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L_p-EQUIVALENCE OF A LINEAR AND A NONLINEAR IMPULSIVE DIFFERENTIAL EQUATION IN A BANACH SPACE

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In the present paper by means of the Schauder–Tychonoff principle sufficient conditions are obtained for L_p-equivalence of a linear and a nonlinear impulsive differential equations.

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1. Introduction

The impulsive differential equations are an adequate apparatus for the mathematical simulation of evolution processes which, during their evolution, are subject to short-time perturbations. They are applied successfully in dynamically developing branches of science and technology such as theoretical physics, ecology, impulse technology, industrial robotics, etc [1]. The impulsive differential equations can be successfully used to the mathematical simulation of biotechnological processes [2]. Consider the equation of Verhulst

$$\frac{dN}{dt}=\frac{\mu N}{K}(K-N)$$

where by $N=N(t)$ the biomass of a given population at the moment $t\geqq 0$ is denoted, K is the capacity of the environment and μ is the difference between the birth-rate and death-rate.

The case when external disturbances act upon the population is often met. We shall consider the cases when the external disturbances take place at fixed moments of time and are expressed as adding to or taking off certain quantities of biomass. The impulsive analogue of the equation of Verhulst in this case has the form

$$\frac{dN}{dt}=\frac{\mu}{K}N(K-N) \qquad (t\neq t_i)$$

$$\Delta N(t)\big|_{t=t_i}=N(t_i+0)-N(t_i-0)=-I_i \qquad (i=1,2,3,\ldots)$$

where $0<t_1<t_2<t_3<\cdots$ are the moments of external effect, I_i, $i=1,2,3,\dots$ are the amounts of biomass added to ($I_i<0$) or taken off ($I_i>0$) at the moments $t_1, t_2, t_3,\dots$

However, the theory of systems with impulse effect develops rather slowly due to the presence of phenomena such as merging of the solutions, bifurcations, dying of the solutions, "beating" of the solutions and loss of the property of autonomy. The beginning of the mathematical theory of these equations was put in 1960 by the work of V. D. Mil'man and A. D. Myshkis [3], and the works [4], [5] mark the beginning of the mathematical theory of these equations in abstract spaces. We shall note that [6] is the first monograph devoted to this subject.

In the present paper sufficient conditions for the L_p-equivalence of the impulsive equations

$$\frac{dx}{dt}=A(t)x \qquad (t\neq t_n, n, 1,2,3,\dots) \tag{1}$$

$$x(t_n+0)=Q_nx(t_n) \qquad (n=1,2,3,\dots) \tag{2}$$

and

$$\frac{dx}{dt}=A(t)x+f(t,x) \qquad (t\neq t_n, n=1,2,3,\dots) \tag{3}$$

$$x(t_n+0)=Q_nx(t_n)+R_nx(t_n) \qquad (n=1,2,3,\dots) \tag{4}$$

and found, where:

$\tilde{\tau}=\{t_n\}_{n=1}^{\infty}$ is a sequence of points satisfying the condition

$$0<t_1<\cdots<t_n<t_{n+1}<\cdots, \lim_{n\to\infty} t_n=\infty \tag{5}$$

$A:[0,\infty)\to L(X)$ is a continuous operator-valued function, where $L(X)$ is the space of linear bounded operators acting in the complex Banach space X,

$\{Q_n\}_{n=1}^{\infty}$ is a sequence of operators such that $Q_n\in L(X)$, $n=1,2,3,\dots$

the function $f:[0,\infty)xX\to X$ is continuous,

$\{R_n\}_{n=1}^{\infty}$ is a sequence of continuous operators acting in X.

We shall note that similar results for equations without impulses were obtained in [7–9].

2. Statement of the problem

Definition 1. The function $\phi(t)\,(0\leqq t<\infty)$ is said to be a *solution* of the impulsive equation (1), (2) if for $t\notin\tilde{\tau}$ it satisfies the ordinary differential equation

$$\frac{dx}{dt} = A(t)x \tag{6}$$

for $t \in \tilde{\tau}$ it satisfies the condition of a "jump" (2), and at the points $t \in \tilde{\tau}$ it is continuous from the left.

In an analogous way the notion of a *solution* of the impulsive equations (3), (4) is introduced.

For the impulsive equation there exists an evolution operator $U(t, \tau)\,(0 \leqq \tau \leqq t < \infty)$ which associates with each element $x_0 \in X$ a solution $x(t) = U(t, \tau)x_0$ of (1), (2) such that $x(\tau) = x_0$.

Lemma 1. *Let the function* $A(t)$ *be continuous for* $0 \leqq t < \infty$. *Then the evolution operator* $U(t, \tau) < 0 \leqq \tau \leqq t < \infty)$ *of the impulsive equation* (1), (2) *has the form*

$$U(t,\tau) = \begin{cases} U_0(t,\tau) & t_n < \tau \leqq t \leqq t_{n+1} \\ U_0(t,t_n)\left(\prod\limits_{j=n}^{k+1} Q_j U_0(t_j, t_{j-1})\right) Q_k U_0(t_k, \tau) & \\ t_{k-1} < \tau \leqq t_k < t_n < t \leqq t_{n+1} & \end{cases}$$

where $U_0(t, \tau)$ *is the evolution operator of equation* (6).

Lemma 1 is proved by a straightforward verification.

