CORRIGENDUM

*to the paper*

# ON SOME CLASSES OF WEIGHTED COMPOSITION OPERATORS

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**0. Abstract.** We provide corrected versions of Theorems 1 and 2 and Corollaries 3(a) and 4 of the paper mentioned in the title.

**I. Basic results.** In our paper [**2**], Theorems 1 and 2 and Corollaries 3(a) and 4 contain errors. Assuming the notation of that paper the following are correct versions of Theorems 1 and 2, respectively.

THEOREM I. *$W_T$ is normal if and only if*
(i) $wE(w)h \circ T = hE(w^2) \circ T^{-1}$ *a.e.*
*and*
(ii) $T^{-1}\Sigma \cap \operatorname{supp} w = \Sigma \cap \operatorname{supp} w$.

THEOREM II. *$W_T$ is quasinormal if and only if $h \circ TE(w^2) = hE(w^2) \circ T^{-1}$ a.e. on the support of $w$.*

REMARKS. 1. For the proofs of these theorems, as well as other examples, see [**3**].

2. The original attempt to prove Theorem 2 in [**2**] yields Theorem II provided one observes a factor of $w$ in each term of $VM$ and $MV$, where $M = |W_T * W_T|^{1/2}$ and $V$ is the partial isometry which gives the unique, canonical polar form $VM = W_T$.

3. The fallacy in the proof of Theorem 1 in [**2**] was the claim that for a normal $W_T$, the set $A$ = support of $w$ (written supp $w$) satisfied $T^{-1}A = A$. This was never proved and may in fact be false; see Example 1 of [**3**], or Example 1 below.

4. All the other results in [**2**] except Corollaries 3(a) and 4, which we deal with below, are correct.

Here is a corrected statement and proof of Corollary 3(a) from [**2**]. For ease of proof we assume $w \geq 0$ a.e. . The general complex case is clear and easily obtained.

COROLLARY 3(a). *Suppose $T$ is a non-invertible, conservative and ergodic measure-preserving transformation. Then $W_T$ is not normal for any (non-zero) choice of $w$.*

*Proof.* Given such a $T$, suppose $W_T$ is normal. It follows (see [**3**]) that $A$, the support of $w$, satisfies $A \subseteq T^{-1}A$. Since $T$ is conservative, it must be the case that $A = T^{-1}A$. Since $T$ is ergodic we have either $\mu(A) = 0$ or $\mu(X \backslash A) = 0$. In the first case $w = 0$. In the second case, by (ii) of the above Theorem II, $T$ must be invertible.

REMARK. In the case of finite measure, every measure-preserving transformation is conservative, and this Corollary was know to Bastian (although his proof rests on different principles; see [**1**]). In any case, it presents an interesting dichotomy for measure-preserving transformations. If a measure-preserving transformation $T$ is invertible, the composition operator it induces is unitary; however if $T$ is not invertible the induced composition operator is not even normal. Corollary 3(a) says that for such a $T$, being conservative and ergodic implies that the composition operator cannot even be *weighted* to become normal. We do not know what is possible for non-conservative $T$.

We turn now to Corollary 4 of **[2]**, which attempts to characterize the Hermitian weighted composition operators. We have the following example.

EXAMPLE 1. Let $X = \{0, 1, 2\}$, $\Sigma = 2^X$, $\mu(x) = 1$ for each $x \in X$. Define $T: X \to X$ by $T(0) = 1$, $T(1) = 2$, and $T(2) = 1$. Set $w = \chi_A$, where $A = \{1, 2\}$. Then direct calculations show that for all $f \in L^2(X)$, $W_T f(0) = 0$, $W_T f(1) = f(2)$, and $W_T f(2) = f(1)$. Also $W_T^* f(0) = h(0)E(wf) \circ T^{-1}(0) = 0$, and similarly $W_T^* f(1) = f(2)$, $W_T^* f(2) = f(1)$, so that $W_T$ is hermitian. However, $T$ is not of period 2 (in fact $T$ is not invertible) and $h\bar{w} \circ T \neq w$ (in fact $w$ is not even $T^{-1}\Sigma$ measurable), contradicting Corollary 4 of **[2]**.

