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On Hilbert Covariants

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Abstract. Let F denote a binary form of order d over the complex numbers. If r is a divisor of d, then the Hilbert covariant $\mathcal{H}_{r,d}(F)$ vanishes exactly when F is the perfect power of an order r form. In geometric terms, the coefficients of $\mathcal{H}$ give defining equations for the image variety X of an embedding $\mathbf{P}^r \hookrightarrow \mathbf{P}^d$. In this paper we describe a new construction of the Hilbert covariant and simultaneously situate it into a wider class of covariants called the Göttingen covariants, all of which vanish on X. We prove that the ideal generated by the coefficients of $\mathcal{H}$ defines X as a scheme. Finally, we exhibit a generalisation of the Göttingen covariants to n-ary forms using the classical Clebsch transfer principle.

1 Introduction

1.1 Let

$$F = \sum_{i=0}^{d} \binom{d}{i} a_i x_1^{d-i} x_2^i, \quad (a_i \in \mathbf{C})$$

denote a binary form of order[1] d in the variables $\mathbf{x} = \{x_1, x_2\}$. Its Hessian is defined to be

$$\mathrm{He}(F) = \frac{\partial^2 F}{\partial x_1^2} \frac{\partial^2 F}{\partial x_2^2} - \left(\frac{\partial^2 F}{\partial x_1 \partial x_2} \right)^2.$$

It is well known that

$$\mathrm{He}(F) = 0 \iff F = (p x_1 + q x_2)^d \quad \text{for some } p, q \in \mathbf{C};$$

that is to say, the Hessian of F vanishes identically exactly when F is the perfect d-th power of a linear form. (The implication $\Leftarrow$ is obvious, and $\Rightarrow$ easily follows by a simple integration—see [26, Proposition 2.23].)

The Hessian is a covariant of binary d-ics, in the sense that its construction commutes with a linear change of variables in the $\mathbf{x}$. More precisely, let $g = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix} \right)$ denote a complex matrix such that $\det g = 1$. Given a binary form $A(x_1, x_2)$, write

$$A^g = A(\alpha x_1 + \beta x_2, \gamma x_1 + \delta x_2).$$

Then we have an identity

$$\mathrm{He}(F^g) = [\mathrm{He}(F)]^g.$$

By definition, $\mathrm{He}(F)$ is of order $2d - 4$ (in the $\mathbf{x}$), and its coefficients are quadratic in the a_i; hence it is said to be a covariant of degree 2 and order $2d - 4$.

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[1] Usually d would be called the degree of F, but 'order' is the common usage in classical invariant theory.

1.2 Now suppose that r is a divisor of d (say $d = r\mu$), and we are looking for a similar covariant which vanishes exactly when F is the perfect μ-th power of an order r form. About a decade ago, the second author had constructed such a covariant using Wronskians. It will be described below in Section 3.2, but tentatively let us denote it by $\mathfrak{G}_{r,d}(F)$. Subsequently, he learnt from the report of a colloquium lecture by Gian-Carlo Rota [28] that Hilbert [19] had already solved this problem. Hilbert's construction (see Section 4.1 below) is based upon an entirely different idea; it will be denoted by $\mathcal{H}_{r,d}(F)$.

In fact, either of the constructions makes sense even if r does not divide d. If we let $e = \gcd(r, d)$ and $d = e\mu$, then we have the property

$$\mathfrak{G}_{r,d}(F) = 0 \iff F = G^\mu \quad \text{for some order } e \text{ form } G \iff \mathcal{H}_{r,d}(F) = 0.$$

Both covariants turn out to be of degree $r + 1$ and order $N = (r + 1)(d - 2)$. This, of course, creates a strong *presumption* that they might indeed be the same. This is our first result.

Theorem 1.1 *There exists a nonzero rational scalar $\kappa_{r,d}$ such that $\mathfrak{G}_{r,d} = \kappa_{r,d}\mathcal{H}_{r,d}$.*

The proof will be given in Section 4. When $r = 1$, either covariant reduces to the Hessian.

1.3 For $p \geqslant 0$, let S_p denote the $(p + 1)$-dimensional space of order p forms in $\mathbf{x}$. We have an embedding

$$\mathbf{P}S_e \longrightarrow \mathbf{P}S_d, \quad [G] \longrightarrow [G^\mu]$$

whose image $X = X_{e,d}$ is the variety of binary d-ics which are perfect μ-th powers of order e forms. (In particular, $X_{1,d}$ is the rational normal d-ic curve.) Let $R = \mathbf{C}[a_0, \dots, a_d]$ denote the co-ordinate ring of $\mathbf{P}S_d \simeq \mathbf{P}^d$. Write

$$\mathcal{H}_{r,d}(F) = \sum_{i=0}^{N} \binom{N}{i} h_i x_1^{N-i} x_2^i,$$

and let $J = (h_0, h_1, \dots, h_N) \subseteq R$ denote the ideal generated by the coefficients of $\mathcal{H}_{r,d}$ (or what is the same, that of $\mathfrak{G}_{r,d}$). By construction, the zero locus of J is precisely X. We show that, when r divides d, the ideal J defines X as a scheme.

Theorem 1.2 *Assume that r divides d. Then the saturation of J coincides with the defining ideal $I_X \subseteq R$.*

The proof will be given in Section 5.4. For the case $r = 1$, this theorem appears in [4].

1.4 In Section 3, we use the plethysm decomposition of SL_2-representations to exhibit $\mathfrak{G}_{r,d}$ as a special case of a family of covariants which vanish on X. We baptise them *Göttingen covariants* in order to commemorate the Göttingen school, of which

Hilbert was a distinguished member for nearly five decades. In Section 3.4, we give an algorithm for the symbolic computation of these covariants. The examples in Sections 5.1–5.3 suggest the conjecture that the coefficients of the Göttingen covariants generate all of I_X, when r divides d.

The J-ideals seem to obey complicated containment relations for varying values of r, and there is much here that we do not understand. We give a preliminary result in this direction in Proposition 5.3. The example in Section 5.7 shows that when r does not divide d, the Hilbert covariants can create interesting nonreduced scheme structures on X.

1.5 The problem discussed in Section 1.2 makes sense in any number of variables. There is a classical construction due to Clebsch called the 'transfer principle', which allows us to lift the binary solution to n-ary forms. We explain this in Section 6, and construct a concomitant $\widetilde{\mathfrak{G}}_{r,d}$ of n-ary d-ics which has exactly the same vanishing property that $\mathcal{H}_{r,d}$ does for binary forms (see Theorem 6.1). For instance, let F denote a quartic form in three variables x_1, x_2, x_3, which we write symbolically as

$$F = a_{\mathbf{x}}^4 = b_{\mathbf{x}}^4 = c_{\mathbf{x}}^4.$$

Then F is the square of a quadratic form, if and only if the concomitant

$$\widetilde{\mathfrak{G}}_{2,4} = (abu)(acu)^2 a_{\mathbf{x}} b_{\mathbf{x}}^3 c_{\mathbf{x}}^2,$$

vanishes on F.

1.6 Although the Hilbert covariants were defined over a century ago, they do not seem to have been studied much in the subsequent years.[2] This may be partly due to Hilbert himself, whose papers around 1890 in the *Mathematische Annalen* changed the texture of modern algebra, and to some extent caused the earlier themes in invariant theory to be seen as *passé* (*cf.* [15, Section II]). We are convinced, however, that these covariants (and their generalisation, namely the Göttingen covariants) encapsulate a large amount of hitherto unexplored algebraic geometry.

2 Preliminaries

In this section we establish notation and explain the necessary preliminaries in the invariant theory of binary forms. Since the latter are less widely known now than they were a century ago, we have included rather more background material. Some of the classical sources for this subject are [17, 18, 20, 30], whereas more modern treatments may be found in [11, 22, 26, 27, 31]. In particular, for explanations pertaining to the symbolic calculus, the reader is referred to [1, Section 2].

[2] There is a later short note by Brioschi [7], but it is mostly a report on Hilbert's original paper and contains little that is new.

2.1 SL_2-representations

The base field will be $\mathbf{C}$. Let V denote a two-dimensional complex vector space with basis $\mathbf{x} = \{x_1, x_2\}$, and a natural action of the group $SL(V) \simeq SL_2$. For $p \geqslant 0$, let $S_p = \mathrm{Sym}^p V$ denote the $(p+1)$-dimensional space of binary p-ics in $\mathbf{x}$. Recall that $\{S_p : p \geqslant 0\}$ is a complete set of finite-dimensional irreducible SL_2-representations, and each finite-dimensional representation is a direct sum of irreducibles. The reader is referred to [16, Section 6] and [21, Section I.9] for the elementary theory of SL_2-representations. For brevity, we will write $S_p(S_q)$ for $\mathrm{Sym}^p(S_q)$, *etc.*

2.2 Transvectants

Given integers $p, q \geqslant 0$, we have a decomposition of representations

$$S_p \otimes S_q \simeq \bigoplus_{k=0}^{\min(p,q)} S_{p+q-2k}.$$

Let A, B denote binary forms in $\mathbf{x}$ of respective orders p, q. The k-th transvectant of A with B, written $(A, B)_k$, is defined to be the image of $A \otimes B$ via the projection map

$$\pi_k \colon S_p \otimes S_q \longrightarrow S_{p+q-2k}.$$

It is given by the formula

$$(A, B)_k = \frac{(p-k)!\,(q-k)!}{p!\,q!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{\partial^k A}{\partial x_1^{k-i} \partial x_2^{i}} \frac{\partial^k B}{\partial x_1^{i} \partial x_2^{k-i}}.$$

Usually k is called the index of transvection. By convention, $(A, B)_k = 0$, if $k > \min(p, q)$. If we symbolically write $A = a_{\mathbf{x}}^p, B = b_{\mathbf{x}}^q$ as in [18, Ch. I], then $(A, B)_k = (ab)^k a_{\mathbf{x}}^{p-k} b_{\mathbf{x}}^{q-k}$. A useful method for calculating transvectants of symbolic expressions is given in [17, Section 3.2.5].

There is a canonical isomorphism of representations

$$S_p \xrightarrow{\sim} S_p^* \big(= \mathrm{Hom}_{SL(V)}(S_p, S_0)\big) \tag{1}$$

that sends $A \in S_p$ to the functional $B \longrightarrow (A, B)_p$. It is convenient to identify each S_p with its dual via this isomorphism, unless it is necessary to maintain a distinction between the two.