The operator-valued function $U(t, \tau)\,(0 \leqq \tau \leqq t < \infty)$ has the semi-group properties

$$U(t,t) = I,\ U(t,\tau) = U(t,s)U(s,\tau) \qquad (0 \leqq \tau \leqq s \leqq t < \infty) \tag{7}$$

and is continuous at the points $(t, \tau) \neq (t_n, \tau)$ and $(t, \tau) \neq (t, t_n)\,(n = 1, 2, 3, \ldots)$. For $n = 1, 2, 3, \ldots$ the following equalities hold

$$U(t_n + 0, \tau) = Q_n U(t_n, \tau) \qquad (0 \leqq \tau \leqq t_n < \infty,\ n = 1, 2, 3, \ldots) \tag{8}$$

$$U'_t(t,\tau) = A(t)U(t,\tau),\ U'_\tau(t,\tau) = U(t,\tau)A(\tau). \tag{9}$$

We shall say that condition (H1) is satisfied if the following condition holds:

H1. The operators Q_n have bounded inverse operators $Q_n^{-1}\,(n = 1, 2, 3, \ldots)$.

Lemma 2. *Let the following conditions be satisfied:*

1. *The function* $A(t)$ *is continuous for* $0 \leqq t < \infty$.
2. *Condition* (*H1*) *holds.*

Then the evolution operator $U(t,\tau)$ *for* $0 \leqq t,\ \tau < \infty$ *has the form*

$$U(t,\tau) = \begin{cases} U_0(t,\tau), & t_n < t \leqq \tau \leqq t_{n+1}; \\ U_0(t,t_n)\left(\prod_{j=n}^{k+1} Q_j U_0(t_j; t_{j-1})\right) Q_k U_0(t_k,\tau), & \\ t_{k-1} < \tau \leqq t_k < t_n < t \leqq t_{n+1}; & \\ U_0(t,t_n)\left(\prod_{j=n}^{k-1} Q_j U_0(t_j, t_j+1)\right) Q_k^{-1} U_0(t_k,\tau), & \\ t_{n-1} < t \leqq t_n < t_k < \tau \leqq t_{k+1}. & \end{cases}$$

Lemma 2 is proved by a straightforward verification.

If condition (H1) holds, the following equalities are valid:

$$U(t,\tau) = U^{-1}(\tau,t),\ U(t,\tau) = U(t,s)U(s,\tau) \qquad (0 \leqq \tau, s, t < \infty) \tag{10}$$

$$U(t_n+0,\tau) = Q_n U(t_n,\tau) \qquad (0 \leqq t_n, s < \infty). \tag{11}$$

We shall say that condition (H2) is satisfied if the following condition holds:

H2. There exists numbers l and λ such that any interval of length l contains not more than λ points of the sequence $\tilde{\tau}$.

Condition (H2) is fulfilled, for instance, if

$$\overline{\lim_{\tau \to \infty}} \sup_{0 \leqq t > \infty} \frac{i(t,t+\tau)}{\tau} < \infty \tag{12}$$

where $i(a,b)$ is the number of the points of the sequence $\tilde{\tau}$ lying in the interval (a,b).

Let $1 \leqq p \leqq \infty$ and let $\Omega \subset [0,\infty)$. Denote by $L_p(\Omega, X)$ the space of functions $f:\Omega \to X$ integrable of power p in the sense of Bochner, i.e.

$$\int_\Omega \|f(t)\|^p \, dt < \infty$$

and with norm $\|f\|_{L_p(\Omega,X)} = (\int_\Omega \|f(t)\|^p \, dt)^{1/p}$, and by $L_\infty([0,\infty), X)$ denote the space of functions$f:\Omega \to X$ which are bounded on Ω with the supremum norm. For $\Omega = [0,\infty)$ we shall write more briefly $L_p(X)$, and for $\Omega = [0,\infty)$ and $X = R^1$ we shall write just L_p.

By $l_p(X)\,(1 \leqq p < \infty)$ we shall denote the space of summable of power p sequences $h = \{h_n\}_{n=1}^{\infty}$ of elements of X, i.e.

$$\sum_{n=1}^{\infty} \|h_n\|^p < \infty$$

and with norm $\|h\|_{l_p(X)} = (\sum_{n=1}^{\infty} \|h_n\|^p)^{1/p}$, and by $l_\infty(X)$ denote the space of bounded sequences of elements of X with the supremum norm. For $X = R^1$ we shall write l_p.

Definition 2. The impulsive equations (1), (2) and (3), (4) are called *L_p-equivalent* if to any bounded solution $x(t)$ of (1), (2) lying in a ball of sufficiently small radius and centre at the zero there corresponds at least one bounded solution $y(t)$ of (3), (4) such that $y(t) - x(t) \in L_p(X)$, and vice versa.

Definition 3. [10]. The linear impulsive differential equation (1), (2) is called *exponentially dichotomous* if there exists a splitting of the space X into a direct sum $X = X_1 \oplus X_2$ of subspaces X_1 and X_2 such that the following inequalities hold

$$\|U(t)P_1U^{-1}(\tau)\| \leqq M\, e^{-\delta(t-\tau)} \qquad (0 \leqq \tau < t < \infty) \tag{13}$$

$$\|U(t)P_2U^{-1}(\tau)\| \leqq M\, e^{\delta(t-\tau)} \qquad (0 \leqq t < \tau < \infty) \tag{14}$$

where $U(t) = U(t, 0)$, P_1 and P_2 are complementary to each other projectors onto X_1 and X_2 respectively, and M and δ are positive numbers.

