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ON THE STABLE CLASSIFICATION OF CERTAIN 4-MANIFOLDS

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We study the s-cobordism type of closed orientable (smooth or PL) 4–manifolds with free or surface fundamental groups. We prove stable classification theorems for these classes of manifolds by using surgery theory.

1. INTRODUCTION

In this paper we shall study closed connected (smooth or PL) 4–manifolds with special fundamental groups as free products or surface groups. For convenience, all manifolds considered will be assumed to be orientable although our results work also in the general case, provided the first Stiefel-Whitney classes coincide. The starting point for classifying manifolds is the determination of their homotopy type. For 4-manifolds having finite fundamental groups with periodic homology of period four, this was done in [**12**] (see also [**1**] and [**2**]). The case of a cyclic fundamental group of prime order was first treated in [**23**]. The homotopy type of 4-manifolds with free or surface fundamental groups was completely classified in [**6**] and [**7**] respectively. In particular, closed 4-manifolds M with a free fundamental group $\Pi_1(M) \cong *_p\mathbb{Z}$ (free product of p factors $\mathbb{Z}$) are classified, up to homotopy, by the isomorphism class of their intersection pairings $\lambda_M : H_2(M;\Lambda) \times H_2(M;\Lambda) \to \Lambda$ over the integral group ring $\Lambda = \mathbb{Z}[\Pi_1(M)]$. For $\Pi_1(M) \cong \mathbb{Z}$, we observe that the arguments developed in [**11**] classify these 4-manifolds, up to TOP homeomorphism, in terms of their intersection forms over $\mathbb{Z}$.

Furthermore, it was proved in [**7**] that a spin connected closed 4-manifold M with $\Pi_1(M) \cong \Pi_1(F)$, F a closed aspherical surface, is homotopy equivalent to a connected sum of $F \times \mathbb{S}^2$ with a simply-connected 4-manifold.

In this paper we shall consider the problem of when the homotopy type determines a classification of manifolds up to s-cobordism or up to a stable homeomorphism. We

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recall that two closed 4-manifolds M and N are said to be *stably homeomorphic* if $M\#k(\mathbb{S}^2\times\mathbb{S}^2)$ is TOP homeomorphic to $N\#\ell(\mathbb{S}^2\times\mathbb{S}^2)$ for some integers $k,\ell\geqslant 0$. It is well-known that TOP s-cobordant 4-manifolds are stably homeomorphic (see for example [**11**, Chapters 7 and 9]).

For either $\Pi_1\cong *_p\mathbb{Z}$, $p\geqslant 2$, or $\Pi_1\cong\Pi_1(F)$, F a closed (orientable) surface of genus at least two, the results of [**11**] are not applicable since the 4-dimensional disc theorem has only been established over elementary amenable groups, the class of groups generated by the class of finite groups and $\mathbb{Z}$ by the operations of extension and increasing union (see [**10**]). However, the above groups are sufficient to get classifications up to TOP s-cobordism by using surgery theory (see [**3**, **11**, **16**, **18**, **23**]) and recent results proved in [**8**] which correct some mistakes of the previous papers.

Our results can be stated as follows.

Theorem 1. *Let M and N be closed connected orientable (smooth) 4-manifolds with a free fundamental group $\Pi_1\cong *_p\mathbb{Z}$, $p\geqslant 1$. Then M is simple homotopy equivalent to N if and only if M is TOP s-cobordant to N.*

In particular, if $H_2(M;\mathbb{Q})=0$ then M is TOP s-cobordant to the connected sum $\#p\,(\mathbb{S}^1\times\mathbb{S}^3)$.

Theorem 1 generalises a well-known result of Wall to a nonsimply-connected case [**22**] and improves Theorem 2.1 of [**3**] for this class of manifolds. Furthermore, the case $H_2(M;\mathbb{Q})=0$ recovers Theorem 2 of [**14**] as a simple corollary.