2.3 The Omega Operator

If $\mathbf{x} = \{x_1, x_2\}$ and $\mathbf{y} = \{y_1, y_2\}$ are two sets of binary variables, then the corresponding Omega operator is defined to be

$$\Omega_{\mathbf{xy}} = \frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}.$$

Given forms A, B as above,

$$(A,B)_k = \frac{(p-k)!(q-k)!}{p!q!}\{\Omega^k_{\mathbf{xy}}[A(\mathbf{x})B(\mathbf{y})]\}_{\mathbf{y}:=\mathbf{x}}.$$

That is to say, change the $\mathbf{x}$ to $\mathbf{y}$ in B, operate k-times by Ω, and then revert back to the $\mathbf{x}$.

2.4 Covariants

We will revive an old notation due to Cayley, and write $(\alpha_0, \dots, \alpha_n \between u, v)^n$ for the expression

$$\sum_{i=0}^{n} \binom{n}{i} \alpha_i u^{n-i} v^i.$$

In particular

$$\mathbb{F} = (a_0, \dots, a_d \between x_1, x_2)^d \tag{2}$$

denotes the *generic* d-ic, which we identify with the natural trace form in $S_d^* \otimes S_d$. Using the duality in (1), this amounts to the identification of $a_i \in S_d^*$ with $\frac{1}{d!}x_2^{d-i}(-x_1)^i$, but it is convenient to think of the $\underline{a} = \{a_0, \dots, a_d\}$ as independent variables. Let R denote the symmetric algebra

$$\bigoplus_{m\geqslant 0} S_m(S_d^*) = \bigoplus_{m\geqslant 0} R_m = \mathbf{C}[a_0, \dots, a_d],$$

so that $\operatorname{Proj} R = \mathbf{P}S_d \simeq \mathbf{P}^d$.

Consider an $\mathrm{SL}(V)$-equivariant embedding

$$S_0 \hookrightarrow R_m \otimes S_q.$$

Let Φ denote the image of 1 via this map, then we may write

$$\Phi = (\varphi_0, \dots, \varphi_q \between x_1, x_2)^q,$$

where each φ_i is a homogeneous degree m form in the $\underline{a}$. One says that Φ is a covariant of degree m and order q (of the generic d-ic $\mathbb{F}$). In other words, the space

$$\operatorname{Span}\{\varphi_0, \dots, \varphi_q\} \subseteq R_m$$

is an irreducible subrepresentation isomorphic to S_q. The *weight* of Φ is defined to be $\frac{1}{2}(dm - q)$ (which is always a nonnegative integer).

In particular, $\mathbb{F}$ itself is a covariant of degree 1 and order d. A covariant of order 0 is called an invariant. Any transvectant of two covariants is also one, hence expressions such as

$$(\mathbb{F},\mathbb{F})_4, \quad \big(\mathbb{F},(\mathbb{F},\mathbb{F})_2\big)_3, \quad \big((\mathbb{F},\mathbb{F})_2,(\mathbb{F},\mathbb{F})_4\big)_5, \quad \dots$$

are all covariants. The Hessian coincides with $(\mathbb{F},\mathbb{F})_2$ up to a scalar. A fundamental result due to Gordan says that each covariant is a $\mathbf{C}$-linear combination of such compound transvectants (see [18, Section 86]). The weight of a compound transvectant is the sum of transvection indices occurring in it; for instance, $\big((\mathbb{F},\mathbb{F})_2,(\mathbb{F},\mathbb{F})_4\big)_5$ is of weight $2+4+5=11$.

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2.5 Recall that a homogeneous form in R is called isobaric of weight w, if for each monomial $\prod a_k^{n_k}$ appearing in it, we have $\sum_k kn_k = w$. If Φ is a covariant of degree-order (m,q), then its coefficient φ_k is isobaric of weight $\frac{1}{2}(dm-q)+k$. For instance, let $d=6$, and $\Phi = \big(\mathbb{F},(\mathbb{F},\mathbb{F})_2\big)_1$, which is a covariant of degree 3, order $3d-6$, and hence weight 3. Its expression begins as

$$\begin{aligned}\Phi = {} & (a_0^2a_3 + 2a_1^3 - 3a_0a_1a_2)x_1^{12} + (12a_1^2a_2 - 15a_0a_2^2 + 3a_0^2a_4)x_1^{11}x_2 \\ & + (15a_1a_2^2 + 3a_0^2a_5 + 18a_0a_1a_4 + 24a_1^2a_3 - 60a_0a_2a_3)x_1^{10}x_2^2 \\ & + (25a_2^3 + 60a_1^2a_4 - 80a_0a_3^2 + a_0^2a_6 - 30a_4a_0a_2 + 24a_1a_0a_5)x_1^9x_2^3 + \cdots,\end{aligned}$$

and one sees that the successive coefficients are isobaric of weights 3, 4, 5, *etc.*

2.6 The Cayley–Sylvester Formula

Let $C(d,m,q)$ denote the vector space of covariants of degree-order (m,q) for binary d-ics; its dimension $\zeta(d,m,q)$ is the same as the multiplicity of S_q in the irreducible decomposition of $R_m \simeq S_m(S_d)$. This number is given by the Cayley–Sylvester formula (see [31, Corollary 4.2.8]). For integers n, k and l, let $\pi(n,k,l)$ denote the number of partitions of n into k parts such that no part exceeds l. Then

$$\zeta(d,m,q) = \dim C(d,m,q) = \pi\Big(\frac{dm-q}{2}, d, m\Big) - \pi\Big(\frac{dm-q-2}{2}, d, m\Big).$$

For instance, $\zeta(6,3,6) = \pi(6,6,3) - \pi(5,6,3) = 7-5 = 2$, and it is easy to check (*e.g.*, by specialising $\mathbb{F}$) that

$$\mathbb{F}(\mathbb{F},\mathbb{F})_6, \quad \big(\mathbb{F},(\mathbb{F},\mathbb{F})_4\big)_2,$$

is a basis of $C(6,3,6)$.

2.7 This is perhaps the correct place to forestall one possible misconception about Theorem 1.1. Recall that $\mathfrak{G}_{r,d}$ and $\mathcal{H}_{r,d}$ both have degree $r+1$ and order $N = (r+1)(d-2)$. If it were the case that

$$\zeta(d, r+1, N) = 1, \tag{3}$$

then one could immediately conclude that the two must be equal up to a scalar. But such may not be the case. For instance, if $r=5, d=15$, then $\zeta(15,6,78)=4$. Hence, Theorem 1.1 does not follow from general multiplicity considerations, but instead requires an explicit hard calculation. However, (3) is true for $r=1,2$. (This can be seen from the plethysm formulae in [24, Section I.8].)

2.8 An alternate equivalent definition of a covariant is as follows. Let $g = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)$, where $\alpha, \dots, \delta$ are regarded as independent indeterminates. Write

$$x_1 = \alpha x_1' + \beta x_2', \quad x_2 = \gamma x_1' + \delta x_2',$$

and substitute into (2). Determine expressions $a_0', \dots, a_d'$ such that we have an equality

$$(a_0', \dots, a_d' \between x_1', x_2')^d = (a_0, \dots, a_d \between x_1, x_2)^d;$$

then each a_i' is a polynomial expression in the $\underline{a}$ and $\alpha, \dots, \delta$. Now suppose $\Phi \in \mathbf{C}[a_0, \dots, a_d; x_1, x_2]$ is a bihomogeneous form of degrees m, q respectively in $\underline{a}, \mathbf{x}$. Then Φ is a covariant if and only if the following identity holds:

$$\Phi(a_0', \dots, a_d'; x_1', x_2') = (\alpha\delta - \beta\gamma)^{\frac{dm-q}{2}} \Phi(a_0, \dots, a_d; x_1, x_2). \tag{4}$$

2.9 Covariants and Differential Operators

Consider the following differential operators:

$$E_+ = \sum_{i=0}^{d-1} (d-i) a_{i+1} \frac{\partial}{\partial a_i}, \quad E_- = \sum_{i=1}^{d} i a_{i-1} \frac{\partial}{\partial a_i}, \quad E_0 = \sum_{i=0}^{d} (2i-d) a_i \frac{\partial}{\partial a_i},$$

and

$$\Gamma_+ = E_+ - x_1 \frac{\partial}{\partial x_2}, \quad \Gamma_- = E_- - x_2 \frac{\partial}{\partial x_1}, \quad \Gamma_0 = E_0 + \left(x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} \right).$$

Proposition 2.1 *A bihomogeneous form Φ is a covariant if and only if*

$$\Gamma_+ \Phi = \Gamma_- \Phi = \Gamma_0 \Phi = 0. \tag{5}$$

A proof is given in [30, Section 149] (also see [31, Section 4.5]), but the central idea is the following: Φ is a covariant exactly when it remains unchanged by an SL_2-action, *i.e.*, when it is annihilated by the Lie algebra $\mathfrak{sl}_2$. Let

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

denote the standard generators of $\mathfrak{sl}_2$. Choose a path $t \longrightarrow g_t$ starting from the identity element in SL_2, and apply condition (4) to g_t. For the three cases $J_\star = [\frac{dg_t}{dt}]_{t=0}$ where $\star \in \{+, -, 0\}$, we respectively get the identities in (5).

The first coefficient φ_0 is called the source (or seminvariant) of Φ. From (5), we get equations

$$E_-(\varphi_0) = 0, \quad \text{and} \quad \varphi_k = \frac{(q-k)!}{q!} E_+^k(\varphi_0) \quad \text{for } 0 \leqslant k \leqslant q.$$

Thus, one can recover the entire covariant from the source alone. Moreover, a homogeneous isobaric form ψ in the $\underline{a}$ can be a source (of some covariant), if and only if it satisfies the condition $E_-(\psi) = 0$. The explicit formula for φ_k is sometimes called Robert's theorem (see [17, Section 8.1.5]).

The commutation relations between the $E_\star$ are parallel to the ones between the standard generators of $\mathfrak{sl}_2$, *i.e.*,

$$[E_+, E_-] = E_0, \quad [E_0, E_+] = 2E_+, \quad [E_0, E_-] = -2E_-.$$

In particular, due to the identity $[\Gamma_+, \Gamma_-] = \Gamma_0$, the condition $\Gamma_0\Phi = 0$ in (5) is automatically satisfied if the other two are.

The following lemma will be needed in Section 4.1.

