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On the Drinfeld discriminant function

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Abstract. The discriminant function Δ is a certain rigid analytic modular form defined on Drinfeld's upper half-plane Ω. Its absolute value $|\Delta|$ may be considered as a function on the associated Bruhat–Tits tree $\mathcal{T}$. We compare $\log|\Delta|$ with the conditionally convergent complex-valued Eisenstein series E defined on $\mathcal{T}$ and thereby obtain results about the growth of $|\Delta|$ and of some related modular forms. We further determine to what extent roots may be extracted of $\Delta(z)/\Delta(nz)$, regarded as a holomorphic function on Ω. In some cases, this enables us to calculate cuspidal divisor class groups of modular curves.

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Notations

We will throughout use the following notation:

$\mathbb{F}_q$ = finite field of characteristic p with q elements
A = $\mathbb{F}_q[T]$ polynomial ring in an indeterminate T
K = $\mathbb{F}_q(T)$ rational function field
K_∞ = $\mathbb{F}_q((\pi))$ completion of K at the infinite place ($\pi := T^{-1}$)
v_∞ = normalized valuation of K_∞
O_∞ = $\mathbb{F}_q[[\pi]]$ integers in K_∞
C = completed algebraic closure of K_∞
$|\,.\,|$ = normalized absolute value on K_∞, extended to C
$|\,.\,|_i$ = 'imaginary part': $C \to \mathbb{R}$, $|z|_i = \inf_{x\in K_\infty}|z-x|$
G = group scheme GL(2)
B = Borel subgroup of upper triangular matrices in G
Z = scalar matrices in G
$\mathcal{K}$ = $G(O_\infty) = \mathrm{GL}(2, O_\infty)$
$\mathcal{I}$ = $\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathcal{K} \mid c \equiv 0 \mod \pi\}$ Iwahori subgroup
Γ = $G(A) = \mathrm{GL}(2, A)$
Ω = $\mathbb{P}^1(C) - \mathbb{P}^1(K_\infty) = C - K_\infty$ Drinfeld upper half-plane
$\mathcal{T}$ = Bruhat–Tits tree of $\mathrm{PGL}(2, K_\infty)$

For any graph $\mathcal{S}$, we let $X(\mathcal{S})$ be its set of vertices, of oriented edges, respectively. For $e \in Y(\mathcal{S})$, $o(e)$, $t(e) \in X(\mathcal{S})$ and $\overline{e} \in Y(\mathcal{S})$ denote its origin, terminus, and inversely oriented edge, respectively. We write 'log' for the logarithm to base q.

0. Introduction

We let $\Gamma = \mathrm{GL}(2, A)$ be the modular group over the rational function field $K = \mathbb{F}_q(T)$ with ring of integers $A = \mathbb{F}_q[T]$. The group Γ acts on Drinfeld's upper half-plane Ω, and the quotient $\Gamma \setminus \Omega$ is canonically identified with the affine line over C = completed algebraic closure of $K_\infty = \mathbb{F}_q((1/T))$.

The isomorphism is given by a j-invariant (the invariant of rank-two Drinfeld A-modules) $j = \frac{g^{q+1}}{\Delta}$, where g and Δ are modular forms on Ω of respective weights $q-1$ and q^2-1. They share a number of properties with their counterparts g_2, g_3 and Δ, respectively, in the classical modular theory: relations with Eisenstein series [13], product formulas [4], expansions around 'infinity' [7]. Further, the C-algebra of modular forms for Γ is generated by g and Δ (or by g and the canonical $(q-1)$th root h of Δ, if a 'nebentype' is admitted).

Surprisingly, not much is known so far about the behavior of the absolute values of Δ, g, h, j, considered as real-valued functions on Ω. Our aim in the present paper is, among others, to fill this gap.

The first main result is Theorem 2.13, where we give a formula for $|\Delta|$ as a function $\Omega \to \mathbb{R}$ that actually factors over the Bruhat–Tits tree $\mathcal{T}$ attached to Ω. Corresponding expressions are given for $|j|$ and $|g|$ (Thm 2.17, Cor. 2.18). The next topic is the (related) question to what extent the functions Δ/Δ_n (where $\Delta_n(z) = \Delta(nz)$, $n \in A$) admit roots in (a) the function field of the modular curve $X_0(n)$ of Hecke type associated with the congruence subgroup $\Gamma_0(n)$ of Γ; (b) the group $\mathcal{O}(\Omega)^*$ of invertible holomorphic functions on Ω. The answer is given in Theorem 3.16 and its corollaries. We further determine the character ω_n of $\Gamma_0(n)$ through which $\Gamma_0(n)$ acts on the 'maximal root' D_n of Δ/Δ_n (Thm 3.20 + Cor. 3.21). These questions are connected with the structure of the cuspidal divisor class group of $X_0(n)$, as is demonstrated in the concluding Examples 3.23 and 3.25.

Our main technical tools are the 'logarithmic derivative' mapping $r\colon \mathcal{O}(\Omega)^* \to \underline{H}(\mathcal{T}, \mathbb{Z})$ (see (1.10)) and Fourier analysis on the tree $\Gamma_\infty \setminus \mathcal{T}$, where $\Gamma_\infty = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Gamma\}$. We compare $r(\Delta)$ with the improper Eisenstein series E on $\mathcal{T}$ defined by complex analytic means [8]. Since $r(\Delta)$ and E agree up to a constant (Cor. (2.8)), we can derive properties of Δ from the properties of E shown in [8]. This way there results e.g. an upper bound for $\max\{i \mid \Delta/\Delta_n$ has an ith root$\}$. We then verify it is sharp by constructing such a root through modular forms.

1. Drinfeld modular forms, logarithmic derivatives, Fourier coefficients

A C-valued function f on the 'upper half-plane' $\Omega = C - K_\infty$ is a modular form of *weight* $k \in \mathbb{N}_0$ and *type* $m \in \mathbb{Z}/(q-1)$ for $\Gamma = \mathrm{GL}(2, A)$ if

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$$\begin{aligned}&\text{(i)}\quad f\left(\frac{az+b}{cz+d}\right)=(cz+d)^k(\det\,\gamma)^{-m}f(z),\\ &\qquad =\gamma\begin{pmatrix}a\,b\\ c\,d\end{pmatrix}\in\Gamma,\ z\in\Omega;\end{aligned}\tag{1.1}$$

(ii) f is holomorphic (in the rigid analytic sense);

(iii) f is holomorphic at infinity.

The description of the analytic structure on Ω is given e.g. in [1, 2, 9, 13]; the meaning of condition (iii) is explained in [7] 5.7. Similarly, we define modular forms for subgroups $\Gamma' \subset \Gamma$ of finite index. Besides the *Eisenstein series*

$$E^{(k)}(z)=\sum_{a,b\in A}{}'(az+b)^{-k}\quad(0<k\equiv 0\mod q-1),$$

which are modular of weight k and type 0 [13], there are three distinguished modular forms $g,\ h,\ \Delta$ which, among others, enjoy the following properties (see [7] for a systematic presentation; here we use the normalization $g=g_{\text{old}}$, $\Delta=\Delta_{\text{old}}$ of *loc. cit.* p. 683, which involves a slight change of constants in some formulas).

Let $M_{k,m}$ be the C-vector space of forms of weight k and type m. Then

$$\begin{aligned}&g=(T^q-T)E^{(q-1)}\in M_{q-1,0}\\ &\Delta=(T^{q^2}-T)E^{(q^2-1)}+(T^{q^2}-T^q)E^{(q-1)^{q+1}}\in M_{q^2-1,0}\\ &\qquad\text{vanishes nowhere on }\Omega\\ &h=g'-\frac{\Delta'}{\Delta}g\in M_{q+1,1}\quad\left(f'=\overline{\pi}\frac{\mathrm{d}f}{\mathrm{d}z}\right)\\ &h^{q-1}=-\Delta\\ &\bigoplus_{k\geqslant 0}M_{k,0}=C[g,\Delta]\quad\text{(David Goss [14])}\\ &\bigoplus_{k\geqslant 0,\ m\in\mathbb{Z}/(q-1)}M_{k,m}=C[g,h].\end{aligned}\tag{1.2}$$

Here $\overline{\pi}\in C$ is some constant analogous with $2\pi i$, with logarithmic absolute value $\log|\overline{\pi}|=\frac{q}{q-1}$. Note that $f\mapsto f'$ is $\overline{\pi}^2$ times the operator θ of [7], which compensates the different normalizations of $g,\ h$, and Δ. These forms naturally appear as the coefficients of Drinfeld modules. Whereas g is similar to the coefficient forms $g_2,\ g_3$ in the theory of elliptic curves, the *Drinfeld discriminant* Δ shares many of the properties of the classical discriminant $\Delta(z)=(2\pi i)^{12}\prod(1-\mathrm{e}^{2\pi i n z})^{24}$.