Corollary 2. *(Hillman [**14**]).*

Let M be a closed connected orientable 4-manifold with the fundamental group $\Pi_1(M)\cong *_p\mathbb{Z}$, $p\geqslant 1$, and Euler characteristic $\chi(M)=2(1-p)$. Then M is TOP s-cobordant to $\#p\,(\mathbb{S}^1\times\mathbb{S}^3)$.

Another partial result can be obtained as follows. Suppose that M is a spin (smooth or PL) 4-manifold, that is, $w_2(M)=0$, where $w_2(.)$ denotes the second Stiefel-Whitney class.

Then one can also define the self-intersection pairing

$$\mu_M : H_2(M;\Lambda)\to\frac{\Lambda}{\{\lambda-\bar{\lambda}:\lambda\in\Lambda\}},$$

where $^-:\Lambda\to\Lambda$ is the canonical anti-involution on $\Lambda=\mathbb{Z}[\Pi_1]$ (see for example [**11**]).

Then the triple $(H_2(M;\Lambda),\lambda_M,\mu_M)$ determines an element of the 4^{th} Wall group $L_4(*_p\mathbb{Z})\cong\mathbb{Z}$ (see [4]). Under this isomorphism the class of $(H_2(M;\Lambda),\lambda_M,\mu_M)$ corresponds to $(1/8)\,\mathrm{sign}\,(M)\in\mathbb{Z}$. Let us denote by M' the simply-connected smooth 4-manifold obtained from M by killing the fundamental group. The intersection forms over $\mathbb{Z}$ of M and M' are the same, as shown in [**5**]. Since $\mathrm{sign}\,(M)=$

sign (M'), the definition of $L_4(\Pi_1)$ yields that the (self-)intersection pairings of M and $M'\#p(\mathbb{S}^1\times\mathbb{S}^3)$ are stably isomorphic over Λ. Thus the manifolds $M\#k(\mathbb{S}^2\times\mathbb{S}^2)$ and $M'\#p(\mathbb{S}^1\times\mathbb{S}^3)\#\ell(\mathbb{S}^2\times\mathbb{S}^2)$ are homotopy equivalent for some $k,\ell\geqslant 0$ (see [**6**, Theorem 1]). Now Theorem 1 above implies that they are TOP s-cobordant and hence stably homeomorphic.

In summary, we have proved the following stable classification result.

Theorem 3. *Let M be a closed connected orientable (smooth or PL) 4-manifold with $w_2(M)=0$ and $\Pi_1(M)\cong *_p\mathbb{Z}$, $p\geqslant 1$. Let M' be the simply-connected manifold obtained from M by killing $\Pi_1(M)$. Then M is stably homeomorphic to $M'\#p(\mathbb{S}^1\times\mathbb{S}^3)$.*

In a special case below, one can apply the Donaldson Theorem (see [**9**]) to obtain the following consequence.

Corollary 4. *Let M be a closed connected orientable (smooth) spin 4-manifold with a definite intersection form over $\mathbb{Z}$. If $\Pi_1(M)\cong *_p\mathbb{Z}$, $p\geqslant 1$, then M is TOP s-cobordant to $\#p(\mathbb{S}^1\times\mathbb{S}^3)$.*

Proof: Let M' be as above. Then the intersection form $\lambda_{M'}$ ($\cong\lambda_M$) is definite. The hypothesis $w_2(M)=0$ and [**9**] imply that $H_2(M;\mathbb{Z})\cong H_2(M';\mathbb{Z})\cong 0$, hence $H_2(M;\Lambda)\cong 0$ as $H_2(M;\Lambda)\otimes_\Lambda\mathbb{Z}\cong H_2(M;\mathbb{Z})$. Therefore we have $\lambda_M\cong 0$ over Λ and so M is simple homotopy equivalent to $\#p(\mathbb{S}^1\times\mathbb{S}^3)$ by Theorem 1 of [**6**]. The result now follows from Theorem 1 above. ∎

For manifolds with surface fundamental groups, we shall prove the following result.