Lemma 2.2 *For $n \geqslant 0$, we have an identity*

$$E_- E_+^{n+1} = E_+^{n+1} E_- - (n+1)E_+^n E_0 - n(n+1)E_+^n.$$

Proof This follows by a straightforward induction on n. ■

2.10 Wronskians

Let $m, n \geqslant 0$ be integers such that $m \leqslant n + 1$. Consider the following composite morphism of representations

$$w\colon \wedge^m S_n \xrightarrow{\sim} S_m(S_{n-m+1}) \longrightarrow S_{m(n-m+1)},$$

where the first map is an isomorphism (described in [3, Section 2.5]) and the second is the natural surjection. Given a sequence of binary n-ics $A_1, \dots, A_m$, define their Wronskian $W(A_1, \dots, A_m)$ to be the image $w(A_1 \wedge \cdots \wedge A_m)$. It is given by the determinant

$$(i, j) \longrightarrow \frac{\partial^{m-1} A_i}{\partial x_1^{m-j} \partial x_2^{j-1}}, \quad (1 \leqslant i, j \leqslant m).$$

The $\{A_i\}$ are linearly dependent over $\mathbf{C}$, if and only if $W(A_1, \dots, A_m) = 0$. (The 'only if' part is obvious. For the converse, see [25, Section 1.1].)

3 The Göttingen Covariants

3.1 Henceforth assume that r, d are positive integers, and let $e = \gcd(r, d)$. Write $d = e\mu$ and $r = e\mu'$. Consider the embedding

$$\mathbf{P}S_e \xrightarrow{\imath} \mathbf{P}S_d, \quad [G] \longrightarrow [G^\mu];$$

and let $X_{e,d}$ denote the image variety. We have a factorisation

$$\begin{tikzcd} & \mathbf{P}S_\mu(S_e) \arrow[d, dotted, "\pi"] \\ \mathbf{P}S_e \arrow[ur, "\nu_\mu"] \arrow[r, "\imath"'] & \mathbf{P}S_d \end{tikzcd}$$

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where ν_μ is the μ-fold Veronese embedding, and π is the projection coming from the surjective map $S_\mu(S_e) \longrightarrow S_{e\mu} = S_d$. Thus $\imath$ corresponds to the incomplete linear series $S_d \subseteq H^0\big(\mathcal{O}_{\mathbf{P}S_e}(\mu)\big)$.

3.2 In this section we will define the covariants $\mathfrak{G}_{r,d}$. For $F \in S_d$, we have a morphism

$$\alpha_F : S_r \longrightarrow S_{r+d-2}, \quad A \longrightarrow (A,F)_1 = \frac{1}{rd}\begin{vmatrix} A_{x_1} & A_{x_2} \\ F_{x_1} & F_{x_2} \end{vmatrix},$$

where A_{x_i} stands for $\frac{\partial A}{\partial x_i}$, *etc.*

Proposition 3.1 *With notation as above,*

$$\ker \alpha_F \neq 0 \iff [F] \in X_{e,d}.$$

Proof Assume $F = G^\mu$ for some e-ic G, then $\alpha_F(G^{\mu'}) = (G^{\mu'}, G^\mu)_1 = 0$.

Alternately, assume that $(A,F)_1 = 0$ for some nonzero A. We will construct a form G such that (up to scalars) $A = G^{\mu'}, F = G^\mu$. Let $\ell \in S_1$ be any linear form which divides either A or F; after a change of variables we may assume $\ell = x_1$. Suppose that a, f are the highest powers of x_1 which divide A, F respectively, and write $A = x_1^a\widetilde{A}, F = x_1^f\widetilde{F}$. Starting from the relation $A_{x_1}F_{x_2} = A_{x_2}F_{x_1}$, after expanding and rearranging the terms, we get

$$x_1^{a+f}(\widetilde{A}_{x_1}\widetilde{F}_{x_2} - \widetilde{A}_{x_2}\widetilde{F}_{x_1}) = x_1^{a+f-1}(-a\widetilde{A}\widetilde{F}_{x_2} + f\widetilde{A}_{x_2}\widetilde{F}),$$

hence x_1 must divide $(-a\widetilde{A}\widetilde{F}_{x_2} + f\widetilde{A}_{x_2}\widetilde{F})$. Thus, either $\widetilde{A}_{x_2} = \widetilde{F}_{x_2} = 0$ (and so $a = r, f = d$), or the terms with highest powers of x_2 in $a\widetilde{A}\widetilde{F}_{x_2}$ and $f\widetilde{A}_{x_2}\widetilde{F}$ cancel against each other. In the latter case,

$$a(d-f) = f(r-a) \implies ad = fr \implies a\mu = f\mu'.$$

In either case, $\mu'|a$ and $\mu|f$. Define G such that x_1 appears in it exactly to the power $\frac{f}{\mu}$, and similarly for all such ℓ. ■

Now consider the composite morphism

$$S_0 \simeq \wedge^{r+1}S_r \xrightarrow{\wedge^{r+1}\alpha_F} \wedge^{r+1}S_{r+d-2} \simeq S_{r+1}(S_{d-2}) \longrightarrow S_{(r+1)(d-2)}.$$

The image of $1 \in S_0$ is the Wronskian $W\left(\alpha_F(x_1^r), \alpha_F(x_1^{r-1}x_2), \ldots, \alpha_F(x_2^r)\right)$, which we define to be $\mathfrak{G}_{r,d}(F)$. To recapitulate, $\mathfrak{G}_{r,d}(F)$ is the determinant of the $(r+1)\times(r+1)$ matrix

$$(6) \qquad (i,j) \longrightarrow \frac{\partial^r C_i}{\partial x_1^{r-j}\partial x_2^j}, \quad (0 \leqslant i, j \leqslant r),$$

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where $C_i = (x_1^{r-i} x_2^i, F)_1$. Then

$$\mathfrak{G}_{r,d}(F) = 0 \iff \ker \alpha_F \neq 0 \iff [F] \in X_{e,d}.$$

Each matrix entry is linear in the coefficients of F, and of order $d-2$ in $\mathbf{x}$, hence $\mathfrak{G}_{r,d}$ has degree $r+1$ and order $N = (r+1)(d-2)$.

In the next section, we will generalise this construction to obtain a family of covariants vanishing on $X_{e,d}$. The reader who is more interested in Hilbert's solution may proceed directly to Section 4.1.

3.3 Let

$$\mathbb{B} = (b_0, b_1, \dots, b_{d-2} \between x_1, x_2)^{d-2},$$

denote a generic form of order $d-2$, with a new set of indeterminates $\underline{b}$. As in Section 2.4, the $\underline{b}$ can be seen as forming a basis of $S_{d-2}^* \simeq S_{d-2}$. Let $\Psi(\underline{b}, \mathbf{x})$ denote a covariant of degree $r+1$ and order q of $\mathbb{B}$. Then Ψ corresponds to an embedding $S_q \longrightarrow S_{r+1}(S_{d-2})$, which can be described as follows: if we realise $S_{r+1}(S_{d-2})$ as the space of degree $r+1$ forms in the $\underline{b}$, then $A(\mathbf{x}) \in S_q$ gets sent to $(A, \Psi)_q$. After dualising, we get a morphism

$$f_\Psi \colon S_{r+1}(S_{d-2}) \longrightarrow S_q.$$

Now consider the composite morphism

$$S_0 \simeq \wedge^{r+1} S_r \xrightarrow{\wedge^{r+1}\alpha_F} \wedge^{r+1} S_{r+d-2} \simeq S_{r+1}(S_{d-2}) \xrightarrow{f_\Psi} S_q,$$

and let $\mathfrak{G}_\Psi(\mathbb{F})$ denote the image of $1 \in S_0$, which will be called the Göttingen covariant of $\mathbb{F}$ associated to Ψ. It is of the same degree and order as Ψ, and hence its weight is $r+1$ more than that of Ψ. In particular, $\mathfrak{G}_{r,d}(\mathbb{F})$ is the same as $\mathfrak{G}_{\mathbb{B}^{r+1}}(\mathbb{F})$. As before,

$$[F] \in X_{e,d} \implies \mathfrak{G}_\Psi(F) = 0.$$

3.4 The Calculation of $\mathfrak{G}_\Psi$

One can calculate $\mathfrak{G}_\Psi(\mathbb{F})$ explicitly by following the sequence of maps above, which amounts to the following recipe:

- Introduce $2(r+1)$ sets of binary variables

$$\mathbf{y}_{(i)} = \{y_{i1}, y_{i2}\}, \quad \mathbf{z}_{(i)} = \{z_{i1}, z_{i2}\} \quad \text{for } 0 \leqslant i \leqslant r;$$

  and let $\Omega_{\mathbf{y}_{(i)}\mathbf{z}_{(i)}}$ be the corresponding Omega operators.
- Let $\mathcal{W}$ denote the determinant

$$(i,j) \longrightarrow \left.\frac{\partial^r (x_1^{r-i} x_2^i, \mathbb{F})_1}{\partial x_1^{r-j} \partial x_2^j}\right|_{\mathbf{x} \longrightarrow \mathbf{y}_{(i)}}, \quad (0 \leqslant i, j \leqslant r).$$

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This is similar to (6), except that the $\mathbf{y}_{(i)}$ variables are used throughout the i-th row. Let $\mathcal{W}^\sharp$ denote the symmetrisation of $\mathcal{W}$ with respect to the sets $\mathbf{y}_{(i)}$, *i.e.*,

$$\mathcal{W}^\sharp = \sum_\sigma \mathcal{W}(\mathbf{y}_{\sigma(0)}, \dots, \mathbf{y}_{\sigma(r)}),$$

the sum quantified over all permutations σ of $\{0, \dots, r\}$. Then $\mathcal{W}^\sharp$ is of degree $r+1$ in $\underline{a}$, and of order $d-2$ in each $\mathbf{y}_{(i)}$.

- Write
$$\Psi = (\psi_0, \dots, \psi_q \between x_1, x_2)^q,$$
where each ψ_i is a degree $r+1$ form in $\underline{b} = \{b_0, \dots, b_{d-2}\}$. Introduce $r+1$ sets of variables $\underline{b}_{(0)}, \dots, \underline{b}_{(r)}$, where
$$\underline{b}_{(i)} = \{b_{i0}, \dots, b_{id-2}\},$$
and let $\widetilde{\Psi}$ be the total polarisation of Ψ with respect to the new variables (see [12, Section 1.1]). Then $\widetilde{\Psi}$ is linear in each set $\underline{b}_{(i)}$.
- Let $\widehat{\Psi}$ denote the form obtained from $\widetilde{\Psi}$ by replacing b_{ik} with $\frac{1}{(d-2)!} z_{i2}^{d-2-k}(-z_{i1})^k$ for $0 \leqslant i \leqslant r$ and $0 \leqslant k \leqslant d-2$. (This is similar to the identification of a_i as in Section 2.4.) Thus $\widehat{\Psi}$ is of order q in $\mathbf{x}$, and of order $d-2$ in each $\mathbf{z}_{(i)}$.
- Finally,
$$\mathfrak{G}_\Psi(\mathbb{F}) = [\Omega^{d-2}_{\mathbf{y}_{(0)}\mathbf{z}_{(0)}} \circ \cdots \circ \Omega^{d-2}_{\mathbf{y}_{(r)}\mathbf{z}_{(r)}}] \widehat{\Psi} \mathcal{W}^\sharp.$$
This removes all the $\mathbf{y}_{(i)}$ and $\mathbf{z}_{(i)}$ variables, which leaves a form of degree $r+1$ in the $\underline{a}$ and order q in $\mathbf{x}$.

