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Infinite Dimensional DeWitt Supergroups and their Bodies

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Abstract. For Dewitt super groups G modeled via an underlying finitely generated Grassmann algebra it is well known that when there exists a body group BG compatible with the group operation on G, then, generically, the kernel K of the body homomorphism is nilpotent. This is not true when the underlying Grassmann algebra is infinitely generated. We show that it is quasi-nilpotent in the sense that as a Banach Lie group its Lie algebra κ has the property that for each $a \in \kappa$, ad_a has a zero spectrum. We also show that the exponential mapping from κ to K is surjective and that K is a quotient manifold of the Banach space κ via a lattice in κ.

1 Introduction

The purpose of this short note is to fill a gap in the literature concerning DeWitt super Lie groups. Standard works on the subject either assume that the Grassmann algebra Λ of supernumbers is finitely generated or it is presumed that the group operations are of class G^ω (sometimes H^ω). This work chooses as underlying Grassmann algebra the infinite dimensional Banach algebra $\Lambda = \Lambda^0 \oplus \Lambda^1$ introduced by Rogers [1] and focusses on DeWitt super groups G whose group operations are G^∞ mappings. We are interested in understanding, as nearly as possible, the weakest conditions under which the body manifold BG of G possesses a group structure induced from that of G. We also want to determine when BG may be identified with a subgroup of G and how its structure is related to that of G.

Due to the brevity of this note we will not present a detailed review of the subject, preferring to reference only those sources we actually utilize here. The author would like to thank Helge Glockner for providing the reference to the paper by Wojtynski and for remarks regarding Banach Lie Groups. He is certainly not responsible in any way for any erroneous understanding of the author.

In the case where the Grassmann algebra is finitely generated, it is known (see [2]) that, generically, the classical Lie group corresponding to a DeWitt super group G is a semidirect product of the body group and a nilpotent group. This is not true in our case, and in fact it is not always true in the case the Grassmann algebra is finitely generated (see the trivial example in Remark 3.2). Our main result is that, subject to certain conditions indicated above, the existence of a body group with group operation induced by that of G implies that the body mapping is a group homomorphism whose kernel K satisfies a condition that we call quasi-nilpotent in that its Lie algebra is quasi-nilpotent in the sense of [3]. Moreover, the exponential mapping from the

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Lie algebra κ of the Banach Lie group K is a surjection from κ onto K. Thus K is a quotient manifold of the Banach space κ and a lattice in κ.

Although the proof is trivial, we also show for completeness that when the body group can be realized as a subgroup of G, then G is a semi-direct product of BG and the kernel of the body mapping.

Observe that the body manifold of a super Lie group is always an ordinary infinite-dimensional Banach manifold, but it need not have a natural super group structure. Moreover, even when the body manifold has a natural group structure, the kernel K of the body mapping is modeled on a subspace of the superspace $\mathbb{R}^{p|q}$ all of whose elements have body zero, so it too does not possess a super group structure. On the other hand, in this case where K is an infinite dimensional Banach Lie group which, however, is generally not nilpotent.

2 Preliminaries

For basic definitions, see the book by Rogers [2]. Unlike Rogers, we denote the space of supernumbers by Λ instead of $\mathbb{R}_S$. Even though we do not use the notation $\mathbb{R}_S$, our underlying field is the field of real numbers. Rogers allows the underlying Grassmann algebra to be either finitely generated or infinitely generated depending on the context. We assume throughout this note that it is infinitely generated and that it has the norm defined in Rogers' book [2, p. 22]. Relative to this norm the space of supernumbers is a Banach algebra. The proof of this fact is also found on page 22 of the same book. The space of supernumbers possesses a $\mathbb{Z}_2$ grading $\Lambda = \Lambda^0 \oplus \Lambda^1$, which is utilized throughout the note. The body mapping from Λ to $\mathbb{R}$ is denoted by b_Λ. If p, q are nonnegative integers, $\mathbb{R}^{p|q}$ denotes, as usual, the set of all $(p+q)$-tuples $(x^1, x^2, \dots, x^p, \theta^1, \theta^2, \dots, \theta^q)$, where $x^i \in \Lambda^0, 1 \le i \le p$ and $\theta^\alpha \in \Lambda^1, 1 \le \alpha \le q$. We denote the body mapping from $\mathbb{R}^{p|q}$ onto $\mathbb{R}^p$ by b. A subset U of $\mathbb{R}^{p|q}$ is a DeWitt open subset of $\mathbb{R}^{p|q}$ if there exists an open subset $W \subseteq \mathbb{R}^p$ such that $U = b^{-1}(W) \subseteq \mathbb{R}^{p|q}$. We will denote such a subset U by $W^{p|q}$.

