $U, Z \in \Gamma^*$.

Suppose now that $\mathcal{T} \in HC^{V[m]}$ is a stack on $\mathcal{Q}$. We then have that $\mathcal{T}$ is according to Λ if and only if for any $\eta \in (\nu, \kappa)$ such that $\mathcal{T} \in HC^{V[l_\eta]}$:

(i) $\mathcal{T}$ does not have a fatal drop and for any limit $\alpha < lh(\mathcal{T})$ letting b be the branch of $\mathcal{T} \restriction \alpha$ the following holds:

  (a) $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ exists if and only if whenever $n \subseteq \mathrm{Coll}(\omega, \delta^{\mathcal{Q}_\eta})$ is $\mathcal{Q}_\eta[\mathcal{T}]$-generic and $x \in \mathcal{Q}_\eta[\mathcal{T}][n]$ is a real coding $\mathcal{M}(\mathcal{T} \restriction \alpha)$, there is $y \in \mathcal{Q}_\eta[\mathcal{T}][n]$ such that $(x, y) \in \pi_\eta(\tau^{\mathcal{Q}}_U)$ and if $\mathcal{M}$ is the mouse coded by y, then $\mathrm{rud}(\mathcal{M}) \vDash$ "$\delta(\mathcal{T} \restriction \alpha)$ is not Woodin".

  (b) $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ does not exist if and only if there is $\sigma \colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{P}_\eta$ such that $\pi_\eta = \sigma \circ \pi^{\mathcal{T}}_{0,\alpha}$.

(ii) $\mathcal{T}$ has a fatal drop at (α, β) and whenever $n \subseteq \mathrm{Coll}(\omega, \delta^{\mathcal{Q}_\eta})$ is $\mathcal{Q}_\eta$-generic, $x \in \mathcal{Q}_\eta[T][n]$ is a real coding $\mathcal{M}^{\mathcal{T}}_\alpha | \beta$ and $y \in \mathcal{Q}_\eta[T][n]$ is a real coding $\mathcal{O}^{\mathcal{M}^{\mathcal{T}}_\alpha}_\beta$, there is $z \in \mathcal{Q}_\eta[T][n]$ such that z codes the part of $\mathcal{T}$ after stage α and $(x, y, z) \in \pi_\eta(\tau^{\mathcal{Q}}_Z)$.

It is not hard to see that if we let ϕ be the formula expressed by the clauses above, then club many hulls of $(H^{V[l]}_\kappa, \Lambda \restriction H^{V[l]}_\kappa, \in)$ are generically correct about $\Lambda \restriction HC^{V[l]}$ and hence, about ϕ. More precisely, in $V[l]$, there is a club of $X \in H^{V[l]}_\kappa$ such that letting $\pi \colon N \to X$ be the transitive collapse of X, then whenever $(n, \mathcal{T})$ are such that n is generic over N and $\mathcal{T} \in N[n]$ is a tree on $\mathcal{Q}$,

$$\mathcal{Q}[n] \vDash \phi[\mathcal{T}] \text{ if and only if } \phi[\mathcal{T}].$$

The claim now follows from [15, Lemma 4.1]. ∎

To finish the proof of (1) we first notice that the claim holds for $\Lambda \restriction HC^{V[l]}$. Let then (T, S) be the κ-complementing trees such that in $V[m]$, $p[T] = \{x : x \text{ codes } \mathcal{T} \text{ on } \mathcal{Q} \text{ such that } \phi[\mathcal{T}]\}$ (see [15, Lemma 4.1]). The proof of the claim then shows that in $V[m]$, $p[T] = \{x : x \text{ codes a tree } \mathcal{T} \text{ according to } \Lambda\}$. It is now easy to modify (T, S) so that they satisfy (1). ∎

5 On the Strength of the Failure of UBH for Tame Trees

In this section, we present the proof of our Main Theorem. For the rest of this section we assume that there is a proper class of strong cardinals. We start by introducing *tame trees*. Recall that we say κ reflects the set of strong cardinals (or κ is a strong reflecting strongs) if for every λ there is an embedding $j\colon V \to M$ witnessing that κ is λ-strong and for any cardinal $\mu \in [\kappa, \lambda)$, $V \vDash$ "μ is strong" if and only if $M \vDash$ "μ is strong".