Next, let $\mathcal{T}$ be the *Bruhat–Tits tree* of $\mathrm{PGL}(2,K_\infty)$. It is a $(q+1)$-regular tree with

$$\begin{aligned}X(\mathcal{T})&=G(K_\infty)/\mathcal{K}\cdot\mathcal{Z}(\mathcal{K}_\infty)\quad\text{(vertices)},\\ Y(\mathcal{T})&=G(K_\infty)/\mathcal{I}\cdot\mathcal{Z}(\mathcal{K}_\infty)\quad\text{(orientied edges)},\end{aligned}$$

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where the canonical map from $Y(\mathcal{T})$ to $X(\mathcal{T})$ associates with each edge e its origin $o(e)$. It is easily verified that

$$S_X := \left\{ \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} k \in \mathbb{Z},\, u \in K_\infty, \\ u \mod \pi^k O_\infty \end{array} \right\}$$

is a set of representatives for $X(\mathcal{T})$. We let $v(k,u)$ be the vertex corresponding to $\left(\begin{smallmatrix} \pi^k & u \\ 0 & 1 \end{smallmatrix}\right)$. By an *end* of $\mathcal{T}$, we understand the equivalence class of an infinite path without backtracking, where two paths that differ in a finite number of edges are identified. The set $\partial\mathcal{T}$ of ends of $\mathcal{T}$ is in $1-1$ correspondence with $\mathbb{P}^1(K_\infty) =$ space of lines in $V = K_\infty^2$. We normalize the bijection such that the end $(v(k,0), v(k-1,0), \ldots)$ corresponds to ∞. It defines an orientation on $\mathcal{T}$, i.e., a decomposition $Y(\mathcal{T}) = Y^+(\mathcal{T}) \,\dot{\cup}\, Y^-(\mathcal{T})$ with $\overline{Y^+(\mathcal{T})} = Y^-(\mathcal{T})$. Namely, $e \in Y(\mathcal{T})$ is *positive* ($\Leftrightarrow e \in Y^+(\mathcal{T}) \Leftrightarrow \mathrm{sgn}(e) = +1$) if it points to the end ∞, end *negative* ($\Leftrightarrow e \in Y^-(\mathcal{T}) \Leftrightarrow \mathrm{sgn}(e) = -1$) otherwise. We thus get a section

$$\begin{array}{rcl} X(\mathcal{T}) & \xrightarrow{\cong} & Y^+(\mathcal{T}) \hookrightarrow Y(\mathcal{T}) \\ v & \longmapsto & e \text{ s.t. } o(e) = v,\ \mathrm{sgn}(e) = +1 \end{array}$$

for the 'origin' map from $Y(\mathcal{T})$ to $X(\mathcal{T})$. Since the reflection e $\mapsto \overline{e}$ on $Y(\mathcal{T})$ group-theoretically is given by

$$\text{class of } g \in G(K_\infty) \mapsto \text{ class of } g \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix},$$

each $e \in Y(\mathcal{T})$ is uniquely represented by

$$\begin{array}{lll} \text{either} & \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} & (\text{if } \mathrm{sgn}(e) = +1, \\ & & \quad \text{in this case we put } e =: e(k,u)) \\ \text{or} & \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix} & (\text{if } \mathrm{sgn}(e) = -1) \end{array} \tag{1.4}$$

with $\left(\begin{smallmatrix} \pi^k & u \\ 0 & 1 \end{smallmatrix}\right) \in S_X$. Now each such element of S_X with $u \notin \pi^k O_\infty$ may be written as

$$\begin{pmatrix} \pi^{a+t}, & \pi^t v \\ 0, & 1 \end{pmatrix} \qquad \text{with } t \in \mathbb{Z}, a \in \mathbb{N} \text{ uniquely determined,}$$

and $v \in O_\infty^*$ uniquely determined modulo $\pi^a O_\infty$. We define the functions $\kappa,\ \tau,\ \alpha$ on $Y(\mathcal{T})$ by

$$\begin{array}{ll} \kappa(e) = k,\ \tau(e) = t,\ \alpha(e) = a, & e \text{ or } \overline{e} \text{ equal to } e(k,u),\ u \notin \pi^k O_\infty, \\ \kappa(e) = \tau(e) = k,\ \alpha(e) = 0, & e \text{ or } \overline{e} \text{ equal to } e(k,0). \end{array} \tag{1.5}$$

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By definition, κ, τ and α are invariant under $e \mapsto \overline{e}$, and κ is invariant under the action of the stabilizer

$$\Gamma_\infty = \Gamma \cap B$$

of the end ∞ in Γ. The intuitive meaning is as follows: Let $A(0,\infty)$ be the *principal axis* of $\mathcal{T}$, i.e., the path $(\ldots, e(k+1,0), e(k,0), e(k-1,0), \ldots)$ from the end 0 to the end ∞ of $\mathcal{T}$. Then $\alpha(e)$ is the distance from e to $A(0,\infty)$, $\tau(e)$ describes the vertex next to e on $A(0,\infty)$, and κ decreases by one on each step towards ∞.

We put $\underline{H}(\mathcal{T},\mathbb{Z})$ for the group (and right $G(K_\infty)$-module) of maps $\varphi\colon Y(\mathcal{T}) \to \mathbb{Z}$ that satisfy

$$\begin{aligned} &\text{(i)} \quad \varphi(e) + \varphi(\overline{e}) = 0, \ e \in Y(\mathcal{T}) \quad (\varphi \text{ alternating}) \\ &\text{(ii)} \sum_{\substack{e \in Y(\mathcal{T}) \\ o(e)=v}} \varphi(e) = 0, \quad v \in X(\mathcal{T}) \quad (\varphi \text{ harmonic}). \end{aligned} \tag{1.6}$$

$\underline{H}(\mathcal{T},\mathbb{Z})$ is called the module of integral-valued harmonic cochains or *currents*. Both Ω and $\mathcal{T}$ are analogues of the complex upper half-plane; they are related by a $G(K_\infty)$-equivariant map $\lambda\colon \Omega \to \mathcal{T}(\mathbb{R})$ (= points of the realization of $\mathbb{R}$) that we will briefly describe. Recall [11] that $\mathcal{T}(\mathbb{R})$ may be canonically identified with the set of equivalence classes of norms on the two-dimensional K_∞-vector space K_∞^2. Then $\lambda(z)$ corresponds to the norm ν_z, where $\nu_z((u,v)) := |uz+v|$. The map λ is onto the rational points $\mathcal{T}(\mathbb{Q})$ of $\mathcal{T}$. We have in $\Omega \subset \mathbb{P}^1(C)$:

$$\lambda^{-1}(\text{vertex}) \cong \mathbb{P}^1(C) - (q+1) \text{ disjoint balls},$$
$$\lambda^{-1}\begin{pmatrix}\text{edge minus}\\ \text{end points}\end{pmatrix} \cong \mathbb{P}^1(C) - \text{ two disjoint balls.}$$

For example,

$$\begin{aligned} \lambda^{-1}(v(0,0)) &= \{z \in C \mid |z| \leqslant 1,\ |z-c| \geqslant 1, \\ &\qquad \forall c \in \mathbb{F}_q\} \\ &= \{z \in C \mid |z| = |z|_i = 1\} \quad \text{and} \\ \lambda^{-1}(e(0,0) - \text{ end points}) &= \{z \in C \mid 1 < |z| < q\}. \end{aligned} \tag{1.7}$$

The relationship between the functions $|.|$, $|.|_i$ on C and the functions κ, τ, α on $\mathcal{T}$ is as follows.