The following definition is elaborated in more detail in [2, pp. 52–53].

Definition 2.1 A DeWitt supermanifold is a set M along with a G^∞ atlas $\mathcal{A}$ such that each chart of the atlas is a bijection from a subset U of M onto a DeWitt open subset $W^{p|q}$ of $\mathbb{R}^{p|q}$ for some pair of nonnegative integers p, q and for some open subset W of $\mathbb{R}^p$.

The following proposition is a consequence of Rogers' proof on page 60 that every DeWitt supermanifold has a body manifold that is an ordinary finite-dimensional C^∞ manifold. Generically it does not follow that when the DeWitt supermanifold is a super group that the body manifold is a group (see Remark 3.2).

Proposition 2.2 *M is a DeWitt supermanifold if and only if it is a C^∞ Banach manifold such that there exists a C^∞ Banach manifold BM and a C^∞ mapping $\beta\colon M \to BM$ subject to the following conditions:*

(i) *M is modeled on the superspace $\mathbb{R}^{p|q}$ and BM is modeled on $\mathbb{R}^p$.*

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(ii) *Each point of M lies in an open subset U of M on which there is defined a chart ϕ_U from U onto a DeWitt open subset $W^{p,q} \subseteq \mathbb{R}^{p|q}$ and a chart $\tilde{\phi}_U$ of BM from $BU \equiv \beta(U)$ onto $W \subseteq \mathbb{R}^p$ such that the following diagram is commutative:*

$$\begin{array}{ccc} U & \xrightarrow{\ \phi_U\ } & W^{p|q} \\ \beta \downarrow & & \downarrow b \\ BU & \xrightarrow[\ \tilde{\phi}_U\ ]{} & W \end{array}$$

where $b\colon \mathbb{R}^{p|q} \to \mathbb{R}^p$ is the body mapping on $\mathbb{R}^{p|q}$.

It is required additionally that if U, V are chart domains of M such that $U \cap V$ is nonempty and subject to the conditions above with $\phi_V, \tilde{\phi}_V$, defined analogously to $\phi_U, \tilde{\phi}_U$, then $\phi_V \circ \phi_U^{-1}$ is a G^∞ mapping with a G^∞ inverse between two appropriate DeWitt open subsets of $\mathbb{R}^{p|q}$ and $\tilde{\phi}_V \circ \tilde{\phi}_U^{-1}$ is a C^∞ diffeomorphism between the corresponding open subsets of $\mathbb{R}^p$.

Remark 2.3 Throughout the paper all our supermanifolds are DeWitt supermanifolds, so we often refer to a DeWitt supermanifold simply as a supermanifold. Given a supermanifold M we denote its underlying Banach manifold by $\mathcal{B}M$.