The aspect of the proof of Corollary 4 in **[2]** which fails is again the fact that the support of the weight function may not be invariant under $T$. We resolve this in the following manner.

Given $W_T$, set $A =$ support of $w$ and define $T_A$ as the restriction of $T$ to $A$. Thus if $B \subseteq A$, $T_A^{-1}B = T^{-1}B \cap A$.

COROLLARY 4. *$W_T$ is Hermitian if and only if*

(i) *$T_A$ is periodic of period* 2

*and*

(ii) $w = hE(\bar{w}) \circ T^{-1}$.

*Proof.* Suppose $W_T$ is Hermitian; then $W_T$ is normal. Thus (see **[3]**) we know that $T$ maps $A$ into $A$ (so that $A \subseteq T^{-1}A$), $L^2(A)$ is reducing for $W_T$, $\ker W_T = L^2(X \backslash A)$, and $A = \operatorname{supp} hE(|w|^2) \circ T^{-1}$.

Setting $W_T = W_T^*$ yields

(1) $wf \circ T = hE(\bar{w}f) \circ T^{-1}$ for all $f \in L^2(X, \Sigma, \mu)$.

Choose an increasing sequence of measurable sets $\{C_n\}$, each of finite measure, whose union is all of $X$. Setting $f = \chi_{C_n}$ in (1) and letting $n \to \infty$ we obtain (ii). Since $W_T^2 = W_T^* W_T$ we obtain

(2) $ww \circ Tf \circ T^2 = hE(|w|^2) \circ T^{-1}f$, for all $f \in L^2(A)$.

Now choose an increasing sequence of measurable subsets $\{C_n\}$ of $A$, each of finite measure, whose union is all of $A$. Setting $f = \chi_{C_n}$ in (2) and letting $n \to \infty$ we obtain

(3) $ww \circ T = hE(|w|^2) \circ T^{-1}$.

Dividing both sides of (2) by $ww \circ T$ (since both sides are supported in $A$ we leave everything 0 off of $A$) yields

(4) $\chi_A f \circ T^2 = f$ for all $f \in L^2(A)$.

In particular, for each measurable subset $C$ of $A$ of finite measure,

(4)′ $\chi_A \chi_C \circ T^2 = \chi_C$,

i.e., $T_A^2 = T_A$.

Conversely, suppose (i) and (ii) hold. Then (i) implies that $T$ maps $A$ into $A$ so that $A \subseteq T^{-1}A$. Combining this with (ii) we have

(5) $ww \circ T = hE(\bar{w}) \circ T^{-1} w \circ T = hE(\bar{w}w \circ T^2) \circ T^{-1} = hE(|w^2|) \circ T^{-1}$,

since $\bar{w}w \circ T^2 = \bar{w}\chi_A w \circ T^2 = \bar{w}w \circ T_A^2 = \bar{w}w = |w|^2$. In particular, $A = \operatorname{supp} hE(|w|^2) \circ T^{-1}$.

But it is always true that $\ker W_T = L^2(X\backslash \operatorname{supp} hE(|w|^2)\circ T^{-1})$, so $W_T f = 0$ for all $f \in L^2(X\backslash A)$. On the other hand, $W_T^* f = hE(\bar{w}f)\circ T^{-1} = 0$ if $f \in L^2(X\backslash A)$. Finally for each $f \in L^2(A)$ we have $W_T f = wf\circ T = hE(\bar{w})\circ T^{-1} f\circ T = hE(\bar{w}f\circ T^2)\circ T^{-1} = hE(\bar{w}f\circ T_A^2)\circ T^{-1} = hE(\bar{w}f)\circ T^{-1} = W_T^* f$, so that $W_T$ is Hermitian.

## REFERENCES


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2. J. T. Campbell and J. E. Jamison, On some classes of weighted composition operators, *Glasgow Math. J.* **32** (1990), 87–94.

3. J. T. Campbell, M. Embryo-Wardrop, R. Fleming and N. Sivaramakrishnan, Normal and quasinormal weighted composition operators, pre-print.



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