1.8 LEMMA. *Let $z \in \Omega$ be such that $\lambda(z) = v \in X(\mathcal{T})$, and let e be the unique positive edge with $o(e) = v$. Then*

$$\log\, |z|_i = -\kappa(e) \quad \text{and} \quad \log\, |z| = -\tau(e).$$

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Proof. Straightforward from (1.7), the $G(K_\infty)$-equivariance of λ, and the formula

$$|\gamma z|_i = |cz+d|^{-2} |\det \gamma| \, |z|_i$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(K_\infty)$. □

1.9 COROLLARY. *For* $z \in \Omega$, *the conditions* (i) $|z| = |z|_i$ *and* (ii) $\lambda(z) \in A(0,\infty)$ *are equivalent.*

Proof. Without restriction, $\lambda(z) \in X(\mathcal{T})$ since both log $|z|$ and log $|z|_i$ factor through λ and are linear on edges of $\mathcal{T}$. Then the assertion is clear from the lemma and (1.5). □

The following construction, due to Marius van der Put, is fundamental for the study of modular forms. Let $\mathcal{O} = \mathcal{O}_\Omega$ be the structure sheaf of the analytic space Ω and $\mathcal{O}(\Omega)^*$ the units of its global sections. Then there is a canonical short exact sequence of $G(K_\infty)$-modules (trivial action on C^*):

$$0 \to C^* \to \mathcal{O}(\Omega)^* \xrightarrow{r} \underline{H}(\mathcal{T},\mathbb{Z}) \to 0. \tag{1.10}$$

It is related with the logarithmic derivative through the commutative diagram

$$\begin{array}{cccc} f & \mathcal{O}(\Omega)^* & \xrightarrow{\;\;r\;\;} & \underline{H}(\mathcal{T},\mathbb{Z}) \\ \downarrow & \downarrow & & \downarrow \text{reduction mod } p \\ f'/f & \mathcal{O}(\Omega) & \xrightarrow{\;\;\text{res}\;\;} & \underline{H}(\mathcal{T},C), \end{array} \tag{1.11}$$

where the lower horizontal map is res: $g(z) \mapsto \text{res } g(z)\, \mathrm{d}z$, $(\text{res } \omega)(e) =$ residue of the differential form ω in the oriented annulus $\lambda^{-1}(e)$. The definition of r is as follows:

$$r(f)(e) = \log \frac{\|f\|_{\lambda^{-1}(t(e))}}{\|f\|_{\lambda^{-1}(o(e))}}, \tag{1.12}$$

where $\|f\|_{\lambda^{-1}(v)}$ denotes the spectral norm of $f \in \mathcal{O}(\Omega)^*$ on $\lambda^{-1}(v)$, i.e., $\sup\{|f(z)| \mid z \in \lambda^{-1}(v)\} = |f(z)|$ $(z \in \lambda^{-1}(v))$ since f is invertible. The fact that r is well-defined (i.e., takes its values in $\underline{H}(\mathcal{T},\mathbb{Z})$) and has the stated properties is proved in [2] and [10]. In particular, we have for $f \in \mathcal{O}(\Omega)^*$, $v_1, v_2 \in X(\mathcal{T})$, $z_1, z_2 \in \Omega$ with $\lambda(z_i) = v_i$:

$$\log \left| \frac{f(z_2)}{f(z_1)} \right| = \int_{v_1}^{v_2} r(f)(e)\, \mathrm{d}e. \tag{1.13}$$

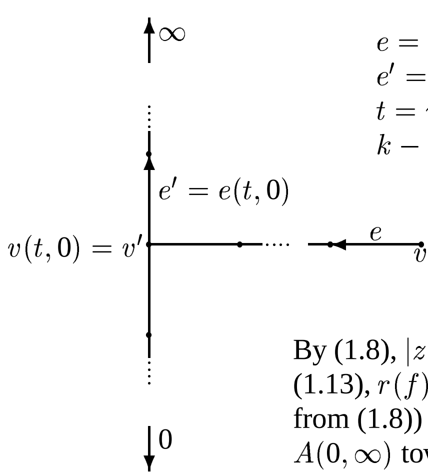
(Recall that 'log'='$\log_q$'. The integral is the sum of $r(f)$ along the unique path from v_1 to v_2.) In view of (1.10) to (1.13), we like to view r as a substitute for the logarithmic derivative map $f \mapsto f'/f$ in the classical theory. The next result calculates $r(f)$ for the most elementary functions $f \in \mathcal{O}(\Omega)^*$.

1.14 PROPOSITION. *Let $a \neq b \in \mathbb{P}^1(K_\infty)$ and $f_{a,b}$ be a rational function on $\mathbb{P}^1(C)$ with a simple zero at a, a simple pole at b and no further zeroes and poles. Let further $A(a,b)$ be the unique path in $\mathcal{T}$ from the end a to the end b, and define $\varphi_{a,b} \in \underline{H}(\mathcal{T},\mathbb{Z})$ by*

$$\varphi_{a,b}(e) = \begin{cases} 1 & e \text{ on } A(a,b) \\ -1 & \overline{e} \text{ on } A(a,b) \\ 1 & \text{otherwise.} \end{cases}$$

Then $r(f_{a,b}) = \varphi_{a,b}$.

Proof. In view of the $G(K_\infty)$-equivariance of r, it suffices to consider the case $(a,b) = (0,\infty)$, i.e., $f(z) = z$ the identity function. Let $z \in \Omega$ be such that $\lambda(z) = v = o(e) \in X(\mathcal{T})$ with some $e \in Y^+(\mathcal{T})$, $e' \in Y^+(\mathcal{T})$ the first edge on the path from v to ∞ lying on the axis $A(0,\infty)$, $v' = o(e')$. Suppose that $e \neq e'$. The picture looks:



By (1.8), $|z|$ is constant on the path from v to v', thus by (1.13), $r(f)(e) = 0$. On the other hand, it is clear (again from (1.8)) that $\log|z|$ increases by one for each step on $A(0,\infty)$ towards ∞, thus $r(f)(e') = +1$. $\square$

Next, we associate Fourier coefficients to Γ_∞-invariant elements φ of $\underline{H}(\mathcal{T},\mathbb{C})$. In view of (1.4) and (1.6)(i), each $\varphi \in \underline{H}(\mathcal{T},\mathbb{C})$ is uniquely determined by its restriction to $Y^+(\mathcal{T}) = B(K_\infty)/(I \cap B(K_\infty))Z(K_\infty)$. We may thus regard $\varphi \in \underline{H}(\mathcal{T},\mathbb{C})^{\Gamma_\infty}$ as a function on $Y^+(\Gamma_\infty \setminus \mathcal{T}) = \Gamma_\infty \setminus Y^+(\mathcal{T})$, the positive edges of the quotient $\Gamma_\infty \setminus \mathcal{T}$ by $\Gamma_\infty = \Gamma \cap B$, and apply the machinery of Fourier analysis. The following is an adaption of [19] Ch. III to our situation. Details are carried out in [8], Sections 2 and 3.

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Let $\mathrm{Div}(K)$ be the multiplicative group of divisors on K and $\mathrm{Div}^+(K) \hookrightarrow \mathrm{Div}(K)$ the monoid of positive divisors. Each $\mathfrak{m} \in \mathrm{Div}(K)$ may uniquely be written as a power ∞^k of the infinite prime ∞ times a finite divisor $\mathfrak{m}_f$ (i.e., $\infty \notin \mathrm{supp}(\mathfrak{m}_f)$). We identify finite positive divisors with ideals of $A = \mathbb{F}_q[T]$. The norm $|\mathfrak{m}|$ of $\mathfrak{m} \in \mathrm{Div}(K)$ is $q^{\deg\,\mathfrak{m}}$. The principal divisor $\mathrm{div}(m)$ of $m \in K^*$ is always understood with its infinity part, so that its degree is zero. For $u = \Sigma u_i \pi^i \in K_\infty$, put $\nu(u) = q - 1$ if $u_1 = 0$ and $\nu(u) = -1$ otherwise. Then we define for each $\varphi \in \underline{H}(\mathcal{T}, \mathbb{C})^{\Gamma_\infty}$ two functions

$$\begin{array}{llll} c_0(\varphi,\cdot): & K_\infty^* & \to \mathbb{C} & \text{(the constant Fourier coefficient of } \varphi) \\ c(\varphi,\cdot): & \mathrm{Div}^+(K) & \to \mathbb{C} & \text{(the nonconstant Fourier coefficient of } \varphi) \end{array}$$

by

$$\begin{aligned} c_0(\varphi, x) &= q^{1-k} \sum_{u \in (\pi)/(\pi^k)} \varphi(e(k,u)) \quad k := v_\infty(x) \geqslant 1 \\ &= \varphi(e(k,0)) \quad k \leqslant 1, \end{aligned} \tag{1.15}$$

$$c(\varphi, \mathfrak{m}) = q^{-1-l} \left[\sum_{\substack{0 \neq u \in (\pi)/(\pi^{2+l}) \\ \text{monic}}} \varphi(e(2+l,u))\nu(-mu) + \varphi(e(2+l,0)) \right]$$

if $\mathfrak{m} = \mathrm{div}(m) \cdot \infty^l$ with some $m \in A$.