Definition 5.1 (Tame iteration tree) A normal iteration tree $\mathcal{T}$ on V is tame if for all $\alpha < \beta < lh(\mathcal{T})$ such that $\alpha = pred_T(\beta + 1)$, we have $\mathcal{M}^{\mathcal{T}}_{\alpha} \vDash$ "$\exists\kappa < \lambda < \mathrm{cp}(E^{\mathcal{T}}_{\beta})$ such that λ is a strong cardinal and κ is strong reflecting strongs".

While our proof will not need the assumption that κ is strong reflecting strongs, we defined tame trees in this particular way because we believe tame failures of UBH give inner models of $\mathbf{AD}_{\mathbb{R}}$ + "Θ is regular". The full proof of this claim will appear in a future publication.

Towards a contradiction, we assume that there is a tame iteration tree $\mathcal{T}$ on V with two cofinal well-founded branches b and c, and the conclusion of the Main Theorem fails. Let $M_b = \mathcal{M}^{\mathcal{T}}_b$, $M_c = \mathcal{M}^{\mathcal{T}}_c$, $M = \mathcal{M}(\mathcal{T})$, $\delta = \delta(\mathcal{T})$, $\delta^+_b = (\delta^+)^{M_b}$, $\delta^+_c = (\delta^+)^{M_c}$, $\pi_b = \pi^{\mathcal{T}}_b$, and $\pi_c = \pi^{\mathcal{T}}_c$. Finally, let $\kappa_0 < \kappa_1 < \kappa_2$ be such that:

- κ_0 is the first strong reflecting strongs in V;
- κ_1 is the first strong above κ_0 in V;
- since $\mathcal{T}$ is tame, we have that all the extenders used in $\mathcal{T}$ have critical point $> \kappa_1$. Hence we can choose an inaccessible $\kappa_2 > \kappa_1$ and κ_2 is below the critical point of any extender used in $\mathcal{T}$.

Suppose $g \subseteq \mathrm{Coll}(\omega, {<}\kappa_1)$ is V-generic. To make the notation as transparent as possible, we will confuse our iteration embeddings that act on V with their extensions that act on $V[g]$. Thus, for instance, $\pi_b\colon V[g] \to M_b[g]$ etc. Working in $V[g]$, fix a hod pair $(\mathcal{P}, \Sigma) \in V[g]$ below κ_1 such that $\mathcal{P} \in HC^{V[g]}$ and $\lambda^{\mathcal{P}} = 0$.

Lemma 5.2 (Key Lemma) *For every hod pair $(\mathcal{P}, \Sigma)$ below κ_1 such that $\lambda^{\mathcal{P}} = 0$, $\mathrm{Proj}(\kappa_1, \kappa_2, \Sigma)$ holds.*

Given the Key Lemma we can easily get a contradiction by using Theorem 4.1 (applied with κ_0 in place of μ, κ_1 in place of κ and κ_2 in place of λ). It is then enough to show that the Key Lemma holds which is what we will do in the next few lemmas. Towards the proof of the Key Lemma, we fix a hod pair $(\mathcal{P}, \Sigma)$ below κ_1. Since clause (b) of $\mathrm{Proj}(\kappa_1, \kappa_2, \Sigma)$ follows from Lemma 2.5, we will only establish clause (a).

We will only verify clause (a) of $\mathrm{Proj}(\kappa_1, \kappa_2, \Sigma)$ for Σ-cmi operators defined according to Definition 3.1(i) as those defined according to Definition 3.1(ii) can be handled in a very similar manner. Let us then fix such a Σ-cmi operator F. Notice that it follows from Lemma 3.5 that for every ξ, both in $M_b[g]$ and in $M_c[g]$, F is ξ-extendable. We then let F_b and F_c be the two Ord-extensions of F in $M_b[g]$ and $M_c[g]$ respectively.

We say F can be *lifted* if for any $x \in H^{M_b[g]}_{\delta^+} \cap H^{M_c[g]}_{\delta^+}$, we have $F_b(x) = F_c(x)$, and $(Lp^{F_b}(x))^{M_b[g]}$ is compatible with $(Lp^{F_c}(x))^{M_c[g]}$ (*i.e.*, one is an initial segment of the other).

We first present a simple lemma which illustrates some of the key ideas that we will use.