3. Main results

Theorem 1. *Let the following conditions be fulfilled:*

1. *The function $A(t)$ is continuous for $0 \leqq t < \infty$.*
2. *Conditions* (H1) *and* (H2) *hold.*
3. *The impulsive equation* (1), (2) *is exponentially dichotomous.*
4. *The function $f: [0, \infty) \to X$ is bounded, continuous for $t \notin \tilde{\tau} = \{t_n\}_{n=1}^{\infty}$, at the points t_n $(n = 1, 2, 3, \ldots)$ it has discontinuities of the first kind and is continuous from the left.*
5. *The sequence $\{h_n\}_{n=1}^{\infty}$ $(h_n \in X, n = 1, 2, 3, \ldots)$ is bounded.*

Then the nonhomogeneous impulsive equation

$$\frac{dx}{dt} = A(t)x + f(t) \qquad (t \neq t_n, n = 1, 2, 3, \ldots) \tag{15}$$

$$x(t_n + 0) = Q_n x(t_n) + h_n \qquad (n = 1, 2, 3, \ldots) \tag{16}$$

has for $0 \leqq t < 0$ a bounded solution $x(t)$ for which the following formula is valid

$$x(t)=U(t)\xi+\int_0^t U(t)P_1U^{-1}(\tau)f(\tau)\,d\tau-\int_t^\infty U(t)P_2U^{-1}(\tau)f(\tau)\,d\tau$$
$$+\sum_{t_n>t} U(t)P_1U^{-1}(t_n+0)h_n-\sum_{t_n\geqq t} U(t)P_2U^{-1}(t_n+0)h_n \tag{17}$$

where $\xi\in X_1$.

Proof. We shall estimate the norms of the integrals and sums in (17):

$$\left\|\int_0^t U(t)P_1U^{-1}(\tau)f(\tau)\,d\tau\right\|\leqq\int_0^t \|U(t)P_1U^{-1}(\tau)\|\cdot\|f(\tau)\|\,d\tau\leqq\frac{M}{\delta}\sup_{0\leqq t>\infty}\|f(t)\|$$

$$\left\|\int_t^\infty U(t)P_2U^{-1}(\tau)f(\tau)\right\|d\tau\leqq\int_t^\infty \|U(t)P_2U^{-1}(\tau)\|\cdot\|f(\tau)\|\,d\tau\leqq\frac{M}{\delta}\sup_{0\leqq t>\infty}\|f(t)\|$$

$$\left\|\sum_{t_j>t} U(t)P_1U^{-1}(t_j+0)h_j\right\|\leqq\sum_{t_j>t}\|U(t)P_1U^{-1}(t_j+0)\|\cdot\|h_j\|\leqq\frac{M\lambda}{1-e^{-\delta l}}\sup_n\|h_n\|$$

$$\left\|\sum_{t\leqq t_j} U(t)P_2U^{-1}(t_j)h_j\right\|\leqq\sum_{t>t_j}\|U(t)P_2U^{-1}(t_j+0)\|\cdot\|h_j\|\leqq\frac{M\lambda}{1-e^{-\delta l}}\sup_n\|h_n\|$$

It is immediately verified that the function $x(t)$ is a solution of the nonhomogeneous impulsive equation (15), (16). □

We write down equality (17) in the form

$$x(t)=U(t)\xi+Gf(t)+\tilde{G}h(t) \tag{18}$$

where G is the linear integral operator

$$Gf(t)=\int_0^\infty g(t,s)f(s)\,ds \tag{19}$$

with kernel—Green's function

$$g(t,s)=\begin{cases}U(t)P_1U^{-1}(\tau) & 0\leqq\tau\leqq t<\infty\\ -U(t)P_2U^{-1}(\tau) & 0\leqq t\leqq\tau<\infty\end{cases} \tag{20}$$

and $\tilde{G}$ is an operator defined in the space $m(X)$ of bounded sequences of elements of X by means of the equality

$$\tilde{G}h(t)=\sum_{n=0}^{\infty} g_n(t)h_n \tag{21}$$

where

$$g_n(t)=\begin{cases} U(t)P_1U^{-1}(t_n+0) & t_n<t \\ -U(t)P_2U^{-1}(t_n+0) & t_n\geqq t. \end{cases} \tag{22}$$

Lemma 3. *Let $1\leqq p\leqq\infty$ and let the following conditions hold:*

1. *The function $A(t)$ is continuous for $0\leqq t\leqq\infty$.*
2. *Condition* (H1) *holds.*
3. *The impulsive equation* (1), (2) *is exponentially dichotomous.*

Then the operator G defined by equality (19) *maps continuously $L_p([0,\infty),X)$ into $L_p([0,\infty),X)\cap L_\infty([0,\infty),X)$ and the following estimates are valid*

$$\|G\|_{L_p(X)\to L_p(X)}\leqq 2M\delta^{-1} \tag{23}$$

$$\|G\|_{L_p(X)\to L_\infty(X)}\leqq\left(\frac{2(p-1)}{\delta p}\right)^{(p-1)/p}M. \tag{24}$$

Proof. If $p'=p/(p-1)$, for $\|Gf(t)\|$ we obtain the estimate

$$\|Gf(t)\|\leqq\int_0^\infty\|g(t,s)\|\,\|f(s)\|\,ds\leqq\left(\int_0^\infty\|g(t,s)\|^{p'}ds\right)^{1/p'}\left(\int_0^\infty\|f(s)\|^p ds\right)^{1/p}$$