The Γ_∞-invariance of φ implies that the summands appearing on the right hand sides only depend on the respective residue classes of $u \in (\pi) = \pi O_\infty$. Some $u \in K_\infty^*$ is monic if its lowest order coefficient in π is one. Again from the Γ_∞-invariance, we could replace the sum over the nonzero monics in $(\pi)/(\pi^{2+l})$ by the sum over the nonzero $u \in [(\pi)/(\pi^{2+l})]/\mathbb{F}_q^*$. The Fourier coefficients satisfy

$$\begin{aligned} &\text{(i)} \quad c_0(\varphi, x) = q^{-v_\infty(x)} c_0(\varphi, 1) = q^{-v_\infty(x)} \varphi(e(0,0)); \\ &\text{(ii)} \quad c(\varphi, \mathfrak{m}_f \cdot \infty^k) = q^{-k} c(\varphi, \mathfrak{m}_f) \quad (k \in \mathbb{N}_0); \\ &\text{(iii)} \quad \varphi(e(k,u)) = c_0(\varphi, \pi^k) \\ &\qquad + \sum_{\substack{m \in A \text{ monic} \\ \deg\, m \leqslant k-2}} c(\varphi, \mathrm{div}(m) \cdot \infty^{k-2}) \nu(mu). \end{aligned} \tag{1.16}$$

Properties (i) and (ii) reflect the harmonicity of φ as a function on $Y^+(\mathcal{T})$, and (iii) is the inversion formula. In (iii) and similar expressions, $c(\varphi, \mathfrak{m}) = 0$ if $\mathfrak{m}$ fails to be positive. Conversely, given functions c_0 and c that satisfy (i) and (ii), the function φ defined by (iii) lies in $\underline{H}(\mathcal{T}, \mathbb{C})^{\Gamma_\infty}$.

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We finally introduce *Hecke operators*. For a function φ on $Y^+(\mathcal{T}) = B(K_\infty)/(\mathcal{I} \cap B(K_\infty))Z(K_\infty)$ and a positive finite divisor $\mathfrak{m}$, we put

$$T_{\mathfrak{m}}\varphi(x) = \sum \varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} x\right) \quad (x \in B(K_\infty)), \tag{1.17}$$

where the sum is over $a, b, d \in A$ such that a, d are monic, $(ad) = \mathfrak{m}$ and deg $b <$ deg d. Then $T_{\mathfrak{m}}\varphi$ is again a function on $Y^+(\mathcal{T})$ (i.e., right $(\mathcal{I} \cap B(K_\infty))Z(K_\infty)$-invariant as a function on $B(K_\infty)$) and even in $\underline{H}(\mathcal{T}, \mathbb{C})^{\Gamma_\infty}$ if φ is. The $T_{\mathfrak{m}}$ have the usual properties, which may be looked up in [19] Ch. VI. We just point out that we can read off from its Fourier coefficients that $\varphi \in \underline{H}(\mathcal{T}, \mathbb{C})^{\Gamma_\infty}$ is an eigenform (*loc. cit.* p. 44).

2. The logarithmic derivative of the discriminant

We now calculate the current $r(\Delta)$ and derive some consequences. The functional equation

$$\Delta(\gamma z) = (cz + d)^{q^2-1}\Delta(z) \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\right)$$

translates to

$$r(\Delta)(\gamma e) = (q^2 - 1)\varphi(e) + r(\Delta)(e),$$

where by (1.13), $\varphi \in \underline{H}(\mathcal{T}, \mathbb{Z})$ equals $\varphi_{-d/c,\infty}$ if $c \neq 0$ and $\varphi = 0$ otherwise. Now

$$\varphi(e) \neq 0 \Leftrightarrow c \neq 0 \quad \text{and} \quad \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}(e) \in A(0, \infty),$$

in which case $\varphi(e) = \operatorname{sgn}(e)$. We therefore define

$$\begin{aligned} S(\gamma, e) &:= \operatorname{sgn}(e), \quad \text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } c \neq 0 \\ &\qquad\qquad \text{and } \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}(e) \in A(0, \infty) \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{2.1}$$

2.2 LEMMA. (i) $S(\gamma, e) = 0 \Leftrightarrow \operatorname{sgn}(e) = \operatorname{sgn}(\gamma e)$.
(ii) $S(\gamma\delta, e) = S(\gamma, \delta e) + S(\delta, e)$, $\gamma, \delta \in \Gamma$.

Proof. (i) [8] (4.5) + (4.6). (ii) Straightforward calculation. □

Using $S(\gamma, e)$, we may thus express the behavior of $r(\Delta)$ under Γ by the functional equation

$$r(\Delta)(\gamma e) = (q^2 - 1)S(\gamma, e) + r(\Delta)(e). \tag{2.3}$$

On the other hand, it is clear that (2.3) characterizes $r(\Delta)$: If φ is another element of $\underline{H}(\mathcal{T}, \mathbb{Z})$ subject to the same transformation rule, the difference $r(\Delta) - \varphi$ is Γ-invariant. But it is well-known that the quotient $\Gamma \setminus \mathcal{T}$ is a half-line

$$\Gamma \setminus \mathcal{T} = \bullet - - - \bullet - - - \bullet - - - \bullet \cdots \tag{2.4}$$

represented by the vertices $v(k, 0)$, $k \leqslant 0$ (e.g. [17] p. 111), and therefore $\underline{H}(\mathcal{T}, \mathbb{Z})^\Gamma = 0$ and $r(\Delta) = \varphi$.

Let now $E \in \underline{H}(\mathcal{T}, \mathbb{C})^{\Gamma_\infty}$ be the current defined through its Fourier coefficients $c_0(.) = c_0(E, .)$, $c(.) = c(E, .)$:

(2.5)

- (i) $c_0(\pi^k) = -\frac{q^2}{q^2-1} q^{-k}$;
- (ii) c_0 is Eulerian ([19] p. 10) at ∞ with Euler factor $(1 - q^{-1}X)^{-1}$;
- (iii) c_0 is Eulerian at finite places $\mathfrak{p}$ of K with Euler factor $(1 - (1 + |\mathfrak{p}|^{-1})X + |\mathfrak{p}|^{-1}X^2)^{-1}$;
- (iv) $c((1)) = 1$.

As can be read off from the Fourier coefficients, E is an eigenform for the Hecke operator $T_\mathfrak{p}$ with eigenvalue $\epsilon_\mathfrak{p} = 1 + |\mathfrak{p}|$. The next result is proved in [8] Theorem 6.1, Corollary 6.2, Proposition 5.8:

2.6 THEOREM. (i) *For each $\gamma \in \Gamma$, E satisfies the functional equation*

$$E(\gamma e) = \frac{q}{q-1} S(\gamma, e) + E(e).$$

(ii)

$$\begin{aligned} E(e(k, 0)) &= -\frac{q^2}{q^2-1} q^{-k} && k \leqslant 1 \\ &= \frac{q^{k+1} - q^2 - q}{q^2 - 1} && k \geqslant 1. \end{aligned}$$

(iii) *The set of values (up to sign) of E on $Y(\mathcal{T})$ is contained in the set of values on $A(0, \infty)$ described by* (ii). *In particular, its values are rational with bounded* *denominator* (q^2-1). □

2.7 *Remark*. E may also be represented, up to a scalar factor, as an improper (= only conditionally convergent) Eisenstein series $\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\varphi(\gamma e)$, where $\varphi(e)=\mathrm{sgn}(e)q^{-\kappa(e)}$, and where the summation has to be taken in a fixed order (*loc. cit.*).

Comparing the constants in the respective functional equations, we have the immediate corollaries.

2.8 COROLLARY.

$$r(\Delta)=\frac{(q^2-1)(q-1)}{q}E$$

and

$$r(h)=\frac{q^2-1}{q}E$$ □

2.9 COROLLARY.

$$\begin{aligned} r(\Delta)(e(k,0)) &= -q^{1-k}(q-1) && k\leqslant 1\\ &= (q^k-q-1)(q-1) && k\geqslant 1 \end{aligned}$$

and this gives (up to sign) all the values of $r(\Delta)$ *on* $Y(\mathcal{T})$. □

2.10 COROLLARY. *Let* r *be the largest number such that there exists an* r*th root of* Δ *in* $\mathcal{O}(\Omega)^*$. *Then* $r=q-1$.