Theorem 5. *Let M and N be closed connected orientable (smooth) 4-manifolds with $\Pi_1(M)\cong\Pi_1(N)\cong\Pi_1(F)$, where F is a closed aspherical (orientable) surface. Then M is simple homotopy equivalent to N if and only if M is TOP s-cobordant to N.*

Theorem 5 together with the results proved in [**7**] imply the following consequence.

Corollary 6. *Let M be as above. Suppose further that M is a spin manifold. Then M is TOP s-cobordant to the connected sum of the product $F\times\mathbb{S}^2$ with a simply-connected 4-manifold.*

2. Results from Surgery Theory

The proofs of our results use surgery theory (see for example [**3, 11, 16, 18, 23**]) as corrected in [**8**]. To make the reading easier, we recall some definitions and results listed in the quoted papers. Note that the Whitehead group Wh(Π_1) vanishes for $\Pi_1\cong *_p\mathbb{Z}$ or $\Pi_1\cong\Pi_1(F)$, F an aspherical surface, hence in our case "s-cobordant" is equivalent to "h-cobordant" (see [**17**]).

Let M^n be any closed connected orientable (H=TOP or PL) n-manifold with the fundamental group $\Pi_1 = \Pi_1(M)$ and let ξ^k be a linear bundle over M. Then $\Omega_n^H(M,\xi)$ denotes the set of bordism classes of normal maps (X,f,b), where X is a (H=TOP or PL) n-manifold, $f : X \to M$ a map of degree one, $b : \nu_X^k \to \xi^k$ a linear bundle map covering f and ν_X^k is the stable normal bundle of $X^n \to \mathbb{S}^{n+k}$, for k sufficiently large with respect to n.

Let $\mathcal{N}_n^H(M)$ be the union of all $\Omega_n^H(M,\xi)$ over all k-plane bundles ξ^k over M, modulo the additional equivalence relation that $(X_0,f_0,b_0) \in \Omega_n^H(M,\xi_0)$ is equivalent to $(X_1,f_1,b_1) \in \Omega_n^H(M,\xi_1)$ if and only if (X_0,f_0,b_0) is normally (H=TOP or PL) cobordant to (X_1,f_1,b_1), for some linear bundle automorphism $\xi_1 \to \xi_0$. The elements of $\mathcal{N}_n^H(M)$ are called the (H=TOP or PL) *normal invariants* of M. In the relative case, we include the condition that the normal map $(X,f,b) \in \mathcal{N}_n^H(M,\partial M)$, represented by $f\colon (X,\partial X) \to (M,\partial M)$, induces a simple homotopy equivalence when restricted to ∂X.

Let $\mathcal{S}_n^H(M)$ be the set of (H=TOP or PL) s-cobordism classes of orientation preserving simple homotopy equivalences $h\colon X \to M$, where X is a compact (H) n-manifold.

Let us denote by $L_n(\Pi_1)$ the n-th Wall group of surgery obstructions for the problem of obtaining simple homotopy equivalences for orientable n-manifolds with the fundamental group Π_1.

Recall that if $h\colon X \to M$ represents an element of $\mathcal{S}_n^H(M)$, then there exists an obvious forgetful map $\eta_n^H\colon \mathcal{S}_n^H(M) \to \mathcal{N}_n^H(M)$ which associates to (X,h) the class of (X,h,h^*) in $\mathcal{N}_n^H(M)$, where h^* is the obvious map on the stable normal bundles, induced by h.

Furthermore, there is a map $\sigma_n^H\colon \mathcal{N}_n^H(M) \to L_n(\Pi_1)$ which associates to any normal invariant (X,f,b) the surgery obstruction (for details see [**16**] and [**23**]).