Remark 2.4 Note that if $W \subseteq \mathbb{R}^p$ is open and $y \in W$, then the set of points of M having body $\tilde{\phi}_U^{-1}(y)$ is precisely the image $\phi_U^{-1}(b^{-1}(y))$ of the set $b^{-1}(y)$ of points of $\mathbb{R}^{p|q}$ having body y. Thus if $x \in BU \subseteq BM$, then the set of points of M having body x is the image under ϕ_U^{-1} of the set of points of $\mathbb{R}^{p|q}$ having body $\tilde{\phi}_U(x)$. Recall that $b^{-1}(y)$ is precisely the subset

$$\left\{ (x^1, x^2, \dots, x^p, \theta^1, \theta^2, \dots, \theta^q) \in \mathbb{R}^{p|q} \mid b_\Lambda(x^i) = y^i, 1 \le i \le p \right\}$$

and that this space is contractible.

3 DeWitt Super Groups and their Bodies

Definition 3.1 We say that G is a DeWitt super Lie group, or simply a super Lie group, if and only if it is a (DeWitt) supermanifold on which one has defined a group product $\mu\colon G \times G \to G$ and a group inversion $\iota\colon G \to G$ that are G^∞ mappings.

Remark 3.2 Every DeWitt super Lie group defines an underlying Banach Lie group $\mathcal{B}G$, since G^∞ mappings are necessarily C^∞ mappings. On the other hand it is not the case that every DeWitt super Lie group induces a Banach Lie group structure on its body. For example, let $G = \mathbb{R}^{1|1}$ and define a group operation on G by

$$(x, \theta) + (y, \phi) = (x + y + \theta\phi, \theta + \phi).$$

It is not difficult to check that G is a DeWitt super Lie group and that its body manifold is $\mathbb{R}$, but the group operation does not close on $\mathbb{R}$.

Notice that in order for a super Lie group to induce a well-defined operation on its body manifold it must be the case that for arbitrary $x, y \in BG$ there must exist a unique $z \in BG$ such that

$$\mu\big(\beta^{-1}(x) \times \beta^{-1}(y)\big) \subseteq \beta^{-1}(z).$$

If we think of $\beta\colon G \to BG$ as a fiber bundle, then this condition states that the product of two fibers must be contained in a single fiber. When this condition holds, we may define an operation μ_β on BG by requiring that $\mu_\beta(x, y)$ be the unique element of BG such that

$$\mu\big(\beta^{-1}(x) \times \beta^{-1}(y)\big) \subseteq \beta^{-1}\big(\mu_\beta(x, y)\big).$$

It is straightforward to check that when this condition is met, $\mu_\beta\colon BG \times BG \to BG$ is a group operation on BG and that β is a group homomorphism from G onto BG. In this case we denote both group operations by juxtaposition, *i.e.,* for $g, h \in G, \quad x, y \in BG, \mu(g, h) = gh$ and $\mu_\beta(x, y) = xy$.

Proposition 3.3 *Suppose G is a DeWitt supermanifold with body map $\beta\colon G \to BG$ and that there is an induced group structure on BG. Then the induced group operation on BG is a C^∞ mapping and β is a C^∞ group homomorphism.*

Proof First recall that the body mapping of a Dewitt supermanifold is known to be a C^∞ mapping (see Rogers [2]). Next choose any $z \in BG$ and $g \in G$ such that $\beta(g) = z$. Choose a DeWitt chart $\phi_U\colon U \to W^{p|q}$ such that $\phi_U(g) = 0$ and $\tilde{\phi}_U(\beta(g)) = 0$. Define a section s of $b\colon W^{p|q} \to W$ by $s(x) = (x, 0)$. Define a local section σ on the neighborhood $\tilde{\phi}_U^{-1}(W)$ of z by $\sigma(q) = \phi_U^{-1}(s(\tilde{\phi}_U(q))), q \in \tilde{\phi}_U^{-1}(W)$. Then σ is a C^∞ mapping defined on an open set about z. Now this is possible for each element of BG, so if $x, y \in BG$ are arbitrary, we can choose C^∞ local sections σ_x, σ_y of β defined near x and y, respectively, and on an appropriate open set of $BG \times BG$ containing (x, y), we have $\mu_\beta = \beta \circ \mu \circ (\sigma_x \times \sigma_y)$. Thus μ_β is a C^∞ mapping. A similar argument shows that inversion is a C^∞ mapping. ■

The following definition is due to [3].