Proof. $\gcd\{\text{values of } r(\Delta)\}=q-1$. □

2.11 COROLLARY. *Let* $\mathfrak{p}=(f)\subset A$ *be a prime,* f *monic. The function* h *(and hence* $\Delta=-h^{q-1}$*) satisfies the functional equation*

$$h(fz)\prod_{\substack{b\in A\\ \deg\, b<\deg\,\mathfrak{p}}} h\left(\frac{z+b}{f}\right)=h^{|\mathfrak{p}|+1}(z).$$

Proof. Let $\tilde{h}$ be the left hand side. Then $r(\tilde{h})=T_{\mathfrak{p}}r(h)=(|\mathfrak{p}|+1)r(h)$, where the first equation is immediate from the definition of Hecke operators, and the second one results from (2.5) and (2.8). Hence $\tilde{h}(z)=\text{const.}\,h^{|\mathfrak{p}|+1}(z)$, and the constant is determined to 1 using the expansion of h around the cusp ∞ ([7] Theorem 9.1). □

2.12 *Remark*. The above formula may be written more suggestively as

$$\frac{h(z)}{h(fz)}=\prod_{\substack{b\in A\\ \deg\, b<\deg\,\mathfrak{p}}}\frac{h(\frac{z+b}{f})}{h(z)},$$

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i.e., as a distribution relation. It is then similar to the distribution relations satisfied by the classical discriminant Δ and related functions (see [15, 16]). Clearly, we can write down relations analogous to (2.11) for not necessarily prime ideals $\mathfrak{m} \subset A$, exploiting the fact that $r(h)$ is an eigenform for $T_{\mathfrak{m}}$.

2.13 THEOREM. *Let $z_k \in \Omega$ be such that $\lambda(z_k) = v(k,0) \in X(\mathcal{T})$. Then $|\Delta(z_k)| = q^{n_k}$ with*

$$\begin{aligned} n_k &= q^2 + q - q^{1-k} && k \leqslant 1 \\ &= q^2 + q + k(q^2-1) - q^{1+k} && k \geqslant 1. \end{aligned}$$

Proof. Let $z_0 \in \Omega$ be any element of $\mathbb{F}_{q^2} - \mathbb{F}_q$. As follows from (1.8), $\lambda(z_0)$ equals the vertex $v(0,0)$. Now $A + Az_0 = \mathbb{F}_{q^2}[T] =: A^{(2)}$, which by the Weierstrass correspondence between lattices and Drinfeld modules corresponds to a rank-one Drinfeld $A^{(2)}$-module Φ. Multiplying $A^{(2)}$ with some constant $\overline{\pi}^{(2)}$ of logarithmic absolute value $\frac{q^2}{q^2-1}$ (cf. [12]) yields the lattice $\overline{\pi}^{(2)}A^{(2)}$ corresponding to the Carlitz $A^{(2)}$-module $\rho^{(2)}$, defined by the operator polynomial $\rho_T^{(2)}(X) = TX + X^{q^2}$ (notations as in [7] Section 4). But $\rho^{(2)}$ may be regarded as a rank-two Drinfeld A-module with complex multiplication, which yields the discriminant

$$\begin{aligned} \Delta(z_0) &= \Delta(A^{(2)}) = (\overline{\pi}^{(2)})^{q^2-1}\Delta(\overline{\pi}^{(2)}A^{(2)}) \\ &= (\overline{\pi}^{(2)})^{q^2-1}\Delta(\rho^{(2)}) = (\overline{\pi}^{(2)})^{q^2-1} \end{aligned}$$

of logarithmic absolute value $\log|\Delta(z_0)| = q^2$. Now the formula comes out by inserting (2.9) into (1.13) and integrating. □

2.14 *Remarks.* (i) Since $r(\Delta)$ is linear on edges, we now know $|\Delta(z)|$ for $z \in \lambda^{-1}(A(0,\infty))$. Referring to [8] 6.5, we may determine $|\Delta(z)|$ for arbitrary $z \in \Omega$, provided the coordinates of $\lambda(z)$ on $\mathcal{T}$ (see (1.5)) are specified.

(ii) The infinite product for $\Delta(z)$ given in [4] doesn't suffice to calculate $|\Delta(z)|$ since it converges only for $|z|_i$ large, i.e., in the relatively uninteresting case where $\lambda(z)$ is 'close to infinity'.

Next, let $j = \frac{g^{q+1}}{\Delta}$ be the Drinfeld j-invariant. It is Γ-invariant and yields an identification $\Gamma \setminus \Omega \xrightarrow{\cong} C$. We have $j(z) = 0 \Leftrightarrow z \in \Gamma(\mathbb{F}_{q^2} - \mathbb{F}_q)$, and all these roots are $q+1$-fold (roots of g are easily verified to be simple: e.g. [7] 5.15). In particular, j is invertible on $\Omega' = \Omega - \lambda^{-1}(\Gamma v(0,0))$. Applying the Definition (1.12) of the map r to j yields some function $r(j) : Y(\mathcal{T}) \to \mathbb{Z}$ which is alternating, Γ-invariant and harmonic (1.6(ii)) at those vertices $v \in X(\mathcal{T})$ which are not Γ-equivalent to $v(0,0)$.

2.15 CLAIM. At $v \in \Gamma v(0,0)$ we have

$$\sum_{\substack{e \in Y(\mathcal{T}) \\ o(e)=v}} r(j)(e) = (q+1)q(q-1),$$

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and therefore $r(j)(e) = q(q-1)$ for such e, since they are all Γ-equivalent. It suffices to verify this for $v = v(0,0)$. In $\lambda^{-1}(v)$, we have the $q(q-1)$ zeroes $z \in \mathbb{F}_{q^2} - \mathbb{F}_q$ of j, each with multiplicity $(q+1)$, and no other zeroes. Then the claim follows from the way $r(j)$ has been constructed, i.e., the residue theorem, see [2] and [10] p. 95. Now recall (2.4) that

$$\Gamma \setminus \mathcal{T} = \overset{v_0}{\bullet} \xrightarrow{e_0} \overset{v_1}{\bullet} \xrightarrow{e_1} \overset{v_2}{\bullet} - \cdots,$$

where v_k is the class of $v(-k,0)$, e_k the class of $e(-k,0)$. Further, for $k \geqslant 0$, the q positive edges of $\mathcal{T}$ meeting $v(-k-1,0)$ and different from $e(-k-1,0)$ are identified under Γ with e_k. Together, this implies

$$r(j)(e_k) = q^{k+1}(q-1). \tag{2.16}$$

Let $z_k \in \Omega$ be as in (2.13), i.e., $\lambda(z_k) = v(k,0)$.

2.17 THEOREM. *For* $0 \neq k \in \mathbb{Z}$ *we have*

$$\log |j(z_k)| = q^{|k|+1}.$$

Proof. Let $z \in \lambda^{-1}(v(0,0)) = \{z \in C \mid |z| = |z|_i = 1\}$. All the terms in $E^{(q-1)}(z) = \sum_{a,b}' \frac{1}{(az+b)^{q-1}}$ are $\leqslant 1$ in absolute value, hence $||E^{(q-1)}||_{\lambda^{-1}(v(0,0))} \leqslant 1$, and the value 1 is attained e.g. for $z \in \mathbb{F}_{q^3} - \mathbb{F}_q$. Consequently, $\log \|g\|_{\lambda^{-1}(v(0,0))} = \log\ |T^q - T| = q$, and from (2.13), $\log\ \|j\|_{\lambda^{-1}(v(0,0))} = q$. Now for $k < 0$,

$$\log\ |j(z_k)| = \int_{v(0,0)}^{v(k,0)} r(j)(e)\ \mathrm{d}e + \log\ \|j\|_{\lambda^{-1}(v(0,0))}$$

(formula (1.13) is not essentially affected from the defect of harmonicity of $r(j)$ in $v(0,0)$)

$$= q^{-k+1} \text{ by } (2.16).$$

For $k > 0$, $|j(z_k)| = |j(z_{-k})|$ since $v(-k,0)$ and $v(k,0)$ are Γ-equivalent. □

2.18 COROLLARY. *With the same notation as above,*

$$\begin{aligned} \log\ |g(z_k)| &= q & k &\leqslant -1 \\ &= q + k(q-1) & k &\geqslant 1. \end{aligned}$$

Proof. $\log\ |g(z_k)| = \frac{1}{q+1}(\log\ |\Delta(z_k)| + \log\ |j(z_k)|)$, which yields the result. □

2.19 *Remark.* As for j, $\log\ |g(z)|$ only depends on $\lambda(z)$ and is linear on edges, as long as $\lambda(z) \notin \Gamma v(0,0)$. The asserted values may be determined directly: The first case $\log|g(z_k)| = q$ for $k \leqslant -1$ reflects that $g = (T^q - T)E^{(q-1)}$ is non-zero at the cusp ∞ with $\log|g(\infty)| = q + \log|E^{(q-1)}(\infty)| = q$, and the second could be seen by inspecting the sum for $E^{(q-1)}(z_k)$. But the crucial point is that $\log|g(z)|$ may be expressed through the corresponding data of Δ and j even if $\lambda(z) \notin A(0,\infty)$, in which case a direct evaluation of $E^{(q-1)}$ seems difficult.