3.5 It may happen that $\mathfrak{G}_\Psi$ is identically zero, even if Ψ is nontrivial. (Hence the implication in (3.3) is not reversible in general.) For instance, recall that a generic binary d-ic has a cubic invariant exactly when d is a multiple of 4. Now let $r = 2$, and assume $d \equiv 2 \pmod 4$. Then $\Psi = \left(\mathbb{B}, (\mathbb{B}, \mathbb{B})_{\frac{d-2}{2}}\right)_{d-2}$ is a nontrivial cubic invariant of $\mathbb{B}$, but $\mathfrak{G}_\Psi$ must vanish identically.

If d is a divisor of r, then $X_{e,d} = \mathbf{P}S_d$, and in that case all $\mathfrak{G}_\Psi$ are identically zero.

N.B. Henceforth, if A, B are two quantities, we will write $A \doteq B$ to mean that $A = cB$ for some unspecified nonzero rational scalar c. This will be convenient in symbolic calculations, where more and more unwieldy scalars tend to accumulate at each stage.

3.6 As an example, we will follow this recipe when $r = 1$ and Ψ is any quadratic covariant. Write symbolically

$$\mathbb{F} = \alpha_{\mathbf{x}}^d = \beta_{\mathbf{x}}^d, \quad \mathbb{B} = p_{\mathbf{x}}^{d-2} = q_{\mathbf{x}}^{d-2}.$$

Every quadratic covariant of $\mathbb{B}$ must be of the form

$$\Psi = (\mathbb{B}, \mathbb{B})_{2n} = (pq)^{2n} p_{\mathbf{x}}^{d-2-2n} q_{\mathbf{x}}^{d-2-2n},$$

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for some n in the range $0 \leqslant n \leqslant \frac{d-2}{2}$. Using α, β for the two rows of $\mathcal{W}$, we get

$$\mathcal{W} \doteq \begin{vmatrix} \alpha_1\alpha_2\alpha_{\mathbf{y}_{(0)}}^{d-2} & \alpha_2^2\alpha_{\mathbf{y}_{(0)}}^{d-2} \\ \beta_1^2\beta_{\mathbf{y}_{(1)}}^{d-2} & \beta_1\beta_2\beta_{\mathbf{y}_{(1)}}^{d-2} \end{vmatrix} = \alpha_2\beta_1(\alpha\beta)\alpha_{\mathbf{y}_{(0)}}^{d-2}\beta_{\mathbf{y}_{(1)}}^{d-2},$$

and hence

$$\mathcal{W}^\sharp \doteq (\alpha\beta)^2\alpha_{\mathbf{y}_{(0)}}^{d-2}\beta_{\mathbf{y}_{(1)}}^{d-2}.$$

Now the symbolic expression for $\widetilde{\Psi}$ is the same as the one for Ψ, provided we make the convention that p, q respectively refer to the $\underline{b}_{(0)}, \underline{b}_{(1)}$ variables. Then

$$\widehat{\Psi} \doteq (\mathbf{z}_{(0)}\mathbf{z}_{(1)})^{2n}(\mathbf{x}\mathbf{z}_{(0)})^{d-2-2n}(\mathbf{x}\mathbf{z}_{(1)})^{d-2-2n};$$

and finally,

$$\mathfrak{G}_\Psi \doteq (\alpha\beta)^{2n+2}\alpha_{\mathbf{x}}^{d-2-2n}\beta_{\mathbf{x}}^{d-2-2n}.$$

We have proved the following.

Proposition 3.2 *If* $\Psi = (\mathbb{B}, \mathbb{B})_{2n}$, *then* $\mathfrak{G}_\Psi \doteq (\mathbb{F}, \mathbb{F})_{2n+2}$.

In particular, $\mathfrak{G}_{1,d} \doteq \mathfrak{G}_{\mathbb{B}^2} = \mathfrak{G}_{(\mathbb{B},\mathbb{B})_0} \doteq (\mathbb{F}, \mathbb{F})_2$ is the Hessian of $\mathbb{F}$. Similar calculations show that

$$\begin{aligned} (7) \qquad & \mathfrak{G}_{2,d} \doteq \big(\mathbb{F}, (\mathbb{F}, \mathbb{F})_2\big)_1, \\ & \mathfrak{G}_{3,d} \doteq 3(2d-3)(\mathbb{F}, \mathbb{F})_2^2 - 2(d-2)\mathbb{F}^2(\mathbb{F}, \mathbb{F})_4, \\ & \mathfrak{G}_{4,d} \doteq 2(3d-4)(\mathbb{F}, \mathbb{F})_2\big(\mathbb{F}, (\mathbb{F}, \mathbb{F})_2\big)_1 - (d-3)\mathbb{F}^2\big(\mathbb{F}, (\mathbb{F}, \mathbb{F})_4\big)_1. \end{aligned}$$

(Such formulae are derived for the $\mathcal{H}_{r,d}$ in [7] and [19], but this makes no difference in view of Theorem 1.1.) However, as r grows, it quickly begins to get more and more tedious to execute this recipe.

4 Hilbert's Construction

In this section we will describe Hilbert's construction of his covariants $\mathcal{H}_{r,d}$, and later prove that the outcome coincides with $\mathfrak{G}_{r,d}$ up to a scalar.

The underlying idea is as follows. Suppose, for instance, that F is an order 10 form such that $F = G^5$ for some quadratic G. Then substituting $x_1 = 1, x_2 = z$, we have

$$\frac{d^3}{dz^3}\sqrt[5]{F(1,z)} = 0.$$

One should like to convert the left-hand side into a covariant condition on F; but this requires some technical modifications. We begin by constructing the source of Hilbert's covariant.

4.1 Define
$$h_0 = a_0^{r+1-\frac{r}{d}} E_+^{r+1}(a_0^{\frac{r}{d}}).$$
This is easily seen to be an isobaric homogeneous form of degree and weight $r+1$ in the $\underline{a}$. For instance,
$$h_0 = \begin{cases} (d-1)(a_0a_2 - a_1^2) & \text{if } r = 1, \\ (2d^2 - 6d + 4)a_0^2 a_3 - (6d^2 - 18d + 12)a_0 a_1 a_2 + (4d^2 - 12d + 8)a_1^3 & \text{if } r = 2. \end{cases}$$

Lemma 4.1 *The form h_0 is a source.*

Proof We need to show that $E_- h_0 = a_0^{r+1-\frac{r}{d}} E_- E_+^{r+1}(a_0^{\frac{r}{d}})$ vanishes. Apply Lemma 2.2 and note that
$$E_0(a_0^{\frac{r}{d}}) = -r a_0^{\frac{r}{d}}, \quad E_-(a_0^{\frac{r}{d}}) = 0,$$
which implies the result. ■

Since h_0 has weight $r+1$, the covariant corresponding to h_0 must have order $N = (r+1)(d-2)$. The Hilbert covariant is defined to be

$$\mathcal{H}_{r,d}(\mathbb{F}) = (h_0, \dots, h_N \between x_1, x_2)^N, \tag{8}$$

where

$$h_k = \frac{(N-k)!}{N!} E_+^k(h_0) \quad \text{for } 0 \leqslant k \leqslant N. \tag{9}$$

4.2 In order to prove Theorem 1.1, it will suffice to show that $\mathcal{H}_{r,d}$ and $\mathfrak{G}_{r,d}$ have the same source up to a scalar. We will avoid writing such scalars explicitly in the course of the calculation, but see formula (16) below.

Let $\underline{a} = (a_0, \dots, a_d)$ denote a $(d+1)$-tuple of complex variables. For $t \in \mathbf{C}$, define
$$\gamma_t \colon \mathbf{C}^{d+1} \longrightarrow \mathbf{C}^{d+1}, \quad (a_0, a_1, \dots, a_d) \longrightarrow \big(a_0(t), a_1(t), \dots, a_d(t)\big),$$
by the formula
$$(a_0, \dots, a_d \between 1, z+t)^d = \big(a_0(t), \dots, a_d(t) \between 1, z\big)^d.$$
It is easy to see that
$$a_i(t) = a_i + (d-i)a_{i+1}t + O(t^2) \quad \text{for } 0 \leqslant i \leqslant d-1.$$
Hence, given an analytic function $\phi \colon \mathbf{C}^{d+1} \longrightarrow \mathbf{C}$, we have an equality
$$E_+\phi = \Big[\frac{d}{dt}\big(\phi(\gamma_t)\big)\Big]_{t=0}.$$

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Iterating this formula,

$$E_+^n\phi = \Big[\frac{\partial^n \phi(\gamma_{t_1+\cdots+t_n})}{\partial t_1\cdots\partial t_n}\Big]_{t_1=\cdots=t_n=0} = \Big[\frac{d^n\phi(\gamma_t)}{dt^n}\Big]_{t=0}. \tag{10}$$

Now write $f(z) = (a_0,\dots,a_d \between 1,z)^d$, and apply this to the function

$$\phi(\underline{a}) = a_0^{\frac{r}{d}} = f(0)^{\frac{r}{d}},$$

which gives the expression

$$h_0 = f(0)^{r+1-\frac{r}{d}}\Big[\frac{d^{r+1}}{dt^{r+1}} f(t)^{\frac{r}{d}}\Big]_{t=0} \tag{11}$$

for the source of $\mathcal{H}_{r,d}$.