Finally, denote by $\omega_n^H\colon L_{n+1}(\Pi_1) \to \mathcal{S}_n^H(M)$ the map induced by the action of $L_{n+1}(\Pi_1)$ on $\mathcal{S}_n^H(M)$ (see for example [**16**]).

The following result is well-known (see [**11**, p.200]).

Theorem 7. *(The surgery sequence).*

Let M^n be a closed connected orientable (H=TOP or PL) n-manifold with the fundamental group Π_1. Then the surgery sequence

$$L_{n+1}(\Pi_1) \xrightarrow{\omega_n^H} \mathcal{S}_n^H(M) \xrightarrow{\eta_n^H} \mathcal{N}_n^H(M) \xrightarrow{\sigma_n^H} L_n(\Pi_1)$$

is exact if $n \geqslant 5$. If $n = 4$, it is also exact provided Π_1 is an elementary amenable group.

For our classes of 4-manifolds we obtain a further result.

THEOREM 8. *Let M be a closed connected orientable (H=TOP or PL) 4-manifold. Suppose that $\Pi_1(M)$ is isomorphic to either $*_p\mathbb{Z}$, $p \geqslant 1$, or $\Pi_1(F_g)$, where F_g is the closed orientable surface of genus $g \geqslant 1$. Then the surgery sequence*

$$\mathcal{N}_5^{\mathrm{TOP}}(M) \xrightarrow{\sigma_5^{\mathrm{TOP}}} L_5(\Pi_1) \xrightarrow{\omega_4^{\mathrm{TOP}}} \mathcal{S}_4^{\mathrm{TOP}}(M) \xrightarrow{\eta_4^{\mathrm{TOP}}} \mathcal{N}_4^{\mathrm{TOP}}(M)$$

is exact.

PROOF: Let k denote either of the integers p or $2g$. We shall prove that every element of $L_5(\Pi_1)$ is realisable by an element in $\Omega_5^{\mathrm{TOP}}(M \times I, M \times \partial I)$, $I = [0,1]$. Thus the result follows from Theorem 6.3 of [**16**] (second part of the statement which is correct; compare also with [**8**]).

Since $\Pi_1(M) \cong *_k\mathbb{Z}$ or $\cong \Pi_1(F_g)$, we have that $L_5(\Pi_1) \cong \oplus_k\mathbb{Z}$ by Theorem 16 of [**4**]. Because M is orientable, any embedded 1-sphere $\tilde{f}\colon \mathbb{S}^1 \to M$ has a trivial normal bundle, that is, $\tilde{f}$ extends to an embedding $f\colon \mathbb{S}^1 \times D^3 \to M$. Let $f_1, f_2, \ldots, f_k\colon \mathbb{S}^1 \times D^4 \to M \times I$ be disjoint embeddings such that

$$\tilde{f}_1 = f_1|_{\mathbb{S}^1 \times 0}, \quad \tilde{f}_2 = f_2|_{\mathbb{S}^1 \times 0}, \quad \ldots, \quad \tilde{f}_k = f_k|_{\mathbb{S}^1 \times 0}$$

represent a set of generators of $\Pi_1(M \times I) \cong \Pi_1(M)$. This is always possible by the general position theorem. Let N_i, $i = 1, 2, \ldots, k$, be the TOP 5-manifold obtained by deleting $f_i\left(\mathbb{S}^1 \times \mathring{D}^4\right)$ from $M \times I$ and substituting $(\mathbb{S}^1 \times \|E_8\|)\backslash\left(\mathbb{S}^1 \times \mathring{D}^4\right)$ by an obvious identification of their boundaries. Here $\|E_8\|$ represents the simply connected TOP 4-manifold realising the form E_8 as constructed in [**10**].