3. Roots of modular units

Let $n \in A$ be monic of degree $\delta > 0$, and let $\Gamma_0(n) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid c \equiv 0 \mod n\}$ be the nth *Hecke congruence* subgroup. An elementary calculation yields that Δ/Δ_n and its $(q-1)$th root h/h_n are modular functions (i.e., invariant) for $\Gamma_0(n)$. Correspondingly, $r(\Delta) - r(\Delta_n)$ is a $\Gamma_0(n)$-invariant current on $\mathcal{T}$. Here of course $\Delta_n(z) = \Delta(nz)$, $h_n(z) = h(nz)$. In case n is prime, we determined in [5] Section 4 to what extent roots may be extracted out of Δ/Δ_n in the function field of the modular curve $X_0(n) = \Gamma_0(n) \setminus \Omega \cup \{\text{cusps}\}$. Here we generalize this result, allowing $n \in A$ arbitrary, and also considering roots in $\mathcal{O}(\Omega)^*$.

Let $\varphi \in \underline{H}(\mathcal{T},\mathbb{Z})^{\Gamma_\infty}$ be given by its Fourier coefficients c_0 and c, and let $\varphi_n := \varphi \circ \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \underline{H}(\mathcal{T},\mathbb{Z})^{\Gamma_\infty}$ be its shift by the matrix $\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

3.1 LEMMA. *φ_n has the Fourier coefficients c_0', c' given by*

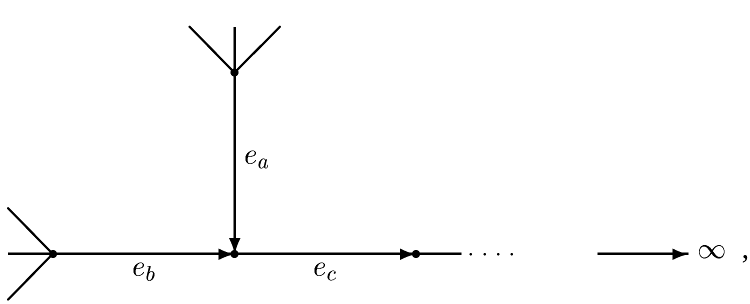
$$c_0'(\pi^k) = q^\delta c_0(\pi^k)$$

$$c'(\mathfrak{m} \cdot \infty^k) = c(\mathfrak{m} \cdot \operatorname{div}(n)_f^{-1} \cdot \infty^k)$$

($\mathfrak{m}$ a positive finite divisor).

Proof. The first formula is immediate from (1.16(i)). The second one results from (1.15) and a change of variables. □

Next, let e_a, e_b, e_c be the edges $e_a = e(2,\pi)$, $e_b = e(2,0)$, $e_c = e(1,0)$ of $\mathcal{T}$. The picture on $\Gamma_\infty \setminus \mathcal{T}$ looks



(3.2)

where all the $e(2, t\pi)$ $(t \in \mathbb{F}_q^*)$ are identified mod Γ_∞ with e_a. Thus for $\varphi \in \underline{H}(\mathcal{T}, \mathbb{Z})^{\Gamma_\infty}$,

$$(q-1)\varphi(e_a) + \varphi(e_b) = \varphi(e_c).$$

On the other hand, some of the Fourier coefficients of φ may directly be evaluated from (1.15):

$$\begin{aligned} c_0(\pi^2) &= q^{-1}((q-1)\varphi(e_a) + \varphi(e_b)) = q^{-1}\varphi(e_c), \\ c((1)) &= q^{-1}(-\varphi(e_a) + \varphi(e_b)). \end{aligned} \tag{3.17}$$

Combining (3.1), (3.3) with (2.6)–(2.9) and solving for the values on the edges e_a, e_b, e_c yields the following table for the functions $\varphi = r(\Delta)$, $r(\Delta)_n = r(\Delta_n)$, $r(\Delta) - r(\Delta_n)$.

3.4 TABLE. $(\delta := \deg n > 0)$

	$r(\Delta)$	$r(\Delta_n)$	$r(\Delta) - r(\Delta_n)$
$c((1))$	$\frac{(q^2-1)(q-1)}{q}$	0	$\frac{(q^2-1)(q-1)}{q}$
$c_0(1)$	$-(q-1)q$	$-(q-1)q^{\delta+1}$	$(q-1)q(q^\delta - 1)$
$\varphi(e_a)$	$-(q-1)q$	$-(q-1)q^{\delta-1}$	$(q-1)q(q^{\delta-2} - 1)$
$\varphi(e_b)$	$(q^2-q-1)(q-1)$	$-(q-1)q^{\delta-1}$	$(q-1)(q^{\delta-1} + q^2 - q - 1)$
$\varphi(e_c)$	$-(q-1)$	$-(q-1)q^\delta$	$(q-1)(q^\delta - 1)$

3.5 COROLLARY. *Let r be the largest integer such that there exists an rth root of Δ/Δ_n in $\mathcal{O}(\Omega)^*$. Then r divides $(q-1)^2$ if $\delta = \deg n$ is odd, and divides $(q-1)(q^2-1)$ if δ is even.*

(We will see in (3.18) that in fact equality holds.)

Proof. As is immediately verified, $\gcd\{\varphi(e_a), \varphi(e_b), \varphi(e_c)\} = (q-1)^2$, $(q-1)(q^2-1)$ if δ is odd, even, respectively, for $\varphi = r(\Delta) - r(\Delta_n)$. □

In order to construct roots of Δ/Δ_n, we have to introduce some more material.

(3.6) Let $X(n) = \Gamma(n) \setminus \Omega \cup \{\text{cusps}\}$ be the Drinfeld modular curve of level n, $\Gamma(n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n\}$. The cusps *cusps*$(\Gamma(n))$ of $X(n)$ correspond bijectively to

$$\Gamma(n) \setminus \Gamma/\Gamma_\infty \xrightarrow{\cong} \Gamma(n) \setminus \mathbb{P}^1(K) \xrightarrow{\cong} [(A/n)^2_{\text{prim}}]/\mathbb{F}_q^*,$$

where $(A/n)^2_{\text{prim}} \hookrightarrow (A/n)^2$ is the set of pairs $\{\binom{a}{c} \mid a, c \in A/n,\ (A/n)a + (A/n)c = A/n\}$, and the identification is induced from $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \binom{a}{c} \bmod n$. We

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simply write $\binom{a}{c}$ for the corresponding cusp of $X(n)$. Similarly, the cusps of $X_0(n)$ are given by

$$\mathit{cusps}(\Gamma_0(n)) \xrightarrow{\cong} \Gamma_0(n) \setminus \Gamma/\Gamma_\infty \xrightarrow{\cong} \left\{ \begin{bmatrix} a \\ c \end{bmatrix} \mid a, c \in A/n \text{ coprime} \right\},$$

where $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ is the equivalence class of $\binom{a}{c} \mod \Gamma_0(n)$. Let

$$n = \prod_{1 \leqslant i \leqslant s} f_i^{r_i} \tag{3.7}$$

be the prime decomposition of n, i.e., the $f_i \in A$ monic, irreducible, of degree d_i, pairwise different, and put $q_i = q^{d_i}$. For $x \in A/n$, we let $\underline{h}(x) = (h_1(x), \ldots, h_s(x))$ be its *height*, where $h_i(x) = \operatorname{ord}_{f_i}(x) \in \{0, 1, \ldots, r_i\}$ is the truncated f_i-adic valuation. In particular, $h_i(0) = r_i$. For $\binom{a}{c} \in \mathit{cusps}(\Gamma(n))$ we put $\underline{h}\binom{a}{c} = \underline{h}(c)$ and $\rho\binom{a}{c} = 1$, if there is an i with $0 < h_i(c) < r_i$, and $\rho\binom{a}{c} = q - 1$ otherwise. Note that $\underline{h}\binom{a}{c}$ and $\rho\binom{a}{c}$ only depend on the class of $\binom{a}{c} \mod \Gamma_0(n)$ and therefore are defined on $cusps(\Gamma_0(n))$. The next lemma (whose proof we omit) follows from calculating the $\Gamma_0(n)$-orbits on *cusps*$(\Gamma(n))$.