Lemma 5.3 *Suppose $x, \mathcal{M} \in M_b \cap M_c$ are such that $\mathcal{M}$ is a sound x-premouse such that $\rho(\mathcal{M}) = x$. Then $\mathcal{M} \trianglelefteq Lp^{M_b}(x)$ if and only if $\mathcal{M} \trianglelefteq Lp^{M_c}(x)$.*

Proof Suppose $\mathcal{N}$ is a countable hull of $\mathcal{M}$ in V. Then by an absoluteness argument using that HC is in both M_b and M_c, $\mathcal{N}$ is a countable hull of $\mathcal{M}$ both in M_b and M_c. Hence, the claim follows. ■

Unfortunately, the lemma does not immediately generalize to F-mice, since the absoluteness used in the proof is not in general true. Fixing an $\mathcal{N}$ as in the proof that is a countable submodel of $\mathcal{M} \trianglelefteq (Lp^{F_b}(x))^{M_b}$, it is still true that $\mathcal{N}$ can be realized as a countable hull of $(Lp^{F_b}(x))^{M_b}$ and $(Lp^{F_c}(x))^{M_c}$ in M_b and M_c via certain embeddings $j_b\colon \mathcal{N} \to \mathcal{M}$ and $j_c\colon \mathcal{N} \to \mathcal{M}$ in M_b and M_c respectively. However, it is not clear, in the case F is an iteration strategy, that $F_b^{j_b}$ and $F_c^{j_c}$ (*i.e.*, the pullbacks of F_b and F_c) are the same strategies. The following lemma in fact shows that they have to be the same.

Lemma 5.4 *F can be lifted.*

Proof We already know that F can be extended to F_b and F_c. It remains to show that whenever $x \in M_b \cap M_c$, $F_b(x) = F_c(x)$ and $(Lp^{F_b}(x))^{M_b}$[17] and $(Lp^{F_c}(x))^{M_c}$ are compatible. We show the second clause as the first is only a special case. Assume towards a contradiction that $(Lp^{F_b}(x))^{M_b}$ and $(Lp^{F_c}(x))^{M_c}$ are not compatible. Let $\mathcal{S}_b = (Lp^{F_b}(x))^{M_b}$ and $\mathcal{S}_c = (Lp^{F_c}(x))^{M_c}$. Fix an elementary $\sigma\colon W \to V_\epsilon[g]$ for some very large ϵ[18] such that W is countable in $V[g]$, $(\mathcal{T}, b, c, \mathcal{P}, F, x) \in \operatorname{rng}(\sigma)$ and if $(\mathcal{U}, d, e, \mathcal{Q}, G, y) = \sigma^{-1}(\mathcal{T}, b, c, \mathcal{P}, F, x)$, then $\sigma[lh(\mathcal{U})]$ is cofinal in $lh(\mathcal{T})$. Let $\eta = |\mathcal{P}|^{M[g]} = |\mathcal{P}|^{V[g]}$. Note that $\eta < \kappa_1$ by the definition of $\mathcal{P}$. By our choice of κ_1, $\operatorname{cp}(\pi_b) > \eta$ and $\operatorname{cp}(\pi_c) > \eta$. Since $\eta \in \operatorname{rng}(\sigma)$, let $\nu = \sigma^{-1}(\eta)$. Also, let $M_d = \mathcal{M}_d^{\mathcal{U}}$, $M_e = \mathcal{M}_e^{\mathcal{U}}$, and $(G_d, G_e) = \sigma^{-1}(F_b, F_c)$. We now have that in W, $(Lp^{G_d}(y))^{M_d}$ is not compatible with $(Lp^{G_e}(y))^{M_e}$.

Let $\sigma_\xi\colon \mathcal{M}_\xi^{\mathcal{U}} \to \mathcal{M}_\xi^{\sigma\mathcal{U}}$ be the copy maps. We have that $\sigma_\beta \in \mathcal{M}_0^{\sigma\mathcal{U}} = V[g]$ and there are $m\colon M_d \to \mathcal{M}_0^{\sigma\mathcal{U}}$ and $n\colon \mathcal{M}_e \to \mathcal{M}_0^{\sigma\mathcal{U}}$ such that $\sigma_0 = m \circ \pi_{\beta,d}^{\mathcal{U}}$ and $\sigma_0 = n \circ \pi_{\beta,e}^{\mathcal{U}}$. Let $H = \sigma_\beta(G^*) \in \mathcal{M}_\beta^{\sigma\mathcal{U}}$, where

$$G^* = \sigma^{-1}\big((\pi_b)^{-1}(F)\big) = \sigma^{-1}\big((\pi_c)^{-1}(F)\big) \in \mathcal{M}_0^{\mathcal{U}}.$$