Using an appropriate normal map

$$\mathbb{S}^1 \times \|E_8\| \to \mathbb{S}^1 \times \mathbb{S}^4,$$

we obtain a normal map of degree one

$$\xi_i\colon N_i \to M \times I = (M \times I)\backslash f_i\left(\mathbb{S}^1 \times \mathring{D}^4\right) \cup_{\mathbb{S}^1 \times \mathbb{S}^3} \mathbb{S}^1 \times \mathbb{S}^4\backslash \mathbb{S}^1 \times \mathring{D}^4,$$

hence $(N_i, \xi_i, \xi_i^*) \in \Omega_5^{\mathrm{TOP}}(M \times I, M \times \partial I, \xi_i)$. Furthermore, the surgery obstruction $\sigma_5(N_i, \xi_i, \xi_i^*)$ is exactly the i-th generator of $L_5(\Pi_1) \cong \oplus_k\mathbb{Z}$. This completes the proof. ∎

We can now apply Lemma 8 of [**13**] to obtain the following consequence.

COROLLARY 9. *Let M be as in Theorem 8. Then the map*

$$\eta_4^{\mathrm{TOP}}\colon \mathcal{S}_4^{\mathrm{TOP}}(M) \to \mathcal{N}_4^{\mathrm{TOP}}(M)$$

is injective.

PROOF: It was proved in [**13**] that the surgery obstruction map

$$\sigma_5^{\mathrm{TOP}} : \mathcal{N}_5^{\mathrm{TOP}}(M \times I, M \times \partial I) \to L_5(\Pi_1)$$

is surjective. This guarantees the exactness of the sequence shown in Theorem 8. But if σ_5^{TOP} is onto, then the map η_4^{TOP} must be injective as claimed. ∎

Now let M^4 be a closed connected orientable (smooth or PL) 4-manifold. Following [**8**], let us denote by $\mathrm{HE}_{\mathrm{Id}}(M)$ the set of homotopy classes of (simple) self-homotopy equivalences of M which induce the identities on Π_1 and on H_*. Recall that any (simple) homotopy equivalence defines a normal invariant. This gives rise to a map

$$n\colon \mathrm{HE}_{\mathrm{Id}}(M) \to \mathcal{N}_4^{\mathrm{PL}}(M).$$

The following result is based upon an argument of Wall (see [**23**, Theorem 16.6] and [**16**, Theorem 6.3]: first part of the statement) as corrected by Cochran-Habegger [**8**].

THEOREM 10. *Let M^4 be a closed connected orientable (smooth or PL) 4-manifold. Suppose that $H_2(\Pi_1(M);\mathbb{Z}_2) \cong 0$. Then the sequence*

$$\mathrm{HE}_{\mathrm{Id}}(M) \xrightarrow{\;n\;} \mathcal{N}_4^{\mathrm{PL}}(M) \xrightarrow{\;\sigma_4^{\mathrm{PL}}\;} L_4(\Pi_1(M))$$

is exact.

PROOF: We first calculate the set of normal maps $\mathcal{N}_4^H(M)$ by the method of Sullivan (see [15] and [**20**]). There is a bijection between $\mathcal{N}_4^H(M)$, H=TOP or PL, and the group $[M, G/H]$ of the homotopy classes of maps $M \to G/H$ (see for example [**16**, Theorem 5.4]). Since $\Pi_2(G/\mathrm{TOP}) \cong \mathbb{Z}_2$, $\Pi_3(G/\mathrm{TOP}) \cong \Pi_5(G/\mathrm{TOP}) \cong 0$ and $\Pi_4(G/\mathrm{TOP}) \cong \mathbb{Z}$ with vanishing k-invariant in $H^5(K(\mathbb{Z}_2,2))$, the Postnikov resolution of G/TOP gives a map $G/\mathrm{TOP} \to K(\mathbb{Z}_2,2) \times K(\mathbb{Z},4)$ which is a 5-equivalence, that is, we can assume that the 5-skeleton of G/TOP is the same as that of $K(\mathbb{Z}_2,2) \times K(\mathbb{Z},4)$. Thus we have

$$\mathcal{N}_4^{\mathrm{TOP}}(M) \cong [M, G/\mathrm{TOP}] \cong [M, K(\mathbb{Z}_2,2) \times K(\mathbb{Z},4)] \underset{S}{\cong} H^2(M;\mathbb{Z}_2) \oplus H^4(M).$$