3.8 LEMMA. *The ramification index* $\operatorname{ram}\binom{a}{c}$ *of the* cusp $\binom{a}{c}$ *of* $X(n)$ *over the* cusp $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ *of* $X_0(n)$ *is given by*

$$\operatorname{ram}\begin{pmatrix} a \\ c \end{pmatrix} = \rho\begin{pmatrix} a \\ c \end{pmatrix} \prod_{1 \leqslant i \leqslant s} q_i^{\inf\{2h_i, r_i\}}.$$

In particular, it depends only on $\underline{h}\binom{a}{c} = (h_1, \ldots, h_s)$. □

3.9 EXAMPLE. Let n be prime of degree δ. There are $(q^{2\delta} - 1)/(q - 1)$ cusps $\binom{a}{c}$ on $X(n)$ and two cusps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ on $X_0(n)$. We have $\operatorname{ram}\binom{1}{0} = (q-1)q^\delta$ and $\operatorname{ram}\binom{0}{1} = q - 1$.

The total ramification index of $\binom{a}{c}$ over the unique cusp '∞' of $X(1)$ equals $(q-1)\Pi q_i^{r_i} = (q-1)|n|$, as follows from the description of *cusps*$(\Gamma(n))$. Since Δ has a simple zero at $\infty \in X(1)$, we get (with zero orders of modular forms defined as in [6])

$$\operatorname{ord}_{\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]} \Delta = \frac{(q-1)|n|}{\operatorname{ram}\binom{a}{c}} = \frac{(q-1)}{\rho\binom{a}{c}} \prod_{1 \leqslant i \leqslant s} q_i^{r_i - \inf\{2h_i, r_i\}}, \; h_i = h_i(c). \tag{3.10}$$

Let $w_n : z \mapsto \frac{1}{nz}$ be the Atkin-Lehner involution on Ω. The matrix $\left(\begin{smallmatrix} 0 & 1 \\ n & 0 \end{smallmatrix}\right)$ normalizes $\Gamma_0(n)$ and thus induces an involution on $X_0(n)$, which interchanges Δ and Δ_n.

Furthermore: If $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ has height $\underline{h} = (h_1, \ldots, h_s)$, the cusp $w_n \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ has height $\underline{h}' = (r_1 - h_1, \ldots, r_s - h_s)$. In view of (3.8), we therefore get

$$\mathrm{ord}_{\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]} \Delta_n = \frac{(q-1)|n|}{\mathrm{ram}(w_n \binom{a}{c})} = \frac{q-1}{\rho\binom{a}{c}} \prod_{1 \leqslant i \leqslant s} q_i^{r_i - \inf\{2(r_i - h_i), r_i\}}. \tag{3.11}$$

For any pair $(u, v) \in A \times A - nA \times nA$, we let $e_{u,v} : \Omega \to C$ be the holomorphic function defined in [3] p. 99, i.e.,

$$e_{u,v}(z) := e_\Lambda \left(\frac{uz + v}{n} \right),$$

where Λ is the A-lattice $Az + A \subset C$ and e_Λ its exponential function. For the moment we are interested in its following properties (*loc. cit.*, in particular Korollar 2.2 and Section 3):

(3.12) (i) $e_{u,v}$ has neither zeroes nor poles on Ω and depends only on the residue class of $(u, v) \mod n$.

(ii) $e_{u,v}(\gamma z) = (cz + d)^{-1} e_{(u,v)\gamma}(z)$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

(iii) The inverse $e_{u,v}^{-1}$ is a holomorphic modular form of weight 1, $e_{u,v}$ itself is a meromorphic modular form of weight -1 for $\Gamma(n)$.

(iv) The zero order of $e_{u,v}^{-1}$ at $\binom{a}{c} \in \mathrm{cusps}(\Gamma(n))$ is given by

$$\mathrm{ord}_{\binom{a}{c}} e_{u,v}^{-1} = |au + cv|_n.$$

Here $|x|_n = |x_0|$ if $x, x_0 \in A$, $x \equiv x_0 \mod n$, deg $x_0 <$ deg n.

3.13 *Remark*. The inverse $e_{u,v}^{-1}$ may also be described as the Eisenstein series

$$e_{u,v}^{-1}(z) = E_{u,v}(z) = \sideset{}{'}\sum_{\substack{a,b \in A \\ (a,b) \equiv (u,v) \bmod n}} (az + b)^{-1}.$$

This will however not be used in this paper.

We now define the functions on Ω

$$\begin{aligned} F(z) &= \prod_{\substack{0 \neq v \in A \\ \deg v < \delta}} e_{0,v}^{-1}, \\ G(z) &= \prod_{\substack{v \text{ monic} \\ \deg v < \delta}} e_{0,v}^{-1}. \end{aligned} \tag{3.14}$$

Then clearly $F = (-1)^\delta G^{q-1}$, and F is modular of weight $|n| - 1$ and type 0 for $\Gamma_0(n)$, as immediately results from the transformation rule of the $e_{u,v}$. Further, its orders at the various cusps of $X_0(n)$ are

$$\mathrm{ord}_{\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]} F = \mathrm{ram}\binom{a}{c}^{-1} \sum_{0 \neq v \in A/n} |vc|_n. \tag{3.15}$$

3.16 THEOREM. *The divisors of* Δ, Δ_n *and* F *on* $X_0(n)$ *are related by*

$$|n| \text{ div } \Delta - \text{div } \Delta_n = (q^2 - 1) \text{ div } F.$$

Proof. All the divisors have their support in *cusps*$(\Gamma_0(n))$, so we have to compare their orders at the different cusps. Those of Δ and Δ_n are given by (3.8), (3.10) and (3.11). They only depend on the height $\underline{h} = (h_1, \ldots, h_s)$ of the cusp $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$, as is the case for $\text{ord}_{\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]} F$. Thus we may without restriction assume that c is a divisor of n, $c = \prod_{1 \leqslant i \leqslant s} f_i^{h_i}$. For such c, we find by an elementary calculation

$$\sum_{0 \neq v \in A/n} |vc|_n = \frac{|n|^2 - |c|^2}{q+1},$$

where $|c| = \prod_{1 \leqslant i \leqslant s} q_i^{h_i}$. Hence for an arbitrary cusp $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ of height $\underline{h}$,

$$\sum_{0 \neq v \in A/n} |vc|_n = \frac{|n|^2 - \Pi q_i^{h_i}}{q+1}.$$

Inserting this into (3.15) yields the result. □

3.17 COROLLARY. *Up to constants we have*

$$\frac{\Delta}{\Delta_n} = \text{const.} \frac{F^{q^2-1}}{\Delta^{|n|-1}}. \qquad \square$$

3.18 COROLLARY. *The estimate given in* (3.5) *for the root number* r *is sharp. that is,* Δ/Δ_n *has an* r*th root in* $\mathcal{O}(\Omega)^*$, *where* $r = (q-1)^2$ *if* δ *is odd and* $r = (q-1)(q^2-1)$ *if* δ *is even, and* r *is maximal.*

Proof. Recall first that $\Delta = -h^{q-1}$ and $F = \text{const.}\ G^{q-1}$ are $(q-1)$th powers. Thus $r \geqslant (q-1)\text{gcd}\{(q^2-1), |n|-1\}$, which is as stated. □

The function G is a modular form of weight $(|n|-1)/(q-1)$ for $\Gamma_1(n) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid a \equiv d \equiv 1,\ c \equiv 0 \mod n\}$ and transforms according to

$$G(\gamma z) = \chi(\gamma)(cz+d)^{(|n|-1)/(q-1)} G(z) \tag{3.19}$$

under $\Gamma_0(n)$, where χ is a character with $\chi^{q-1} = 1$, i.e., $\chi\colon \Gamma_0(n) \to \mathbb{F}_q^*$. We will next determine χ. Recall that $n = \Pi f_i^{r_i}$ as usual. For $1 \leqslant i \leqslant s$, we let

$$N_i\colon (A/n)^* \to (A/f_i)^* \to \mathbb{F}_q^*$$

be the canonical projection followed by the norm.