From inequalities (13), (14) for $\int_0^\infty\|g(t,s)\|^{p'}ds$ we obtain

$$\int_0^\infty\|g(t,s)\|^{p'}ds\leqq M^{p'}\int_0^t e^{-\delta p'(t-s)}ds+M^{p'}\int_t^\infty e^{-\delta p'(s-t)}ds$$

$$=M^{p'}e^{-\delta p't}\int_0^t e^{\delta p's}ds+M^{p'}e^{\delta p't}\int_t^\infty e^{-\delta p's}ds$$

$$\leqq M^{p'}e^{-\delta p't}\frac{1}{\delta p'}e^{\delta p't}$$

$$+M^{p'}e^{\delta p't}\frac{1}{\delta p'}e^{-\delta p't}=\frac{2M^{p'}}{\delta p'}$$

hence

$$\|Gf(t)\| \leqq \left(\frac{2}{\delta p'}\right)^{1/p'} M \|f\|_{L_p}(x)$$

which implies estimate (24).

Consider the inequality

$$\|Gf(t)\| \leqq \alpha_1(t) + \alpha_2(t)$$

where

$$\alpha_1(t) = M \int_0^t e^{-\delta(t-s)} \|f(s)\| \, ds, \ \alpha_2(t) = M \int_t^\infty e^{\delta(t-s)} \|f(s)\| \, ds.$$

We estimate $\alpha_2(t)$ by means of Hölder's inequality:

$$\alpha_2(t) = M \int_t^\infty e^{\delta(t-s)/p'} e^{\delta(t-s)/p} \|f(s)\| \, ds \leqq M \left(\int_t^\infty e^{\delta(t-s)} \, ds \right)^{1/p'} \left(\int_t^\infty e^{\delta(t-s)} \|f(s)\|^p \, ds \right)^{1/p}$$

$$= M\delta^{-1/p'} \left(\int_t^\infty e^{\delta(t-s)} \|f(s)\|^p \, ds \right)^{1/p}.$$

Then

$$\left(\int_0^\infty \alpha_2^p(t) \, dt \right)^{1/p} \leqq M\delta^{-1/p'} \left(\int_0^\infty \left(\int_t^\infty e^{\delta(t-s)} \|f(s)\|^p \, ds \right) dt \right)^{1/p}.$$

Applying Funini's theorem, we obtain

$$\left(\int_0^\infty \alpha_2^p(t) \, dt \right)^{1/p} \leqq M\delta^{-1} \left(\int_0^\infty \|f(s)\|^p \, ds \right)^{1/p}.$$

Analogously for $\alpha_1(t)$ we obtain

$$\left(\int_0^\infty \alpha_1^p(t) \, dt \right)^{1/p} \leqq M\delta^{-1} \left(\int_0^\infty \|f(s)\|^p \, ds \right)^{1/p}.$$

Hence

$$\|Gf(t)\| \leqq 2M\delta^{-1} \|f\|_{L_p}(x)$$

which implies estimate (23). □

Lemma 4. *Let* $1 \leqq p \leqq \infty$ *and let condition* (H2) *hold. Then the operator* G *defined by* *equality* (21) *maps continuously* $l_p(X)$ *into* $L_p([0,\infty),X)\cap L_\infty([0,\infty),X)$ *and the following estimates are valid:*

$$\|\tilde{G}\|_{l_p(X)\to L_p(X)}\leqq(2\lambda(1-e^{-\delta\lambda})^{-1})^{(p-1)/p}\delta^{-(1/p)}M \tag{25}$$

$$\|\tilde{G}\|_{l_p(X)\to L_\infty(X)}\leqq(2\lambda(1-e^{-\delta p' l})^{-1})^{1/p'}M \tag{26}$$

where

$$\frac{1}{p}+\frac{1}{p'}=1.$$

Proof. Let $h=\{h_n\}_{n=1}^\infty$ be an arbitrary sequence of $l_p(X)$. Then from (21), (22) and Hölder's inequality there follow the inequalities

$$\|\tilde{G}h(t)\|\leqq\sum_{n=0}^\infty\|g_n(t)\|\cdot\|h_n\|\leqq\left(\sum_{n=0}^\infty\|g_n(t)\|^{p'}\right)^{1/p'}\left(\sum_{n=0}^\infty\|h_n\|^p\right)^{1/p}.$$

From (13) and (14) there follows the estimate

$$\|\tilde{G}\|_{l_p(X)\to L_\infty(X)}\leqq M\left(\sum_{n=0}^\infty e^{-\delta p'|t-t_n|}\right)^{1/p'}. \tag{27}$$

Condition (H2) implies the estimate

$$\sum_{t_n\geqq t} l^{-\delta p'|t-t_n|}=\sum_{k=0}^\infty\sum_{lk\leqq t-t_n\leqq l(k+1)}e^{-\delta p'(t_n-t)}\leqq\lambda\sum_{k=0}^\infty e^{-\delta p' lk}=\frac{\lambda}{1-e^{-\delta p' l}}.$$

Analogously we obtain the estimate

$$\sum_{t_n>t}e^{-\delta p'|t-t_n|}\leqq\frac{\lambda}{1-e^{-\delta p' l}}.$$

From the last two estimates we deduce

$$\sum_{n=0}^\infty e^{-\delta p'|t-t_n|}\leqq\frac{2\lambda}{1-e^{-\delta p' l}}. \tag{28}$$

Inequality (26) follows from inequalities (27) and (28).