4.3 We make a small digression to prove that $\mathcal{H}_{r,d}$ has the required vanishing property.

Proposition 4.2 *Let F be a d-ic. Then*

$$\mathcal{H}_{r,d}(F) = 0 \iff [F] \in X_{e,d}.$$

Proof This will, of course, follow from Theorem 1.1, but even so, we include an independent proof. After a change of variables, we may assume $a_0 \neq 0$. Write

$$f(t)^{\frac{r}{d}} = a_0^{\frac{r}{d}}\Big(1 + \sum_{m\geqslant 1}\frac{\theta^m}{m!}t^m\Big). \tag{12}$$

Using the reformulation of E_+ above, we have

$$a_0^{\frac{r}{d}}\theta_{m+1} = E_+(a_0^{\frac{r}{d}}\theta_m).$$

Now a simple induction shows that there are identities

$$a_0^{r+1}\theta_{r+1} = h_0, \quad a_0^{r+2}\theta_{r+2} \doteq a_0h_1 + \square_1 h_0,$$

and in general

$$a_0^{r+1+k}\theta_{r+1+k} \doteq a_0^k h_k + \sum_{i=1}^{k}\square_i h_{k-i},$$

for some homogeneous polynomials $\square_i(\underline{a})$ of degree k and weight i. (Here we have set $h_i = 0$ for $i > N$.) If $h_0, h_1, \dots,$ *etc.*, all vanish, then so do θ_k for $k \geqslant r+1$, and the power series in (12) becomes a polynomial of degree $\leqslant r$. Thus $f(t)$ reduces to a perfect μ-th power. Conversely, if $f(t) = g(t)^\mu$, then $f^{\frac{r}{d}} = g^{\mu'}$ is of degree $\leqslant r$, and hence $h_0 = h_1 = \cdots = 0$. ∎

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4.4 We should like to calculate the source of $\mathfrak{G}_{r,d}$ as defined by the determinant in (6). However, the dehomogenisation in the previous section is with respect to the other variable, so a preparatory step is needed. The Wronskian construction is equivariant, hence in the notation of Section 1.1, we have an identity

$$W(C_0,\dots,C_r)(x_1,x_2) = W(C_0^g,\dots,C_r^g)(x_2,-x_1),$$

for $g = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. If $z = -x_2/x_1$, then (up to a scalar) the right-hand side becomes

$$(-x_1)^N \times \begin{vmatrix} v_0^{(r)} & \cdots & v_0 \\ \vdots & & \vdots \\ v_r^{(r)} & \cdots & v_r \end{vmatrix},$$

where

$$v_i(z) = C_i^g(z,1) = C_i(-1,z), \quad \text{and} \quad v_i^{(k)} = \frac{d^k}{dz^k} v_i.$$

Substituting $x_1 = 1, x_2 = 0$,

$$g_0 = \text{source of } \mathfrak{G}_{r,d} \doteq \begin{vmatrix} v_0^{(r)}(0) & \cdots & v_0(0) \\ \vdots & & \vdots \\ v_r^{(r)}(0) & \cdots & v_r(0) \end{vmatrix}.$$

By definition,

$$C_i \doteq \frac{\partial[x_1^{r-i}x_2^i]}{\partial x_1}\frac{\partial F}{\partial x_2} - \frac{\partial[x_1^{r-i}x_2^i]}{\partial x_2}\frac{\partial F}{\partial x_1}.$$

Using $F(x_1,x_2) = x_1^d f(\frac{x_2}{x_1})$, this can be rewritten as

$$C_i \doteq \left(i d x_1^{d+r-i-1} x_2^{i-1} f\left(\frac{x_2}{x_1}\right) - r x_1^{d+r-i-2} x_2^i f'\left(\frac{x_2}{x_1}\right)\right),$$

and therefore

$$v_i(z) \doteq \left(id(-1)^{d+r-i-1} z^{i-1} f(-z) - r(-1)^{d+r-i-2} z^i f'(-z)\right).$$

Let

$$b_i(z) = idz^{i-1} f(z) - rz^i f'(z),$$

so that $v_i(-z) = (-1)^{d+r} b_i(z)$. After reordering the columns,

$$g_0 \doteq \begin{vmatrix} b_0(0) & \cdots & b_0^{(r)}(0) \\ \vdots & & \vdots \\ b_r(0) & \cdots & b_r^{(r)}(0) \end{vmatrix}.$$

Now the key step is to write

$$b_i \doteq f(z)^{1+\frac{r}{d}} \underbrace{\frac{d}{dz}\big(z^i f(z)^{-\frac{r}{d}}\big)}_{c_i},$$

which is similar to the idea of an integrating factor in the theory of ordinary differential equations. Then

$$g_0 \doteq (a_0^{1+\frac{r}{d}})^{r+1} \times \begin{vmatrix} c_0(0) & \cdots & c_0^{(r)}(0) \\ \vdots & & \vdots \\ c_r(0) & \cdots & c_r^{(r)}(0) \end{vmatrix}.$$

Now let $\mu_i = z^i$ and $\omega = f(z)^{-\frac{r}{d}}$, so that $c_i = (\mu_i \omega)'$. By the Leibniz rule,

$$c_i^{(j)} = (\mu_i \omega)^{(j+1)} = \sum_{k=0}^{r+1} \binom{j+1}{k} \mu_i^{(k)} \omega^{(j+1-k)} \quad \text{for } 0 \leqslant j \leqslant r,$$

with the convention that $\binom{j+1}{k} = 0$ if $k > j+1$. Since $\mu_i^{(r+1)} = 0$, we can stop the summation at $k = r$, and factor the determinant in (4.4) as

$$\begin{vmatrix} \mu_0(0) & \cdots & \mu_0^{(r)}(0) \\ \vdots & & \vdots \\ \mu_r(0) & \cdots & \mu_r^{(r)}(0) \end{vmatrix} \times \Delta, \tag{13}$$

where

$$\Delta = \begin{vmatrix} \binom{1}{0}\omega' & \binom{2}{0}\omega'' & \cdots & \cdots & \binom{r+1}{0}\omega^{(r+1)} \\ \binom{1}{1}\omega & \binom{2}{1}\omega' & \cdots & \cdots & \binom{r+1}{1}\omega^{(r)} \\ 0 & \binom{2}{2}\omega & \cdots & \cdots & \binom{r+1}{2}\omega^{(r-1)} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \binom{r}{r}\omega & \binom{r+1}{r}\omega' \end{vmatrix}$$

evaluated at $z = 0$. The first determinant in (13) is a pure rational constant, so it only remains to calculate Δ.

4.5 Let $\nu = \omega^{-1} = f(z)^{\frac{r}{d}}$. For $f(0) \neq 0$, these are holomorphic functions of z near the origin. From

$$\omega\nu = (\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots)(\nu_0 + \nu_1 z + \nu_2 z^2 + \cdots) = 1,$$

we get the linear system

$$\omega_0 \nu_0 = 1, \qquad \sum_{i=0}^{k} \omega_i \nu_{k-i} = 0 \quad \text{for } 1 \leqslant k \leqslant r+1.$$

Solving for ν_{r+1} by Cramer's rule,

(14)
$$\nu_{r+1} = \frac{1}{\omega_0^{r+2}} \times \begin{vmatrix} \omega_0 & 0 & \cdots & \cdots & 1 \\ \omega_1 & \omega_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \omega_r & & & \omega_0 & 0 \\ \omega_{r+1} & \cdots & \cdots & \omega_1 & 0 \end{vmatrix} = \frac{(-1)^{r+1}}{\omega_0^{r+2}} \times \begin{vmatrix} \omega_1 & \omega_2 & \cdots & \cdots & \omega_{r+1} \\ \omega_0 & \omega_1 & \cdots & \cdots & \omega_r \\ 0 & \omega_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \omega_0 & \omega_1 \end{vmatrix},$$

where the second determinant is obtained by expanding the first by its last column and transposing.

4.6 On the other hand,

$$\Delta = \left| \binom{j+1}{k} \omega^{(j+1-k)} \right|_{0 \leqslant k, j \leqslant r} = \left| \frac{(j+1)!}{k!} \mathbb{1}_{\{k \leqslant j+1\}} \omega_{j+1-k} \right|_{0 \leqslant k, j \leqslant r},$$

where the characteristic function $\mathbb{1}_{\{k \leqslant j+1\}}$ assumes the value 1 if $k \leqslant j+1$, and 0 otherwise. This is the same as the rightmost determinant in (14), hence

(15)
$$g_0 \doteq (a_0^{1+\frac{r}{d}})^{r+1} \Delta \doteq (a_0^{1+\frac{r}{d}})^{r+1} \omega_0^{r+2} \nu_{r+1}.$$

Now recall that
$$\omega_0 = \omega(0) = a_0^{-\frac{r}{d}},$$
and
$$\nu_{r+1} = \frac{1}{(r+1)!} \nu^{(r+1)}(0) = \frac{1}{(r+1)!} \left[\frac{d^{r+1}}{dt^{r+1}} f(t)^{\frac{r}{d}} \right]_{t=0}.$$

The exponent of a_0 reduces to

$$\left(1 + \frac{r}{d}\right)(r+1) - \frac{r}{d}(r+2) = r + 1 - \frac{r}{d}.$$

Hence, by comparing (11) with (15), we get

$$g_0 \doteq h_0,$$

which completes the proof of Theorem 1.1. ■

If we keep track of the unwritten scalars in the intermediate stages, the connecting relation is seen to be

(16)
$$g_0 = \left\{ \frac{\prod_{i=0}^{r} i!\,(d+i-2)!}{[r \times (d-2)!]^{r+1}} \right\} h_0.$$

This, of course, implies a parallel relation between $\mathfrak{G}_{r,d}$ and $\mathcal{H}_{r,d}$.

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4.7 One can give a formula for the Hilbert covariant directly, without constructing its source first. Introduce binary variables $\mathbf{y} = \{y_1, y_2\}$.

Proposition 4.3 *We have an identity*

$$\mathcal{H}_{r,d}(\mathbb{F}) \doteq \frac{\mathbb{F}(x_1,x_2)^{r+1-\frac{r}{d}}}{(x_1y_2 - x_2y_1)^{r+1}} \Big[\Big(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} \Big)^{r+1} \mathbb{F}(x_1,x_2)^{\frac{r}{d}} \Big].$$

This is merely the homogenised version of the formula (9) combined with (10), so we will omit the proof.

5 The Three Ideals

Let $X = X_{e,d}$ be as in Section 3.1, with $I_X \subseteq R$ its homogeneous defining ideal. Let J (respectively $\mathfrak{g}$) denote the ideal in R generated by the coefficients of $\mathfrak{G}_{r,d}$ (respectively all possible $\mathfrak{G}_\Psi$). In other words, $\mathfrak{g}$ is the ideal generated by the maximal minors of a matrix representing the morphism $\alpha_{\mathbb{F}}\colon S_r \longrightarrow S_{r+d-2}$ from Section 3.2. There are inclusions

$$J \subseteq \mathfrak{g} \subseteq I_X.$$

The zero locus of each of these ideals is X, but depending on the values of r and d, either of these inclusions may be proper. Since I_X has nonzero elements in degree $e+1$ (arising from the coefficients of $\mathfrak{G}_{e,d}$), we must have a proper containment $\mathfrak{g} \subsetneq I_X$, whenever r does not divide d.