3.20 THEOREM. *The 'nebentype' χ of G is given by*

$$\begin{aligned}\chi\colon \Gamma_0(n) &\to \mathbb{F}_q^* \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \Pi N_i(d)^{-r_i}.\end{aligned}$$

Proof. First note that for $a \in \mathbb{F}_q^*$

$$e_{0,av}(z) = a \cdot e_{0,v}(z). \tag{1}$$

Let $S \subset A/n - \{0\}$ be the set of monics, which is a set of representatives for $(A/n - \{0\})/\mathbb{F}_q^*$, as is d$S$ if $d \in (A/n)^*$. Hence for each $v \in S$ there are unique $a_v \in \mathbb{F}_q^*$, $v' \in S$ such that $dv = a_v \cdot v'$, d being fixed. In view of (1) and the definition of G, we will have

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \prod_{v \in S} a_v^{-1}. \tag{2}$$

For each height vector $\underline{h} = (h_1, \ldots, h_s) < \underline{r} = (r_1, \ldots, r_s)$, we let $(A/n)(\underline{h})$ be the elements of A/n of height $\underline{h}$ and $S(\underline{h})$ the monics in $(A/n)(\underline{h})$. We calculate the contribution of $S(\underline{h})$ to (2).

Let $m = m(\underline{h}) := \Pi f_i^{r_i - h_i}$. Then the group $(A/m)^*$ acts faithfully and simply transitively on $(A/n)(\underline{h})$, that is, for $v, v' \in (A/n)(\underline{h})$ there exists a unique $x \in (A/m)^*$ such that $xv = v'$, labelled $(\frac{v'}{v})$. Then

$$\prod_{v \in S(\underline{h})} a_v = \prod_{v \in S(\underline{h})} \left(\frac{dv}{v}\right) = (d \mod m)^{\varphi(m)/(q-1)}, \tag{3}$$

since $dS(\underline{h})$ and $S(\underline{h})$ are representatives for $(A/n)(\underline{h})/\mathbb{F}_q^*$. Here $\varphi(m) := \sharp(A/m)^*$, and the right hand term lies in $\mathbb{F}_q^* \hookrightarrow (A/m)^*$. Now

$$(A/m)^* \xrightarrow{\cong} \prod_{i \text{ s.t. } h_i < r_i} (A/f_i^{r_i - h_i})^*,$$

which implies $(d \mod m)^{\varphi(m)/(q-1)} = 1$ if m fails to be primary, i.e., if there are at least two i with $h_i < r_i$. If $m = f_i^{r_i - h_i}$ is primary, $(A/m)^*$ decomposes canonically into $(A/f_i)^*$ and its p-Sylow group of order $q_i^{r_i - h_i - 1}$, and $(d \mod m)^{\varphi(m)/(q-1)} = (d \mod f_i)^{(q_i - 1)/(q-1)} = N_i(d)$. Hence

$$\begin{aligned}\prod_{v \in S(\underline{h})} a_v &= N_i(d), \quad \exists!\, i \text{ s.t. } h_i < r_i \\ &= 1 \qquad\quad \text{otherwise.}\end{aligned} \tag{4}$$

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Inserting (4) into (2) finishes the proof. □

For $\delta = \deg n$ even (odd), let D_n be the function $G \cdot h^{-(|n|-1)/(q^2-1)}$ $(G^{q+1} \cdot h^{-(|n|-1)/(q-1)})$, respectively, i.e., $\Delta/\Delta_n = \text{const.}\ D_n^r$ with r as in (3.18).

3.21 COROLLARY. *The function D_n transforms under $\Gamma_0(n)$ according to the character $\omega_n := \chi \cdot \det^{\delta/2}$ if δ is even and $\omega_n = \chi^2 \cdot \det^\delta$ if δ is odd.*

Proof. For δ odd,

$$D_n(\gamma z) = \frac{\chi^{q+1}(\gamma)}{\det(\gamma)^{-(|n|-1)/(q-1)}} D_n(z)$$

(by the theorem and the definition of D_n)

$$= \chi^2(\gamma) \det(\gamma)^\delta D_n(z),$$

since $\frac{|n|-1}{q-1} = \frac{q^\delta - 1}{q-1} \equiv \delta \mod q-1$. A similar consideration gives the result for δ even. □

Let $o(\omega_n)$ be the order of ω_n. Then $D_n^{o(\omega_n)}$ is the least power of D_n which is $\Gamma_0(n)$-invariant, and $r/o(\omega_n)$ is the largest number k such that Δ/Δ_n has a kth root in the field of modular functions for $\Gamma_0(n)$.

3.22 PROPOSITION.

$$o(\omega_n) = \frac{q-1}{\gcd(q-1, r_1, \ldots, r_s, \delta/2)}, \quad \delta \textit{ even}$$

$$= \frac{q-1}{\gcd(q-1, r_1, \ldots, r_s, \delta)}, \quad \delta \textit{ odd}.$$

Proof. For any of the characters $N_i^{-r_i}$ (see (3.20)), χ, det, ω_n, its order is the size of its image in $\mathbb{F}_q^*$. E.g. for $N_i^{-r_i}$, it equals $\frac{q-1}{\gcd(q-1, r_i)}$ since $N_i \colon \Gamma_0(n) \to \mathbb{F}_q^*$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto N_i(d)$, is surjective. The assertion now follows from the Chinese remainder theorem. □

In the concluding corollaries, we let '0' = $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ '∞' = $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ be the distinguished cusps of $X_0(n)$.

3.23 COROLLARY (see [5]). *Let n be prime of degree δ. The cuspidal divisor class group $\mathcal{C}$ of $X_0(n)$ is cyclic of order $(|n|-1)/(q^2-1)$ if δ is even and $(|n|-1)/(q-1)$ if δ is odd.*

Proof. By (3.9), $\mathcal{C}$ is the group generated by the class of $[(0) - (\infty)]$. The divisor of Δ/Δ_n is $(|n|-1)[(0) - (\infty)]$, and the character ω_n of D_n has exact order

$(q-1)$. Hence D_n^{q-1} but no smaller power of D_n is invariant under $\Gamma_0(n)$, and the class of $[(0)-(\infty)]$ has the asserted order. □

3.24 *Remark.* In the above situation, let t be the gcd of $q-1$ and $\sharp\mathcal{C}$. Then yet the divisor of $D_n^{(q-1)/t}$ on $X(n)$ comes from a divisor on $X_0(n)$, and $(q-1)/t$ is minimal with that property. As we will show in subsequent work, this implies that the kernel of the canonical map from $\mathcal{C}$ to the group Φ_∞ of connected components of the Néron model $/K_\infty$ of the Jacobian $\mathrm{Jac}(X_0(n))$ is the subgroup of order t in $\mathcal{C}$. Hence the picture differs significantly from the one at the finite place (n), where the corresponding mapping $\mathcal{C}\to\Phi_{(n)}$ is always bijective [5]. This gives a negative answer to a question raised by J. Teitelbaum ([18] p. 283).

3.25 COROLLARY. *Let $n=f^2$, f prime. The divisor class of $[(0)-(\infty)]$ in $X_0(n)$ has order $(|n|-1)/(q^2-1)$ is q is even or $\deg f$ is odd, and $(|n|-1)/2(q^2-1)$ if q is odd and $\deg f$ is even.*

Proof. Besides 0 and ∞, there are $(|f|-1)/(q-1)$ cusps $s=\begin{bmatrix}u\\f\end{bmatrix}$ of height 1 (u monic of degree $<\deg f$). Now $\mathrm{ord}_s\,\Delta=\mathrm{ord}_s(\Delta_n)=q-1$ for such s, and thus $\mathrm{div}(\Delta/\Delta_n)=(|n|-1)[(0)-(\infty)]$. We conclude with (3.22). □

We believe that an extension of the preceding arguments eventually will lead to the determination of the cuspidal divisor class groups $\mathcal{C}$ of all the curves $X_0(n)$, $X_1(n)$, $X(n)$, where n is a not necessarily prime element of A. A first step has been carried out in [3], from whose results upper estimates for $\sharp\mathcal{C}$ may be derived.

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