Using again Hölder's inequality, we obtain the inequalities

$$\|\tilde{G}h(t)\|\leqq\sum_{n=0}^\infty\|g_n(t)\|\,\|h_n\|\leqq M\sum_{n=0}^\infty e^{-\delta|t-t_n|}\|h_n\|$$

$$\leqq M \sum_{n=0}^{\infty} e^{-(\delta/p')|t-t_n|} e^{-(\delta/p)|t-t_n|} \|h_n\|$$

$$\leqq M \left(\sum_{n=0}^{\infty} e^{-\delta|t-t_n|} \right)^{1/p'} \left(\sum_{n=0}^{\infty} e^{-\delta|t-t_n|} \|h_n\|^p \right)^{1/p}.$$

Making use of the estimate

$$\sum_{n=0}^{\infty} e^{-\delta|t-t_n|} \leqq \frac{2\lambda}{1-e^{-\delta l}} \tag{29}$$

which is analogous to estimate (28), we obtain

$$\int_0^{\infty} \|\tilde{G}h(t)\|^p \, dt \leqq \left(\frac{2\lambda}{1-e^{-\delta l}} \right)^{p-1} M^p \int_0^{\infty} \sum_{n=0}^{\infty} e^{-\delta|t-t_n|} \|h_n\|^p \, dt$$

$$= \left(\frac{2\lambda}{1-e^{-\delta l}} \right)^{p-1} M^p \sum_{n=0}^{\infty} \left(\int_0^{\infty} e^{-\delta|t-t_n|} \, dt \right) \|h_n\|^p. \tag{30}$$

Because of

$$\int_0^{\infty} \mathrm{e}^{-\delta|t-t_n|} \, dt \leqq \frac{1}{\delta} (n=1,2,3,\ldots)$$

from (30) there follows the inequality

$$\int_0^{\infty} \|\tilde{G}h(t)\|^p \, dt \leqq \frac{1}{\delta} \left(\frac{2\lambda}{1-e^{-\delta l}} \right)^{p-1} M^p \sum_{n=0}^{\infty} \|h_n\|^p$$

which implies (25). □

Further on we shall consider some questions related to the compactness of the operators defined by equalities (19) and (21).

Lemma 5. *Let the following conditions be fulfilled:*

1. $1 \leqq p \leqq \infty$.
2. *The function* $A(t)$ *is continuous for* $0 \leqq t < \infty$.
3. *Conditions* (H1) *and* (H2) *hold.*

Then for any nonnegative function $w \in L_p$ *and for any sequence of nonnegative numbers* $\{w_n\}_{n=1}^{\infty}$ *the set of functions*

$$A=\{Gf+\tilde{G}h: \|f(t)\| \leqq w(t), \|h_n\| \leqq w_n (n=1,2,3,\ldots)\}$$

is uniformly bounded and equicontinuous on each interval $(t_n, t_{n+1}]$ $(n=1,2,3,\ldots)$.

Proof. The sets

$$B=\{f(t): \|f(t)\| \leqq w(t)\}, \tilde{B}=\{\{h_n\}_{n=1}^{\infty}: \|h_n\| \leqq w_n (n=1,2,3,\ldots)\}$$

are bounded respectively in $L_p([0,\infty), X)$ and $l_p(X)$. From Lemmas 3 and 4 it follows that the set $A \subset GB+\tilde{G}\tilde{B}$ is bounded in $L_\infty([0,\infty), X)$, hence the set of the restrictions of the functions of A to $(t_{n-1}, t_n]$ is uniformly bounded for $n=1,2,3,\ldots$ From (20), Hölder's inequality and Lebesgue's theorem there follow the equalities

$$\lim_{t'-t''\to 0} \sup_{t_n<t',t''\leqq t_{n+1}} \int_0^\infty \|g(t',\tau)-g(t'',\tau)\| w(\tau)\, d\tau = 0 \qquad (n=1,2,3,\ldots)$$

$$\lim_{t'-t''\to 0} \sup_{t_n<t',t''<t_{n+1}} \sum_{k=0}^\infty \|g_k(t')-g_k(t'')\| w_n = 0 \qquad (n=1,2,3,\ldots).$$

The equicontinuity follows from the estimates

$$\|Gf(t')-Gf(t'')\| \leqq \int_0^\infty \|g(t',\tau)-g(t'',\tau)\| w(\tau)\, d\tau \qquad (f\in B)$$

$$\|\tilde{G}h(t')-\tilde{G}h(t'')\| \leqq \sum_{k=0}^\infty \|g_k(t')-g_k(t'')\| w_k \qquad (h\in\tilde{B})$$

where $t', t'' \in (t_n, t_{n+1}]$ $(n=1,2,3,\ldots)$. □

By $\tilde{C}=\tilde{C}([0,\infty), X, \tilde{\tau})$ we shall denote the space of functions $f:[0,\infty)\to X$ which are continuous for $t\neq t_n$ and at the points $t=t_n$ they have discontinuities of the first kind and are continuous from the left. With respect to the metric

$$\rho(x,y)=|||x-y|||$$

where

$$|||z|||=\sum_{n=1}^\infty 2^{-n} \frac{\sup_{t_n<t\leqq t_{n+1}} \|z(t)\|}{1+\sup_{t_n<t\leqq t_{n+1}} \|z(t)\|}$$

the space $\tilde{C}$ is locally convex.

Lemma 6. *Let the following conditions be fulfilled:*

1. *Conditions* 1, 2, *and* 3 *of Lemma* 5 *hold.*
2. *The set* K *is a centrally symmetric convex compact subset of* X.