The injection $\mathcal{N}_4^{\mathrm{PL}}(M) \to \mathcal{N}_4^{\mathrm{TOP}}(M)$ yields the isomorphisms (use the Wu formula)

$$\begin{aligned} S\colon \mathcal{N}_4^{\mathrm{PL}}(M) &\cong [M, G/\mathrm{PL}] \cong \{(a,b) \in H^2(M;\mathbb{Z}_2) \oplus H^4(M) \colon a^2 \equiv b \bmod 2\} \\ &\cong w_2(M)^{\perp} \oplus H^4(M) \cong \operatorname{Ker} w_2(M) \oplus H^4(M) \cong \operatorname{Hom}(\operatorname{Ker} w_2(M), \mathbb{Z}_2) \oplus H^4(M), \end{aligned}$$

where $w_2(M)^{\perp} = \{a \in H^2(M;\mathbb{Z}_2) \colon a^2 \equiv 0 \bmod 2\}$.

The map $\mathcal{N}_4^H(M) \to H^4(M) \to L_4(1) \cong \mathbb{Z} \subset L_4(\Pi_1)$ is given by the surgery obstruction (see [**23**, p.237]). Thus we obtain $\operatorname{Ker}\sigma_4^{\mathrm{TOP}} \subset H^2(M;\mathbb{Z}_2)$ and $\operatorname{Ker}\sigma_4^{\mathrm{PL}} \subset \operatorname{Hom}(\operatorname{Ker} w_2(M), \mathbb{Z}_2)$. Now by [**21**] we can represent a basis of $\operatorname{Ker} w_2(M)$ by characteristic 2-submanifolds V_i^2 of M. From our hypothesis, that $H_2(\Pi_1;\mathbb{Z}_2) \cong 0$, we are going to prove that any assignment of elements of $\mathbb{Z}_2$ to the classes $[V_i^2] \in \operatorname{Ker} w_2(M)$ is induced by a self-homotopy equivalence in $\mathrm{HE}_{\mathrm{Id}}(M)$. In particular, we have $\operatorname{Ker}\sigma_4^{\mathrm{PL}} = w_2(M)^{\perp} = \operatorname{Hom}(\operatorname{Ker} w_2(M), \mathbb{Z}_2)$. Suppose $[V] \in \operatorname{Ker} w_2(M)$ is one of the previous characteristic submanifolds. The exact sequence

$$\Pi_2(M) \to H_2(M;\mathbb{Z}_2) \to H_2(\Pi_1(M);\mathbb{Z}_2) \cong 0$$

implies that there exists a map $x\colon \mathbb{S}^2 \to M$ which is sent to $[V]$ under the surjection $\Pi_2(M) \to H_2(M;\mathbb{Z}_2)$. Choose an embedding $D^4 \subset M$. If D^4 is shrunk to a point, then the result is homeomorphic to M. Shrink instead ∂D^4 to a point to give a map $c\colon M \to M \vee \mathbb{S}^4$. Now let $\eta\colon \mathbb{S}^3 \to \mathbb{S}^2$ be the Hopf map, $\Sigma\eta\colon \mathbb{S}^4 \to \mathbb{S}^3$ its suspension and $\eta^2\colon \mathbb{S}^4 \to \mathbb{S}^2$ the composition $\eta^2 = \eta \circ \Sigma\eta$. Let $f\colon M \to M$ be the composite map

$$M \xrightarrow{\ c\ } M \vee \mathbb{S}^4 \xrightarrow{1\vee\eta^2} M \vee \mathbb{S}^2 \xrightarrow{1\vee x} M.$$