5.1 Suppose $r = 1$, so that X is the rational normal d-ic curve. We have a decomposition

$$R_2 \simeq S_2(S_d) \simeq \bigoplus_{n=0}^{\lfloor \frac{d}{2} \rfloor} S_{2d-4n},$$

where the summand S_{2d-4n} is spanned by the coefficients of $(\mathbb{F},\mathbb{F})_{2n}$. It is classically known that I_X is minimally generated in degree 2, and $(I_X)_2 \simeq \bigoplus_{n \geqslant 1} S_{2d-4n} \subseteq R_2$ (see [9]). By Proposition 3.2, we have $\mathfrak{g} = I_X$. Moreover, J and $\mathfrak{g}$ coincide for $d \leqslant 3$ and differ afterwards.

5.2 Assume $r = 3, d = 6$. One can explicitly calculate the ideal of $X = X_{3,6}$ using the following elimination-theoretic technique. Let $Q = (q_0, q_1, q_2, q_3 \between x_1, x_2)^3$, where the q_i are independent indeterminates. Write

$$(a_0, \dots, a_6 \between x_1, x_2)^6 = Q^2$$

and equate the corresponding coefficients on both sides. This gives expressions $a_i = f_i(q_0, \dots, q_3)$, defining a ring homomorphism

$$\mathfrak{f}\colon R \longrightarrow \mathbf{C}[q_0, \dots, q_3], \quad a_i \longrightarrow f_i(q_0, \dots, q_3).$$

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Then I_X is the kernel of $\mathfrak{f}$. We carried out this computation in the computer algebra system Macaulay-2 (henceforth M2); it shows that I_X is minimally generated by a 45-dimensional subspace of R_4.

In order to determine the piece $(\mathfrak{g})_4$, we need to list the degree 4 covariants of a generic binary quartic $\mathbb{B}$. By the Cayley–Sylvester formula,

$$S_4(S_4) = S_{16} \oplus S_{12} \oplus S_{10} \oplus (S_8 \otimes \mathbf{C}^2) \oplus (S_4 \otimes \mathbf{C}^2) \oplus S_0.$$

It is classically known (see [18, Section 89]) that each covariant of $\mathbb{B}$ is a polynomial in these fundamental covariants:

$$C_{1,4} = \mathbb{B}, \qquad C_{2,4} = (\mathbb{B},\mathbb{B})_2, \qquad C_{2,0} = (\mathbb{B},\mathbb{B})_4,$$
$$C_{3,6} = \big(\mathbb{B},(\mathbb{B},\mathbb{B})_2\big)_1, \qquad C_{3,0} = \big(\mathbb{B},(\mathbb{B},\mathbb{B})_2\big)_4,$$

where $C_{m,q}$ is of degree-order (m,q). Hence, the space of degree 4 covariants of $\mathbb{B}$ is spanned by

$$\Psi_{4,16} = C_{1,4}^4, \qquad \Psi_{4,12} = C_{1,4}^2 C_{2,4}, \qquad \Psi_{4,10} = C_{1,4}C_{3,6}, \qquad \Psi_{4,8}^{(1)} = C_{2,4}^2,$$
$$\Psi_{4,8}^{(2)} = C_{1,4}^2 C_{2,0}, \qquad \Psi_{4,4}^{(1)} = C_{1,4}C_{3,0}, \qquad \Psi_{4,4}^{(2)} = C_{2,4}C_{2,0}, \qquad \Psi_{4,0} = C_{2,0}^2.$$

We have calculated $\mathfrak{G}_\Psi$ in each case using the recipe of Section 3.4. It turns out that the ones coming from $\Psi_{4,16}, \Psi_{4,12}, \Psi_{4,0}$ are nonzero, whereas $\mathfrak{G}_{\Psi_{4,10}}$ vanishes identically. Moreover, we have identities

$$6\mathfrak{G}_{\Psi_{4,8}^{(1)}} = \mathfrak{G}_{\Psi_{4,8}^{(2)}}, \quad 29\mathfrak{G}_{\Psi_{4,4}^{(1)}} = 36\mathfrak{G}_{\Psi_{4,4}^{(2)}};$$

that is to say, both $\Psi_{(4,8)}^{(i)}$ lead to the same Göttingen covariant (up to a scalar), and similarly for $\Psi_{(4,4)}^{(i)}$. Hence

$$(\mathfrak{g})_4 \simeq S_{16} \oplus S_{12} \oplus S_8 \oplus S_4 \oplus S_0,$$

which is exactly 45-dimensional; this forces $\mathfrak{g} = I_X$.

5.3 Assume $r = 2$, and d even. Now [2, Theorem 7.2] says that I_X is minimally generated by cubic forms, and its generators are explicitly described there. If $d = 4$, then $(I_X)_3 \simeq S_6$, with the only piece coming from $\mathfrak{G}_{2,4}$. If $d = 6$, then

$$(I_X)_3 \simeq S_{12} \oplus S_8 \oplus S_6.$$

The three summands are respectively generated by the coefficients of:

$$\Phi_{3,12} = (\mathbb{F}^2,\mathbb{F})_3, \quad \Phi_{3,8} = (\mathbb{F}^2,\mathbb{F})_5, \quad \Phi_{3,6} = 33(\mathbb{F}^2,\mathbb{F})_6 - 250\big(\mathbb{F},(\mathbb{F},\mathbb{F})_2\big)_4.$$

Now, following the recipe of Section 3.4, one finds that

$$\mathfrak{G}_{\mathbb{B}^3} \doteq \Phi_{3,12}, \quad \mathfrak{G}_{\mathbb{B}(\mathbb{B},\mathbb{B})_2} \doteq \Phi_{3,8}, \quad \mathfrak{G}_{(\mathbb{B},(\mathbb{B},\mathbb{B})_2)_1} \doteq \Phi_{3,6};$$

and hence $\mathfrak{g} = I_X$ once again.

We have calculated several such examples, which suggest the following pair of conjectures.

Conjecture 5.1 *Assume that r divides d. Then*

(c1) *the ideal I_X is always minimally generated in degree $r+1$, and*
(c2) $\mathfrak{g} = I_X$.

At least for $r = 1, 2$, something much stronger than (c1) is true; namely I_X has Castelnuovo regularity $r+1$, and its graded minimal resolution is linear (see [2, Theorem 1.4]). We do not know of a counterexample to this when $r > 2$.

Referring to the diagram at the beginning of Section 3, note that the ideal of the Veronese embedding is generated by quadrics; but the projection π (which implicitly involves elimination theory) will tend to increase the degrees of the defining equations of its image.

5.4 The Saturation of J

In this section we will prove Theorem 1.2. We want to show that the ideal J defines X scheme-theoretically when r divides d, that is to say,

$$\operatorname{Proj} R/I_X \longrightarrow \operatorname{Proj} R/J$$

is an isomorphism of schemes. The following example should convey the essential idea behind the proof.

Assume $r = 2, d = 6$. Write $t_i = a_i/a_0$ for $1 \leqslant i \leqslant 6$. Let $A = \mathbf{C}[t_1, \dots, t_6]$, and consider the corresponding degree zero localisation $\mathfrak{a} = (J_{a_0})_0 \subseteq A$. The zero locus of $\mathfrak{a}$ is $X \setminus \{a_0 = 0\} \simeq \mathbb{A}^2$. Since the question is local on X, it would suffice to show that $A/\mathfrak{a}$ is isomorphic to a polynomial algebra $\mathbf{C}[v_1, v_2]$.

Now $\mathcal{H}_{2,d} \doteq \big(\mathbb{F}, (\mathbb{F}, \mathbb{F})_2\big)_1$, and we have explicitly written down its first few terms in Section 2.5. Note that the monomial $a_0^2 a_3$ occurs in its source, and similarly $a_0^2 a_4, a_0^2 a_5, a_0^2 a_6$ occur in the successive coefficients. Hence, modulo $\mathfrak{a}$, we have identities of the form

$$t_k = \text{a polynomial expression in } t_1, t_2, \dots, t_{k-1}, \quad \text{for } 3 \leqslant k \leqslant 6.$$

Thus we have a surjective ring morphism

$$\mathbf{C}[v_1, v_2] \longrightarrow A/\mathfrak{a}, \quad v_i \longrightarrow t_i.$$

Since Krull-dim$A/\mathfrak{a} = 2$, this must be an isomorphism.

For the general case, write $\mathcal{H}_{r,d}$ as in (8), and recall that $h_{k-(r+1)}$ is isobaric of weight k.

Lemma 5.2 *The coefficient of $a_0^r a_k$ in $h_{k-(r+1)}$ is nonzero for $r+1 \leqslant k \leqslant d$.*

Proof The monomial $a_0^r a_{r+1}$ can appear in h_0 only by one route, namely by applying the sequence

$$\Big[(d-r)a_{r+1}\frac{\partial}{\partial a_r}\Big] \circ \cdots \circ \Big[(d-1)a_2\frac{\partial}{\partial a_1}\Big] \circ \Big[d a_1 \frac{\partial}{\partial a_0}\Big]$$

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to $a_0^{\frac{r}{d}}$, and then multiplying by $a_0^{r+1-\frac{r}{d}}$. Hence its coefficient is nonzero. Now $a_0^r a_k$ can appear in $h_{k-(r+1)} \doteq E_+ h_{k-1-(r+1)}$ only by applying $(d-k+1)a_k \frac{\partial}{\partial a_{k-1}}$ to $a_0^r a_{k-1}$, so we are done by induction. ■

We can always change co-ordinates such that $a_0 \neq 0$ at any given point of X. Write $t_i = a_i/a_0$ and $\mathfrak{a} < A = \mathbf{C}[t_1, \dots, t_d]$ as above. By the lemma, each of $t_{r+1}, \dots, t_d$ is a polynomial in $t_1, \dots, t_r$ modulo $\mathfrak{a}$. This gives a bijection

$$\mathbf{C}[v_1, \dots, v_r] \longrightarrow A/\mathfrak{a}, \quad v_i \longrightarrow t_i,$$

which shows that the scheme $\operatorname{Proj} R/J$ is locally isomorphic to the affine space $\mathbb{A}^r$, and hence $J_{\text{sat}} = I_X$. This completes the proof of Theorem 1.2. ■

5.5 It follows that J and I_X coincide in sufficiently large degrees. Let $\mathfrak{S}(r,d)$ denote the saturation index of J, namely it is the smallest integer m_0 such that

$$J_m = (I_X)_m \quad \text{for all } m \geqslant m_0.$$

It would be of interest to have a bound on this quantity in either direction. It is proved in [9] that

$$\frac{1}{d-2}\sqrt{\frac{(d-1)(d^2-2)}{2}} \leqslant \mathfrak{S}(1,d) \leqslant d+2;$$

but those techniques do not seem to generalise readily to the case $r > 1$. We have obtained the following few values by explicit calculations in M2:

$$\begin{array}{lllll} \mathfrak{S}(2,4) = 3, & \mathfrak{S}(2,6) = 7, & \mathfrak{S}(2,8) = 9, & \mathfrak{S}(2,10) = 9, & \mathfrak{S}(2,12) = 10, \\ \mathfrak{S}(3,6) = 9, & \mathfrak{S}(3,9) = 11, & \mathfrak{S}(4,8) = 13. & & \end{array}$$

A similar (but larger) table for $r = 1$ is given in [9], where the value of $\mathfrak{S}$ is related to transvectant identities involving the Hessian.