Then for any nonnegative function $w \in L_p$ *and any sequence of nonnegative numbers* $\{w_n\}_{n=1}^{\infty} \in l_p$ *the set of functions*

$$A(K)=\{Gf+\tilde{G}h: w^{-1}(t)f(t)\in K(0\leqq t<\infty), w_n^{-1}h_n \in K(n=1,2,3,\ldots)\}$$

is compact in $\tilde{C}$.

Proof. From Ascoli–Arzella's theorem, Lemma 5, and the boundedness of the set K it follows that it suffices to show that for any fixed $t\in(t_{n-1},t_n]$ $(n=1,2,3,\ldots)$ the set of values of the functions $x\in A(K)$ at the point t is a relatively compact subset of X. Let $t\in(t_{n-1},t_n]$ be fixed and let $\varepsilon>0$ be arbitrarily chosen. For sufficiently large values of T and N the inequalities

$$\left\| Gf(t)-\int_0^T g(t,s)f_N(s)\,ds\right\| <\varepsilon$$

are valid, where $f_N(s)=\begin{cases} f(s) & w(s)\geqq N \\ 0 & w(s)>N \end{cases}$ and, analogously

$$\left\| \tilde{G}h(t)-\sum_{k=0}^{T} g_n(t)h_n^{(N)}\right\| <\varepsilon,$$

where $h_n^{(N)}=\begin{cases} h_n & \|h_n\|\leqq N \\ 0 & \|h_n\|>N. \end{cases}$

We shall prove the compactness of the set

$$A_{T,N}(K,t)=\left\{\int_0^T g(t,s)f(s)\,ds+\sum_{k=0}^{T} g_k(t)h_k: f(s)\in NK(0\leqq s<\infty), h_k\in NK(k=1,2,3,\ldots)\right\} \tag{31}$$

From the elementary properties of the integrals and the sums it follows that for $f(s)\in NK$ and $h_k\in NK$ the following relations are valid

$$\int_0^T g(t,s)f(s)\,ds\in TN\bigcup_{0\leqq s\leqq T} g(t,s)K$$

$$\sum_{k=0}^{T} g_k(t)h_k\in TN\bigcup_{0\leqq k\leqq T} g_k(t)K.$$

The sets at the right-hand sides of the above inclusions are compact in X, which implies that the set defined by (31) is also compact. From Hausdorff's theorem there follows the compactness of the set $A(K)$. □

We shall note that for $\dim X < \infty$, K can be taken to be an arbitrary ball with centre at the zero. For $\dim X = \infty$ the condition of compactness of the set K can be essentially weakened. It suffices to assume compactness of K in some "quasi"-weak topology $\mathscr{T}_0$ for which the sets of the form

$$B = \{f(t) : \|f(t)\| \leqq w(t)\} \; (\tilde{B} = \{\{h_n\}_{n=1}^{\infty} : \|h_n\| \leqq w_n\})$$

are $\mathscr{T}$-weakly ($\tilde{\mathscr{T}}$-weakly) compact in $L_p([0,\infty), X) \, (l_p(X))$, thus "quasi"-functionals of the form

$$\phi(f) = \int_0^{\infty} g(t) f(t) \, dt, \; \psi(h) = \sum_{n=0}^{\infty} g_n h_n$$

with bounded functions $g(t) \, (0 \leqq t < \infty)$, respectively sequences $\{g_n\}_{n=1}^{\infty}$ are $\mathscr{T}$-weakly ($\tilde{\mathscr{T}}$-weakly) continuous in $L_p([0,\infty), X) \, (l_p(X))$.

Theorem 2. *Let the following conditions be fulfilled:*

1. *The functions* $A(t)$ *is continuous for* $0 \leqq t < \infty$.

2. *Conditions* (H1), (H2) *hold.*

3. *The impulsive equation* (1), (2) *is exponentially dichotomous and inequalities* (13), (14) *hold.*

4. *The inequality*

$$\|f(t,x)\| \leqq \psi_r(t) \, (\|x\| \leqq r, \, 0 \leqq t < \infty) \tag{32}$$

is satisfied, where $\psi_r(t) \in L_p$ *and the set*

$$K(r) = \{\psi_r^{-1}(t) f(t,x) : 0 \leqq t < \infty, \, \|x\| \leqq r\} \tag{33}$$

is relatively compact for any r small enough (*the last condition is automatically satisfied for* $\dim X < \infty$).

5. *The sequence* $\{R_n\}_{n=1}^{\infty}$ *satisfies the inequalities*

$$\|R_n x\| \leqq \chi_n(r) \qquad (\|x\| \leqq r, \, n = 1, 2, 3, \ldots) \tag{34}$$

where $\{\chi_n(r)\} \in l_p$ *and the set*

$$\tilde{R}(r) = \{\chi_n^{-1}(r) R_n x : n = 1, 2, 3, \ldots, \|x\| \leqq r\} \tag{35}$$

is relatively compact in X for any r small enough (the last condition is also automatically satisfied for $\dim X < \infty$).

6. *The inequality*

$$\left(\frac{2}{\delta p'}\right)^{1/p'} \|\psi_{r+\rho}(t)\|_{L_p} + \left(\frac{2\lambda}{1-e^{-\delta p' l}}\right)^{1/p'} \|\{\chi_n(r+\rho)\}_{n=1}^{\infty}\|_{l_p} \leqq \frac{\rho}{M} \tag{36}$$

holds for some $r, \rho > 0$.