It is easy to see that f induces the identities on Π_1 and on H_*, hence f is a homotopy equivalence by Theorem 5.5 of [**16**]. To compute the splitting invariant of the map f along V, we assume V disjoint from D^4. Then $f^{-1}(V) = V \cup W$ with W framed in D^4 and the splitting invariant is the Arf invariant of W. We may always assume that $x\colon \mathbb{S}^2 \to M$ is an immersion which is transverse to V in M, that is, $x(\mathbb{S}^2)$ meets V transversely in n_V points. Then W is the union of n_V preimages (torii) in $\mathbb{S}^4$ (top cell) of oriented points in $\mathbb{S}^2$ under the map $\eta^2\colon \mathbb{S}^4 \to \mathbb{S}^2$. The associated quadratic form defined on $\operatorname{Ker}\bigl(f_*\colon H_1\bigl(f^{-1}(V)\bigr) \to H_1(V)\bigr)$ is the direct sum of its restrictions to $H_1(T)$ for each component torus T so that it suffices to compute the appropriate Arf invariant for a single torus T. Since T maps to a point in V, it follows by [**8**] that the quadratic form on $H_1(T)$ has Arf invariant $1 + w_2([x]) = 1 + w_2([V]) = 1$ as $[V] \in \operatorname{Ker} w_2(M)$. Hence each of the preimages in W has Arf invariant one. Thus the required $f \in \mathrm{HE}_{\mathrm{Id}}(M)$ can be constructed if $[x] \in \Pi_2(M)$ is dual to a mod 2 cohomology class which assignes to each V_i^2 the given corresponding element of $\mathbb{Z}_2$. Because $\Pi_2(M) \to H_2(M;\mathbb{Z}_2)$ is surjective such an x exists. This proves the statement. ∎

Proof of Theorem 1:

Let $h\colon N \to M$ be a simple homotopy equivalence. Since $H_2(\Pi_1(M);\mathbb{Z}_2) = H_2(*_p\mathbb{Z};\mathbb{Z}_2) \cong 0$, there exists a simple self-homotopy equivalence $f \in \mathrm{HE}_{\mathrm{Id}}(M)$ such that $n(f) = n(h) \in \mathcal{N}_4^{\mathrm{PL}}(M)$ by Theorem 10. Hence it follows that $\eta_4^{\mathrm{TOP}}(f) =$

$\eta_4^{\mathrm{TOP}}(h) \in \mathcal{N}_4^{\mathrm{TOP}}(M)$. By Corollary 9 f is TOP s-cobordant to h in $\mathcal{S}_4^{\mathrm{TOP}}(M)$, that is, M is TOP s-cobordant to N. The second part of the statement follows in the same way by using Theorem 1 of [**6**]. ∎

We conclude the section with a related computation.

Proposition 11. *Let M^4 be a closed connected orientable (TOP or PL) 4-manifold with fundamental group $\Pi_1 \cong *_p\mathbb{Z}$, $p \geqslant 1$. Then there is a bijection between $\mathcal{S}_5^{\mathrm{TOP}}(M \times I, M \times \partial I)$ and $H^1(M;\mathbb{Z}_2)$, hence the number of distinct topological 5-manifolds homotopy equivalent to $(M \times I, M \times \partial I)$ is at most 2^p.*