Definition 3.4 A Banach Lie algebra κ is quasi-nilpotent if and only if for each $a \in \kappa$, the adjoint mapping ad_a has spectrum zero. A C^∞ Banach Lie group is quasi-nilpotent if and only if its corresponding Lie algebra is quasi-nilpotent.

Theorem 3.5 *Let G be a DeWitt super Lie group such that there is an induced group structure on BG. Let $\beta\colon G \to BG$ denote the induced group homomorphism and K its kernel.*

(i) *K is a Banach Lie group whose Lie algebra κ is a freely finitely generated Λ^0 left module.*
(ii) *The Lie module κ is quasi-nilpotent and consequently the Baker–Campbell–Hausdorff formula holds globally on it.*

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(iii) *The exponential mapping* $\exp\colon \kappa \to K$ *is a smooth surjective mapping that is locally injective. Consequently, K is diffeomorphic to a manifold that is the quotient of a Banach space modulo a lattice.*

Proof Proof of (i). Since β is a C^∞ homomorphism clearly its kernel is a normal Banach sub-Lie group of G. By Remark 2.4, we know that if we choose a chart $\phi = (x, \theta)$ of G near the identity $e \in G$ such that $\phi(e) = 0$, then its restriction to K is a C^∞ diffeomorphism from $K = \beta^{-1}(e)$ onto

$$\widetilde{\mathbb{R}^{p|q}} \equiv \left\{ (\bar{x}, \bar{\theta}) \in \mathbb{R}^{p|q} \mid b_\Lambda(\bar{x}^i) = 0, 1 \le i \le p \right\}.$$

Thus the manifold structure of the Banach Lie group K may be modeled by an atlas having a single global chart with values in $\widetilde{\mathbb{R}^{p|q}}$. Thus neither K nor BG are super manifolds in the sense of Rogers' definition given above. On the other hand if we choose a chart $\phi = (x, \theta)$ of G near the identity e and define $y^i = x^i - b_\Lambda(x^i)$, then $z = (y^i|K, \theta^\alpha|K)$ is a chart on K such that every even tangent vector to K at e is a linear combination of the tangent vectors $\{\partial/\partial y^i, \partial/\partial\theta^\alpha\}$ over Λ^0. It follows that κ is a freely finitely generated Λ^0 module as asserted.

Proof of (ii). Recall that the Lie algebra of a Banach Lie group may be identified with the space of tangent vectors at the identity of the group. For $a, b \in T^0_e K$ the Lie bracket $[a, b]$ may be defined as follows. Let X^a, X^b denote the left-invariant vector fields of K defined by a and b, respectively. Then $[a, b] = [X^a, X^b](e)$. We claim that for $a \in T^0_e K$, ad_a is Λ^0 left linear. To see this, first notice that if $u \in K$, then the left-translation mapping L_u on G has the property that $d_e L_u\colon T^0_e G \to T^0_e G$ is left Λ^0 linear, since G is a Lie super group. Moreover if for $a, b \in T^0_e K$, we may extend X^a, X^b to all of G by left translation of a and b to all of G. Now for $a, b \in G$ and $\lambda \in \Lambda^0$, $[a, \lambda b] = [X^a, X^{\lambda b}](e) = [X^a, \lambda X^b](e) = \lambda[X^a, X^b] = \lambda[a, b]$. Here we have used the fact that G is a super Lie group. Now, left-translation on K is the restriction of left-translation on G, so it follows that if $a, b \in \kappa$, then $\lambda b \in \kappa$ and $\mathrm{ad}_a(\lambda b) = [a, \lambda b] = \lambda[a, b] = \lambda\, \mathrm{ad}_a(b)$, as required.