Let $\mathcal{R}_d = (Lp^{G_d}(y))^{M_d}$ and $\mathcal{R}_e = (Lp^{G_e}(y))^{M_e}$. Finally, let $\mathcal{W}_d = m(\mathcal{R}_d)$ and $\mathcal{W}_e = n(\mathcal{R}_e)$. Notice that $\sigma_\beta \restriction \mathcal{Q} = m \restriction \mathcal{Q} = n \restriction \mathcal{Q}$ and $m(G_d), n(G_e)$ both extend H. But now, in $V[g] = \mathcal{M}_0^{\sigma\mathcal{U}}$, $\mathcal{R}_d$ and $\mathcal{R}_e$ can be compared as the both are G^+-iterable where G^+ is σ_0-pullback of H. ■

Next, we show that $M[g] \vDash$ "$\mathcal{M}_1^{\#,F}$ exists and is $<\delta$-iterable". This will complete the proof of the Key Lemma. Suppose not. Without loss of generality, assume that $\delta_b^+ \leq \delta_c^+$. By our assumption, in $M[g]$, the F-closed core model K^F derived from a

[17]This means that whenever $\pi\colon (\mathcal{N}, \mathcal{P}^*, x^*) \to (\mathcal{M}, \mathcal{P}, x)$ is such that $\mathcal{M} \triangleleft Lp^{F_b}(x)$ and $\mathcal{N}$ is countable transitive, then $\mathcal{N}$ has a unique $\omega_1 + 1$ Λ-strategy where Λ is such that whenever $\mathcal{R}$ is an iterate of $\mathcal{N}$ and $\mathcal{U} \in \mathcal{N}$ is a tree on $\mathcal{P}^*$ according to Λ then $\Lambda(\mathcal{U}) = F(\pi\mathcal{U}) \in \mathcal{R}$.

[18]We will confuse this $V_\epsilon[g]$ with $V[g]$ during the proof.

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$K^{c,F}$ that is constructed (up to δ) using extenders with critical point $> \kappa_2$ exists and is 1-small.[19] The following claim then gives us a contradiction.

Claim $Lp^{F_b}(K^F) \models \delta$ is Woodin.

Proof Recall that we assume $(\delta^+)^{M_b} \leq (\delta^+)^{M_c}$ and this along with the proof of [16, Claim 4] in turns imply that $Lp^{F_b}(K^F) \trianglelefteq Lp^{F_c}(K^F)$ and hence $Lp^{F_b}(K^F) \in M_b \cap M_c$. By the proofs of [16, Theorem 4.1] and [3, Theorem 2.2], $Lp^{F_b}(K^F) \models \delta$ is Woodin. ■

The claim together with the fact that there is a proper class of strong cardinals (in $M[g]$) imply that $M[g] \models$ "$\mathcal{M}_1^{\sharp,F}$ exists and is $<\delta$-iterable." By the agreement between V and M, we have that $V[g] \models$ "$\mathcal{M}_1^{\sharp,F}$ exists and is $<\kappa_2$-iterable." This finishes the proof of the Main Theorem.

6 On the Strength of ¬UBH without Strongs

It is possible to prove a similar lower bound for ¬UBH by somewhat strengthening the hypothesis yet dropping the assumption that there are proper class of strong cardinals. In this section, we state the result. Its proof is mostly due to the second author and will appear elsewhere.

Given an iteration tree $\mathcal{T}$ of limit length and $\alpha < lh(\mathcal{T})$, we let $\mathcal{T}_{\geq\alpha}$ be $\mathcal{T}$ starting from α and $\mathcal{T}_{\leq\alpha} = \mathcal{T} \restriction \alpha + 1$. Similarly, we define $\mathcal{T}_{<\alpha}$ and $\mathcal{T}_{>\alpha}$.