Then the nonlinear impulsive equation (3), (4) *is* L_p*—equivalent to the linear impulsive differential equation* (1), (2).

Proof. Each bounded solution $y(t)$ of the linear impulsive equation (1), (2) has the form

$$y(t) = U(t)\xi \tag{37}$$

where $\xi \in X_1$ (the subspace X_1 consists of all ξ for which the function $U(t)\xi$ is bounded for $0 \leqq t < \infty$). Let $x(t)\,(0 \leqq t < \infty)$ be a bounded solution of (3), (4). Then $x(t)$ is a solution of the nonhomogeneous linear equation (15), (16) for $f(t) = f(t, x(t))$ and $h_n = R_n x(t_n)$. That is why the function satisfies the nonlinear integral equation

$$x(t) = U(t)\xi + Gf(t, x(t)) + \tilde{G}(R_n x(t_n))\,(t). \tag{38}$$

Conversely, each bounded solution of the nonlinear integral equation (38) is a solution of the nonlinear impulsive equation (3), (4).

We set

$$z(t) = x(t) - U(t)\xi \tag{39}$$

and rewrite equation (38) in the form

$$z(t) = Gf(t, U(t)\xi + z(t)) + \tilde{G}(R_n(U(t_n)\xi + z(t_n))\,(t). \tag{40}$$

We shall show that for sufficiently small $\xi \in X_1$ equation (40) has a bounded solution $z(t)$. For this purpose we shall investigate the operator defined by the formula

$$Fz(t) = Gf(t, U(t)\xi + z(t)) + \tilde{G}(R_n(U(t_n)\xi + z(t_n)) \tag{41}$$

on the set

$$D(\rho) = \{z(t) \in \tilde{C}([0, \infty), X, \tau), \|z(t)\| \leqq \rho, 0 \leqq t < \infty\}. \tag{42}$$

In view of conditions 4 and 5 of Theorem 2 for $\|U(t)\xi\| \leqq r$ and $z \in D(\rho)$ we obtain the estimates

$$\|Fz(t)\| \leqq \|G\|_{L_p \to L_\infty} \|\psi_{r+\rho}\|_{L_p} + \|\tilde{G}\|_{l_p \to L_\infty} \|x_n(r+\rho)\|_{l_p}.$$

From Lemmas 3 and 4 and condition 6 of Theorem 2 we obtain the estimate

$$\|Fz(t)\| \leqq \rho \ (0 \leqq t < \infty, z \in D(\rho)).$$

Hence the set $D(\rho)$ is invariant with respect to the operator F. By means of the Schauder–Tychonoff theorem we shall show that the operator F has a fixed point in the set $D(\rho)$. The set $D(\rho)$ is a closed and convex subset of the space $\tilde{C}([0,\infty), X, \tilde{\tau})$. From Lemmas 5 and 6, and the compactness of the sets $K(r)$ and $\tilde{R}(r)$ defined respectively by (33) and (35) there follows the compactness of the set $FD(\rho)$ in the space $\tilde{C}([0,\infty), X, \tilde{\tau})$.

We shall establish the continuity of the operator F. Let $\{z_k(t)\}_{k=1}^{\infty} \subset D(\rho)$ be a sequence tending to $z(t) \in D(\rho)$ in the space $\tilde{C}([0,\infty), X, \tilde{\tau})$. Then the sequence $u_k(t) = \{f(t, U(t)\xi + z_k(t))\}_{k=1}^{\infty}$ tends to $f(t, U(t)\xi + z(t))$ for any t, and the sequence $v_k = \{R_n(U(t_n)\xi + z_k(t_n))\}_{n=1}^{\infty}$ tends to the sequence $\{R_n(U(t_n)\xi + z(t_n))\}_{n=1}^{\infty}$ coordinate-wise $(n = 1, 2, 3, \ldots)$. From (32) and (34) there follow the inequalities

$$\|u_k(t)\| \leqq \psi_{r+\rho}(t) \qquad (0 \leqq t < \infty)$$
$$\|v_k(t)\| \leqq \chi_n(r+\rho) \qquad (n = 1, 2, 3, \ldots).$$

Using the theorem of Lebesgue to pass to the limit under the sign of the integral, we obtain that the sequence of functions $Gu_k(t)$ tends for $t \in [0,\infty)$ to the function $Gu(t)$. By the analogue of the theorem of Lebesgue for series we obtain that the sequence of functions $\tilde{G}v_k(n)$ tends to the function $\tilde{G}v(n)$. Since the functions $Gu_k(t) + \tilde{G}v_k(t)$ lie in a compact set, they tend to the function $G_u(t) + \tilde{G}v(t)$ in the metric of the space $\tilde{C}([0,\infty), X, \tilde{\tau})$ too.

Let $z_*(t)$ be a fixed point of the operator F. Then from (32) and (34) it follows that the function u_* and the sequence $\{v_*^{(n)}\}_{n=1}^{\infty}$ defined respectively by the formulae

$$u_*(t) = f(t, U(t)\xi + z_*(t)), \ v_*^{(n)} = R_n(u(t_n)\xi + z_*(t_n))$$

lie respectively in $L_p([0,\infty), X)$ and $l_p(X)$.