Now let $\{e^i, f^\alpha\}$ denote a finite basis of $T_e K$ that freely generates $T^0_e K$ over Λ^0 where the e_i are even and the f_α are odd. For $a \in T^0_e K$, represent ad_a as a matrix relative to this basis. If

$$t = \sum_i a_i e^i + \sum_\alpha b_\alpha f^\alpha \quad \text{and} \quad \mathrm{ad}_a(t) = t' = \sum_i a'_i e^i + \sum_\alpha b'_\alpha f^\alpha,$$

then the matrix M_a of ad_a sends $(a_i, b_\alpha) \in \widetilde{\mathbb{R}^{p|q}}$ to $(a'_i, b'_\alpha) \in \widetilde{\mathbb{R}^{p|q}}$ (observe that the vectors (a_i, b_α) and (a'_i, b'_α) are in fact in $\widetilde{\mathbb{R}^{p|q}}$, since the differential of a chart on K such as that described in the proof of (i) maps $T^0_e K$ onto $\widetilde{\mathbb{R}^{p|q}}$). Thus the bodies of the coefficients of the matrix M_a are zero. It follows that M_a is never invertible; moreover, if $\lambda \in \mathbb{C}$, the matrix $\lambda Id - M_a$ is invertible if and only if λ is not zero.Thus the spectrum of M_a and therefore of ad_a is $\{0\}$. It follows that ad_a is quasi-nilpotent. By a Theorem of Wojtynski [3], the global Baker–Campbell–Hausdorff formula holds for κ. It follows that $\overline{K} = \{\exp(a) \mid a \in \kappa\}$ is closed under the group operation of K and consequently is a subgroup of K. Since exp is a diffeomorphism on a neighborhood

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of the identity, $\overline{K}$ is open. Thus $\overline{K}$ is both open and closed and is the component of the identity in K. Since there is a global chart from K onto $\mathbb{R}^{p|q}$, K is connected and so is equal to $\overline{K}$. It follows that exp is a smooth surjective mapping from κ onto K. Since exp is a local diffeomorphism, it follows that K is diffeomorphic to a manifold which is the quotient of a vector space modulo a lattice. ■

In a large number of special cases the body group BG, when it exists, is a subgroup of the supergroup G itself. This is the case, for example, when $G = Gl(\Lambda, p, q)$ the group of $(p+q) \times (p+q)$ nonsingular graded matrices.

Definition 3.6 Suppose G is a Dewitt super Lie group. We say that G contains its body group BG if and only if there exists a C^∞ homomorphism $\eta \colon BG \to G$ such that $\eta \circ \beta$ is the identity mapping on $\eta(BG)$, where β is the body mapping.

Proposition 3.7 *Suppose the Dewitt super Lie group G contains its body group. Then* $\mathcal{B}(G) = \eta(BG) \rtimes K$, *where* $K = ker\beta$.

Proof Define an operation on $\eta(BG) \times K$ by

$$\big(\eta(b_1), k_1\big) \cdot \big(\eta(b_2), k_2\big) = \big(\eta(b_1)\eta(b_2), \eta(b_2)^{-1} k_1 \eta(b_2) k_2\big).$$

This product defines a semi-direct product structure on $\eta(BG) \times K$. The mappings $\psi \colon G \to \eta(BG) \times K$ and $\mu \colon \eta(BG) \times K \to G$ defined by $\psi(g) = (\eta(\beta(g)), \eta(\beta(g))^{-1} g)$ and $\mu(\tilde{b}, k) = \tilde{b}k$ respectively are C^∞ group homomorphisms and inverse functions. ■

Remark 3.8 Observe that BG does not possess a G^∞ structure, and so it is not meaningful to ask whether or not η is a G^∞ mapping. Thus $BG \cong \eta(BG)$ does not have a superstructure. Consequently, one cannot expect that BG be a sub-super group of G, nor can one expect that the semi-direct product makes sense in the G^∞ category. In the finite dimensional case K is a nilpotent Lie group, but this is not the case in our context.

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