Theorem 6.1 *Suppose $\mathcal{T}$ is a normal tree on V with two well-founded branches b and c such that if $\alpha = \sup(b \cap c)$, then $\delta(\mathcal{T}) \in \operatorname{rng}(\pi^{\mathcal{T}}_{\alpha,b}) \cap \operatorname{rng}(\pi^{\mathcal{T}}_{\alpha,c})$ and $\mathcal{T}_{\geq\alpha} \in M^{\mathcal{T}}_\alpha$. Then in some homogenous extension of V there is a transitive model M such that $\mathbb{R}, \mathrm{Ord} \subseteq M$ and $M \models$ "$\mathbf{AD}^+ + \theta_0 < \Theta$". In particular, there is a non-tame mouse.*

The hypothesis of Theorem 6.1 includes, among other trees, alternating chains.

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References

[1] T. Jech, *Set theory*. The third millennium ed., revised and expanded, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[2] R. O. Ketchersid, *Toward* $\mathbf{AD}_{\mathbb{R}}$ *from the continuum hypothesis and an* ω_1*-dense ideal*. Ph.D. Thesis, University of California, Berkeley, ProQuest LLC, Ann Arbor, MI, 2000.

[19] Because we are assuming that there are proper class of strong cardinals, if such a $K^{c,F}$ construction reaches a Woodin cardinal, then it also reaches $\mathcal{M}_1^{\#,F}$. If such a $K^{c,F}$ construction reaches $\mathcal{M}_1^{\#,F}$, then it must be κ_2-iterable as countable submodels of such a $K^{c,F}$ are κ_2-iterable.

[3] D. A. Martin and J. R. Steel, *Iteration trees*. J. Amer. Math. Soc. **7**(1994), no. 1, 1–73. http://dx.doi.org/10.1090/S0894-0347-1994-1224594-7

[4] W. J. Mitchell and J. R. Steel, *Fine structure and iteration trees*. Lecture Notes in Logic, 3, Springer-Verlag, Berlin, 1994.

[5] I. Neeman, *Inner models in the region of a Woodin limit of Woodin cardinals*. Ann. Pure Appl. Logic **116**(2002), no. 1–3, 67–155. http://dx.doi.org/10.1016/S0168-0072(01)00103-8

[6] G. Sargsyan, *Descriptive inner model theory*. Bull. Symbolic Logic, to appear. http://math.rutgers.edu/~gs481. http://dx.doi.org/10.2178/bsl.1901010

[7] ———, *On the strength of* **PFA** *I*. http://math.rutgers.edu/~gs481.

[8] ———, *A tale of hybrid mice.* http://math.rutgers.edu/~gs481/.

[9] R. Schindler and J. R. Steel, *The core model induction*. http://math.berkeley.edu/~steel.

[10] F. Schlutzenberg and N. Trang, *Scales in* $Lp^{\Sigma}(\mathbb{R})$. http://math.cmu.edu/~namtrang.

[11] J. R. Steel, **PFA** *implies* $\mathbf{AD}^{L(\mathbb{R})}$. J. Symbolic Logic **70**(2005), no. 4, 1255–1296. http://dx.doi.org/10.2178/jsl/1129642125

[12] ———, *Derived models associated to mice*. In: Computational prospects of infinity. Part I. Tutorials, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 14, World Sci. Publ., Hackensack, NJ, 2008, pp. 105–193. http://dx.doi.org/10.1142/9789812794055_0003

[13] ———, *Scales in* $\mathbf{K}(\mathbb{R})$. In: Games, scales, and Suslin cardinals. The Cabal Seminar. Vol. I, Lect. Notes Log., 31, Assoc. Symbol. Logic, Chicago, IL, 2008, pp. 176–208. http://dx.doi.org/10.1017/CBO9780511546488.011

[14] ———, *Scales in* $\mathbf{K}(\mathbb{R})$ *at the end of a weak gap*. J. Symbolic Logic, **73**(2008), no. 2, 369–390. http://dx.doi.org/10.2178/jsl/1208359049

[15] ———, *The derived model theorem*. In: Logic Colloquium 2006, Lect. Notes Log., Assoc. Symbol. Logic, Chicago, IL, 2009, pp. 280–327. http://dx.doi.org/10.1017/CBO9780511605321.014

[16] ———, *Core models with more Woodin cardinals*. J. Symbolic Logic, **67**(2002), no. 3, 1197–1226. http://dx.doi.org/10.2178/jsl/1190150159

[17] W. H. Woodin, *Suitable extender models I*. J. Math. Log. **10**(2010), no. 1–2, 101–339. http://dx.doi.org/10.1142/S021906131000095X

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