5.6 Suppose $e_i = \gcd(r_i, d)$ for $i = 1, 2$. Then $X_{e_1,d} \subseteq X_{e_2,d}$ exactly when $e_1 | e_2$. However, the containment relations between the ideals $J_{r_i,d}$ are not altogether obvious. For $J_{r_1,d} \supseteq J_{r_2,d}$ to be true, it is necessary that $r_1 \leqslant r_2$ and $e_1 | e_2$, but these conditions are not sufficient. For instance, we have obtained the following miscellaneous data by calculating these ideals in M2:

$$\begin{array}{lll} J_{2,5} \not\supseteq J_{3,5}, & J_{3,5} \not\supseteq J_{4,5}, & J_{2,5} \supseteq J_{4,5}, \\ J_{4,5} \not\supseteq J_{6,5}, & J_{2,4} \supseteq J_{6,4}, & J_{6,4} \supseteq J_{10,4}, \end{array}$$

which at least shows that the general pattern is not so easily guessed. Nevertheless, we have the following modest result.

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Proposition 5.3 *There are inclusions $J_{1,d} \supseteq J_{r,d}$ for arbitrary d, and $r = 2, 3, 4$.*

Proof It is clear from the formula for a transvectant (see Section 2.2), that if the coefficients of A belong to an ideal, then all the coefficients of $(A, B)_k$ also belong to this ideal. Hence, given any covariants $\Phi_1, \dots, \Phi_n$ of $\mathbb{F}$, all the coefficients of any transvectant of the form

$$\Big(\dots\big(((\mathfrak{G}_{1,d}, \Phi_1)_{k_1}, \Phi_2\big)_{k_2}, \dots, \Phi_n\Big)_{k_n}$$

are in $J_{1,d}$. Thus the result would follow if we could obtain $\mathfrak{G}_{r,d}$ as a linear combination of such expressions.

Observe the formulae in (7). It is clear that $\mathfrak{G}_{2,d}$ is itself such an expression. Let $r = 4$, then this is also true of the first term in $\mathfrak{G}_{4,d}$. Now the so-called Gordan syzygies give relations between cubic covariants of $\mathbb{F}$. In particular, the syzygy which is written as $\left(\begin{smallmatrix} \mathbb{F} & \mathbb{F} & \mathbb{F} \\ d & d & d \\ 0 & 1 & 4 \end{smallmatrix}\right)$ in the notation of [18, Ch. IV], gives an identity

$$\big(\mathbb{F}, (\mathbb{F}, \mathbb{F})_4\big)_1 = \frac{2(2d-5)}{d-4}\big(\mathbb{F}, (\mathbb{F}, \mathbb{F})_2\big)_3,$$

for any $d \geqslant 5$. Hence the same follows for the second term. (If $d \leqslant 4$, then the second term is identically zero.) This proves the result for $r = 4$.

The argument for $r = 3$ is similar. The first term in $\mathfrak{G}_{3,d}$ is already of the required form. Moreover, we have an identity

$$\mathbb{F}^2(\mathbb{F}, \mathbb{F})_4 = \frac{d(2d-5)}{(d-3)(2d-1)}(\mathbb{F}, \mathbb{F})_2^2 + \frac{2(2d-5)}{d-3}\big(\mathbb{F}^2, (\mathbb{F}, \mathbb{F})_2\big)_2,$$

for $d \geqslant 4$. (This can be shown by a routine but tedious symbolic calculation as in [18, Chapter IV-V].) Hence the same is true of the second term, which completes the proof. ∎

Since the argument depends on specific features of these formulae, it seems unlikely that this technique will generalise substantially. Even so, we suspect that the proposition may well be true of all r.

5.7 The Twisted Cubic Curve

Assume $d = 3$, and r arbitrary (but not divisible by 3). Then $X \subseteq \mathbf{P}^3$ is the twisted cubic curve. Since $\mathbb{B}$ is a linear form, the only possibility for Ψ is $\mathbb{B}^{r+1}$, hence $J = \mathfrak{g}$. It follows that the Hilbert–Burch complex (see [13, Section 20]) of $\alpha_{\mathbb{F}}$ gives a resolution

$$0 \leftarrow R/J \leftarrow R \leftarrow R(-r-1) \otimes S_{r+1} \leftarrow R(-r-2) \otimes S_r \leftarrow 0.$$

Its first syzygy shows that we have an identity $(\mathfrak{G}_{r,3}, \mathbb{F})_2 = 0$. (The correspondence between syzygies and transvectant identities is discussed in [8, Section 4].) The scheme $\operatorname{Proj} R/J$ has degree $\binom{r+2}{2}$, that is to say, it is a nonreduced $\frac{(r+1)(r+2)}{6}$-fold structure on X for $r > 1$.

We have $\sqrt{J_{r,3}} = I_X$ for any r. Some experimental calculations in M2 suggest the following narrow but interesting conjecture.

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Conjecture 5.4 *There is an inclusion $(I_X)^r \subseteq J_{r,3}$, and moreover r is the smallest such power.*

This problem is related to identities between the covariants of a generic cubic form. For instance, we have an identity

$$\mathfrak{G}_{1,3}^2 = -\frac{1}{2}(\mathbb{F}, \mathfrak{G}_{2,3})_1,$$

which can be verified by a direct symbolic computation. This immediately shows that $(I_X)^2 \subseteq J_{2,3}$. (Compare the argument of Proposition 5.3 above.)

In general, if r and d are coprime, then $\mathfrak{g}$ is a perfect ideal of height $d-1$, which is resolved by the Eagon–Northcott complex (see [13, Appendix 2]) of $\alpha_{\mathbb{F}}$. By the Porteous formula (see [5, Chapter II, Section 4]), the scheme $\operatorname{Proj} R/\mathfrak{g}$ supported on the rational normal d-ic curve has degree $\binom{r+d-1}{d-1}$.

6 The Clebsch Transfer Principle

In this section we generalise the Göttingen covariants to n-ary forms.

6.1 Let W be an n-dimensional complex vector space with basis $\mathbf{x} = \{x_1, \dots, x_n\}$, and a natural action of the group $\mathrm{SL}(W)$. Given an n-tuple of nonnegative integers $I = (i_1, \dots, i_n)$ adding up to d, let

$$\binom{d}{I} = \frac{d!}{\prod\limits_k i_k!}, \qquad x^I = \prod x_k^{i_k}.$$

We write a generic form of order d in the $\mathbf{x}$ as

$$\Gamma = \sum_I \binom{d}{I} a_I x^I,$$

where the a_I are independent indeterminates. As in the binary case, the $\{a_I\}$ can be seen as forming a basis of $S_d W^*$. Define the symmetric algebra

$$\mathcal{A} = \bigoplus_{m \geqslant 0} S_m(S_d W^*) = \mathbf{C}[\{a_I\}]$$

so that $\operatorname{Proj} \mathcal{A} = \mathbf{P} S_d \simeq \mathbf{P}^{\binom{d+n-1}{d}-1}$ is the space of n-ary d-ics.

6.2 Each irreducible representation of $\mathrm{SL}(W)$ is a Schur module of the form $S_\lambda = S_\lambda W$, where λ is a partition with at most $n-1$ parts (see [16, Section 15]). Moreover, we have an isomorphism

$$S_{(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, 0)} W \simeq S_{(\lambda_1, \lambda_1 - \lambda_{n-1}, \dots, \lambda_1 - \lambda_2, 0)} W^*.$$

An inclusion $S_\lambda W^* \subseteq \mathcal{A}_m$ corresponds to a morphism

$$S_0 \hookrightarrow \mathcal{A}_m \otimes S_\lambda W,$$

and then the image of $1 \in S_0$ will be called a concomitant of Γ of degree m, and type λ.

6.3 In the case of ternary forms, for $\lambda = (\lambda_1, \lambda_2)$, we have an embedding (see [16, Section 15])

$$S_\lambda \subseteq S_{\lambda_2}(\wedge^2 W) \otimes S_{\lambda_1 - \lambda_2}.$$

Using the basis $u_1 = x_1 \wedge x_2, u_2 = x_2 \wedge x_3, u_3 = x_3 \wedge x_1$ for $\wedge^2 W \simeq W^*$, we can write the concomitant as a form of degree m in the a_I, degree $\lambda_1 - \lambda_2$ in $\mathbf{x}$, and degree λ_2 in $\mathbf{u}$. For instance, assume $m = 2, d = 3$. We have a plethysm decomposition $S_2(S_3^*) \simeq S_6^* \oplus S_{4,2}^*$, and hence (up to a scalar) a unique morphism

$$S_0 \hookrightarrow \mathcal{A}_2 \otimes S_{4,2}.$$

If we symbolically write $\Gamma = a_{\mathbf{x}}^3 = b_{\mathbf{x}}^3$, then this concomitant is $(abu)^2 a_{\mathbf{x}} b_{\mathbf{x}}$. We refer the reader to [18, Chapter XII] or [23] for the symbolic calculus of n-ary forms and their concomitants.

6.4 The "Clebsch transfer principle" is a type of construction used to lift a binary covariant to a concomitant of n-ary forms in a geometrically natural way. As such, it comes in many flavours depending on the specifics of the geometric situation in play. (See [6, Section 4], [12, Section 3.4.2] or [18, Section 215] for various descriptions of this principle.) Clebsch's own statement of this technique may be found in [10, p. 28], but Cayley and Salmon seem to have been aware of it earlier (see [29, p. 28]).