From Lemmas 3 and 4 it follows that the function $z_*(t) = Gu_*(t) + \tilde{G}(v_*(n))(t)$ also belongs to the space $L_p([0,\infty), X)$. Thus the difference $z_*(t)$ between the bounded solution $x_*(t) = U(t)\xi + z_*(t)$ of the nonlinear impulsive equation (3), (4) and the bounded solution $U(t)\xi$ of the linear impulsive equation in the space $L_p([0,\infty), X)$. □

Theorem 3. *Let the following conditions be fulfilled:*

1. *The function* $A(t)$ *is continuous for* $0 \leqq t < \infty$.
2. *Conditions* (H1), (H2) *hold.*
3. *The linear impulsive equations is exponentially dichotomous and inequalities* (13), (14) *hold.*
4. *The inequality*

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$$\|f(t,x)\| \leqq \psi_r(t)(\|x\| \leqq r, 0 \leqq t < \infty)$$

holds, where $\psi_r(t) \in L_p$ *and the set*

$$K(r) = \{\psi_r^{-1}(t) f(t,x) : 0 \leqq t < \infty, \|x\| \leqq r\}$$

is relatively compact for any r *small enough.*

5. *The sequence* $\{R_n\}_{n=1}^{\infty}$ *satisfies the inequality*

$$\|R_n x\| \leqq X_n(r) \qquad (n = 1, 2, 3, \ldots)$$

where $\{X_n(r)\} \in l_p$ *and the set*

$$\tilde{R}(r) = \{\chi_n^{-1}(r) R_n x : n = 1, 2, 3, \ldots, \|x\| \leqq r\}$$

is relatively compact in X *for any* r *small enough.*

6. *The operators* $Q_n + R_n (n = 1, 2, 3, \ldots)$ *are continuously invertible in some neighbourhood of the zero.*

7. *The function* $f(t,x)$ *is Lipschitz continuous with respect to its second argument.*

Then the nonlinear impulsive equation (3), (4) *is* L_p*-equivalent to the linear impulsive differential equation* (1), (2).

Proof. From the conditions of Theorem 3 it follows that for sufficiently small $r, \rho > 0$ there exists a number $T > 0$ for which the following inequality is valid

$$\left(\frac{2}{\delta p'}\right)^{1/p'} \left(\int_T^{\infty} \psi_{r+\rho}^p(t)\, dt\right)^{1/p} + \left(\frac{2\lambda}{1 - e^{-\delta p' l}}\right)^{1/p} \left(\sum_{t_n \geqq T} X_n^p(r+\rho)\right)^{1/p} \leqq \frac{\rho}{M}. \tag{43}$$

This allows us to apply Theorem 2 to the impulsive equations (1), (2) and (3), (4) on the interval $[T, \infty)$. Let $x_0(t) = U(t)\xi$ be a bounded solution of (1), (2) for which $\|x(t)\| \leqq r (T \leqq t < \infty)$ and let $x_*(t) = U(t)\xi + z_*(t)$ be a solution of (3), (4), where $z_* \in L_p([T, \infty), X)$. From conditions 6 and 7 of Theorem 3 it follows that the solution $x_*(t)$ constructed on $[T, \infty)$ can be continued as a solution defined on $[0, \infty)$ (see [**11**]). Because of the continuous dependence of the impulsive equation (3), (4) on the initial condition, the solution continued on $[0, T]$ will slightly differ from the solution $x_0(t) = U(t)\xi$ of the impulsive equation (1), (2). Since the two solutions $x_*(t)$ and $x_0(t) = U(t)\xi$ are bounded on $[0, T]$, then their difference $z_*(t)$ lies in the space $L_p([0, \infty), X)$. □

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REFERENCES

1. D. D. Bainov and P. S. Simeonov, *Stability Theory of Differential Equations with Impulse Effect: Theory and Applications* (Ellis Horwood, 1989).

2. D. D. Bainov and A. B. Dishliev, Population dynamics control in regard to minimizing the time necessary for the regeneration of a biomass taken away from the population, *Math. Model. and Numer. Anal.* **24** (1990), 681–692.

3. V. D. Mil'man and A. D. Myshkis, On the stability of motion in the presence of impulses, *Siberian Math. J.* **1** (1960), 223–237 (in Russian).

4. D. D. Bainov, P. P. Zabreiko and S. I. Kostadinov, Stability of the general exponent of nonlinear impulsive differential equations in a Banach space, *Internat. J. Theoret. Phys.* **27** (1988), 374–380.

5. D. D. Bainov, P. P. Zabreiko and S. I. Kostadinov, Characteristic exponents of differential equations with impulse effect in a Banach space, *Internat. J. Theoret. Phys.* **27** (1988), 731–743.

6. A. M. Samoilenko and N. A. Perestyuk, *Differential Equations with Impulse Effect* (Kiev, Višča Skola, 1987, p. 283, in Russian).

7. J. Futak, Asymptotic and integral equivalence of functional and ordinary differential equations, *Fasc. Math.* **15** (1984), 97–108.

8. F. Brauer, On the asymptotic relationships between solutions of two systems of ordinary differential equations, *J. Differential Equations* **6** (1969), 527–543.

9. A. Haščak, Integral equivalence between a nonlinear system and its nonlinear perturbation, *Math. Slovaca* **34** (1984), 393–404.

10. D. D. Bainov, S. I. Kostadinov and P. P. Zabreiko, Exponential dichotomy of linear differential equations in a Banach space, *Internat. J. Theoret. Phys.* **28** (1989), 797–814.

11. P. P. Zabreiko and A. I. Smirnov, On the solvability of the Cauchy problem for ordinary differential equations in Banach spaces, *Diff. Uravn.* **19** (1979), 2085–2086 (in Russian).

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