The following example should convey an idea of how the transfer principle is used. Let $n = 3$ and $d = 4$, so that $\mathbf{P}S_4 \simeq \mathbf{P}^{14}$ is the space of quartic plane curves. Let $Z \subset \mathbf{P}^{14}$ be the 5-dimensional subvariety of double conics, *i.e.*,

$$Z = \{[\Gamma] \in \mathbf{P}S_4 : \Gamma = Q^2 \text{ for some ternary quadratic } Q\}.$$

A line L in the plane $\mathbf{P}W^* \simeq \mathbf{P}^2$ will intersect a general quartic curve $\Gamma(x_1, x_2, x_3) = 0$ in four points, which become two double points when $\Gamma \in Z$. With the identification $L \simeq \mathbf{P}^1$, let $\Gamma|_L$ denote the "restriction" of Γ to L, regarded as a binary quartic form. Hence the "function"

$$L \longrightarrow \mathfrak{G}_{2,4}(\Gamma|_L)$$

should vanish identically when $\Gamma \in Z$.

In order to make this precise, write $p = [p_1, p_2, p_3], q = [q_1, q_2, q_3]$, where p_i, q_i are indeterminates. We think of a generic L as spanned by the points $p, q \in \mathbf{P}^2$, and thus L has line co-ordinates

$$u_1 = \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}, \quad u_2 = \begin{vmatrix} p_2 & q_2 \\ p_3 & q_3 \end{vmatrix}, \quad u_3 = \begin{vmatrix} p_3 & q_3 \\ p_1 & q_1 \end{vmatrix}.$$

Introduce binary variables $\lambda = \{\lambda_1, \lambda_2\}$, and substitute $x_i = \lambda_1 p_i + \lambda_2 q_i$ in Γ to get a form Θ (which represents the restriction). Now evaluate $\mathfrak{G}_{2,4}$ on Θ by regarding the latter as a binary form in the λ; then the final result is the required lift $\widetilde{\mathfrak{G}}_{2,4}$. The actual symbolic calculation proceeds as follows. Let

$$\Gamma = a_{\mathbf{x}}^4 = b_{\mathbf{x}}^4 = c_{\mathbf{x}}^4,$$

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where $a_{\mathbf{x}} = a_1x_1 + a_2x_2 + a_3x_3$, *etc.* After substitution, $a_{\mathbf{x}}$ becomes $\lambda_1 a_p + \lambda_2 a_q$, which we rewrite as

$$\alpha_\lambda = \alpha_1\lambda_1 + \alpha_2\lambda_2 \quad \text{where } \alpha_1 = a_p, \alpha_2 = a_q,$$

and similarly $b_{\mathbf{x}} = \beta_\lambda, c_{\mathbf{x}} = \gamma_\lambda$. Thus $\Theta = \alpha_\lambda^4 = \beta_\lambda^4 = \gamma_\lambda^4$. Recall from Section 3.6 that

$$\mathfrak{G}_{2,d}(\Theta) \doteq \big(\Theta, (\Theta, \Theta)_2\big)_1 \doteq (\alpha\beta)^2(\alpha\gamma)\alpha_\lambda\beta_\lambda^2\gamma_\lambda^3. \tag{17}$$

Now

$$(\alpha\beta) = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = a_pb_q - b_pa_q = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = (abu),$$

and similarly for the bracket factor $(\alpha\gamma)$. Hence we arrive at the expression

$$\widetilde{\mathfrak{G}}_{2,4}(\Gamma) = (abu)^2(acu)a_{\mathbf{x}}b_{\mathbf{x}}^2c_{\mathbf{x}}^3, \tag{18}$$

which is a concomitant of degree 3 and type $(9, 3)$. We have the property

$$[\Gamma] \in Z \iff \widetilde{\mathfrak{G}}_{2,4}(\Gamma) \text{ vanishes identically as a polynomial in } \mathbf{x} \text{ and } \mathbf{u}.$$

The implication $\Rightarrow$ follows by construction. The converse says that if $\Gamma = 0$ were not a double conic, then a line could be found which does not intersect it in two double points. This is clear on geometric grounds.

6.5 The case of a general Göttingen covariant is similar. Assume that $\mathfrak{G}_\Psi$ is of degree $r+1$, order q, and weight $w = \frac{(r+1)d-q}{2}$. Let $p = [p_1, \dots, p_n]$ and $q = [q_1, \dots, q_n]$, substitute

$$x_i = \lambda_1 p_i + \lambda_2 q_i, \quad (1 \leqslant i \leqslant n), \tag{19}$$

into Γ, and evaluate $\mathfrak{G}_\Psi$ on the new binary form in the λ variables. The resulting concomitant $\widetilde{\mathfrak{G}}_\Psi$ is of degree $r+1$ and type $(q+w, w)$. If $\Gamma = G^\mu$, then $\widetilde{\mathfrak{G}}_\Psi(\Gamma)$ vanishes identically for the same reason as above.

If $\mathfrak{G}_\Psi$ is written as a symbolic expression in $r+1$ binary letters $a, b, \dots$ and their brackets (ab), *etc.*, then $\widetilde{\mathfrak{G}}_\Psi$ is obtained by simply treating them as n-ary letters and replacing the corresponding brackets by (abu), *etc.* This follows immediately by tracing the passage from (17) to (18). In particular, the concomitant in Section 6.3 is the Clebsch transfer of the Hessian of a binary cubic. The formal symbolic expression for $\widetilde{\mathfrak{G}}_\Psi$ does not depend on n, although its interpretation certainly does.

Theorem 6.1 *Let Γ be an n-ary d-ic. Then $\widetilde{\mathfrak{G}}_{r,d}(\Gamma)$ is identically zero, if and only if $\Gamma = G^\mu$ for some n-ic G of order e.*

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Proof The "if" part follows from the discussion above. Let $\Gamma = \prod H_i^{\nu_i}$ be the prime decomposition, where H_i is an irreducible form of degree c_i. A general line L will intersect each hypersurface $H_i = 0$ in c_i distinct points. If Γ cannot be written as G^μ, then at least one ν_i is not divisible by μ. Altogether L intersects $\Gamma = 0$ in $c_1 + c_2 + \cdots$ points, at least one of which occurs with multiplicity not divisible by μ. Thus $\widetilde{\mathfrak{G}}_{r,d}(\Gamma)$ will not vanish if the $\mathbf{u}$ variables in it are specialised to the Plücker co-ordinates of a general L. ■

6.6 This is a continuation of Section 6.4. We have calculated the homogeneous defining ideal of Z using a procedure similar to the one in Section 5.2, and it turns out that I_Z is minimally generated by a 218-dimensional space of forms in degree 3. We have a plethysm decomposition

$$\mathcal{A}_3 = S_3(S_4^*) \simeq S_{12}^* \oplus S_{10,2}^* \oplus S_{9,3}^* \oplus S_{8,4}^* \oplus S_6^* \oplus S_{6,3}^* \oplus S_{6,6}^* \oplus S_{4,2}^* \oplus S_0^*,$$

where the summands are of respective dimensions

$$91, 162, 154, 125, 28, 64, 28, 27, 1.$$

Now $(I_Z)_3$ is a subdirect sum of the above, and we already know that $S_{9,3}^*$ is one of its pieces. This forces $(I_Z)_3 \simeq S_{9,3}^* \oplus S_{6,3}^*$ on dimensional grounds. Hence there is a concomitant of type $(6,3)$ vanishing on Z. We have checked by a direct calculation that it can be written as

$$(abc)(abu)^2(acu)b_{\mathbf{x}}c_{\mathbf{x}}^2.$$

In fact, all that needs to be checked is that this symbolic expression is not identically zero, which can be done by specialising Γ. This suffices, since we have, up to a scalar, only one concomitant of this type in degree 3.

Recall from Section 5.3 that for $r = 2, d = 4$, that there are no Göttingen covariants other than $\mathfrak{G}_{2,4}$. Hence we have found a concomitant vanishing on Z which is not the Clebsch transfer of any binary covariant.

Let $J \subseteq \mathcal{A}$ denote the ideal generated by the coefficients of $\widetilde{\mathfrak{G}}_{2,4}$. We have checked using M2 that the saturation of J is I_Z, and moreover the two ideals coincide in degrees $\geqslant 7$. But in general, we do not know whether there is an analogue of Theorem 1.2 in the n-ary case.

6.7 We end with an example which is at least a pleasing curiosity. Assume that $\Gamma = 0$ is a *nonsingular* plane quartic curve. A line $L \subset \mathbf{P}^2$ with co-ordinates $[u_1, u_2, u_3]$ passes through the points $p = [u_3, 0, -u_1], q = [u_2, -u_1, 0]$, and moreover these points are distinct (and well-defined) when $u_1 \neq 0$. Now make substitutions into $\widetilde{\mathfrak{G}}_{2,4}(\Gamma)$ as in (19) to get a binary sextic $\mathcal{E}_1(\lambda)$; it represents the binary form $\mathfrak{G}_{2,4}$ as living on $L \simeq \mathbf{P}^1$. (This is no longer correct if $u_1 = 0$, hence in order to avoid spurious solutions, we also need to consider the forms $\mathcal{E}_2(\lambda), \mathcal{E}_3(\lambda)$ similarly obtained from

$$p = [0, u_3, -u_2], q = [u_2, -u_1, 0], \quad p = [0, u_3, -u_2], q = [u_3, 0, -u_1].)$$

Now, all the $\mathcal{E}_i(\lambda)$ are identically zero exactly when $\{\Gamma = 0\} \cap L$ represents two double points, *i.e.*, when L is a bitangent to the curve defined by Γ. Let $B = \mathbf{C}[u_1, u_2, u_3]$ denote the co-ordinate ring of the dual plane, and $\mathfrak{b}_\Gamma \subseteq B$ the ideal generated by the coefficients of all the monomials in λ for $\mathcal{E}_i(\lambda)$, $i = 1, 2, 3$. Then the zero locus of $\mathfrak{b}_\Gamma$ is the set of 28 points (see [12, Chapter 6]) corresponding to the bitangents of the curve. We have verified in M2 that $\mathfrak{b}_\Gamma$ is not saturated, but its saturation has resolution

$$0 \leftarrow B/(\mathfrak{b}_\Gamma)_{\text{sat}} \leftarrow B \leftarrow B(-7)^8 \leftarrow B(-8)^7 \leftarrow 0,$$

which is characteristic of 28 general points in the plane (see [14, Chapter 3]). In much the same way, the concomitant in Section 6.3 can be used to give equations for the nine inflexional tangents of a nonsingular plane cubic curve.

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