Proof: Since the surgery obstruction map σ_5^{TOP} is onto (see [**13**, Lemma 8]), the sequence

$$\mathcal{S}_5^{\mathrm{TOP}}(M \times I, M \times \partial I) \xrightarrow{\eta_5^{\mathrm{TOP}}} \mathcal{N}_5^{\mathrm{TOP}}(M \times I, M \times \partial I) \xrightarrow{\sigma_5^{\mathrm{TOP}}} L_5(\Pi_1) \longrightarrow 0$$

is exact (see [**16**, Theorem 5.11]). We are going to prove that η_5^{TOP} is injective. Since $L_6(*_p\mathbb{Z}) \cong \mathbb{Z}_2$ (see [**4**, Theorem 1.6]), the map $L_6(1) \cong \mathbb{Z}_2 \to L_6(*_p\mathbb{Z}) \cong \mathbb{Z}_2$ is an isomorphism, hence one can represent the nontrivial element of L_6 by a degree one normal map $(\mathbb{S}^3 \times \mathbb{S}^3, f, b)$ with $f\colon \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^6$ (see [**16**] and [**23**]). Then the action of $L_6(*_p\mathbb{Z}) \cong \mathbb{Z}_2$ on $\mathcal{S}_5^{\mathrm{TOP}}(M \times I, M \times \partial I)$ is defined by taking an element $h\colon (K, \partial K) \to (M \times I, M \times \partial I)$ in $\mathcal{S}_5^{\mathrm{TOP}}(M \times I, M \times \partial I)$ and forming the connected sum in the interior $(h \times 1)\# f\colon K \times I \# \mathbb{S}^3 \times \mathbb{S}^3 \to M \times I^2 = M \times I^2 \# \mathbb{S}^6$. Using the additivity of surgery obstructions and the fact $\sigma_6(h \times 1) = 0$, we have that $\sigma_6(h \times 1 \# f) = \sigma_6(f)$ so the action of $L_6(*_p\mathbb{Z})$ on $\mathcal{S}_5^{\mathrm{TOP}}(M \times I, M \times \partial I)$ is trivial. Thus η_5^{TOP} is injective as claimed. By Sullivan [**20**], there is a bijection between $\mathcal{N}_5^{\mathrm{TOP}}(M \times I, M \times \partial I)$ and the group $[(M \times I, M \times \partial I), (G/\mathrm{TOP}, *)]$.

Hence we have

$$\begin{aligned} \mathcal{N}_5^{\mathrm{TOP}}(M \times I, M \times \partial I) &\cong [M \times I / M \times \partial I, K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}, 4)] \\ &\cong H^2(\Sigma M; \mathbb{Z}_2) \oplus H^4(\Sigma M) \cong H^1(M; \mathbb{Z}_2) \oplus H^3(M) \cong \oplus_p \mathbb{Z}_2 \oplus \oplus_p \mathbb{Z}, \end{aligned}$$

where ΣM denotes the suspension of M. Putting all these facts together we have the exact sequence

$$0 \to \mathcal{S}_5^{\mathrm{TOP}}(M \times I, M \times \partial I) \xrightarrow{\eta_5^{\mathrm{TOP}}} \oplus_p \mathbb{Z}_2 \oplus \oplus_p \mathbb{Z} \xrightarrow{\sigma_5^{\mathrm{TOP}}} L_5(\Pi_1) \cong \oplus_p \mathbb{Z} \to 0.$$

Thus we obtain that

$$\operatorname{Ker} \sigma_5^{\mathrm{TOP}} \cong \operatorname{Im} \eta_5^{\mathrm{TOP}} \cong \oplus_p \mathbb{Z}_2 \cong H^1(M; \mathbb{Z}_2) \cong \mathcal{S}_5^{\mathrm{TOP}}(M \times I, M \times \partial I)$$

as required. ∎

3. Free groups

In this section we shall present a simple alternative proof of Theorem 1 which is essentially based upon some recent results due to Cochran and Habegger (see [8]).

Let M^4 be a closed connected orientable (smooth or PL) 4-manifold with fundamental group $\Pi_1(M) \cong *_p\mathbb{Z}$, $p \geqslant 1$. Choose embeddings $\varphi_i : \mathbb{S}^1 \times D^3 \to M$, $i = 1, 2, \ldots, p$, such that

$$\varphi_1|_{\mathbb{S}^1\times 0}, \varphi_2|_{\mathbb{S}^1\times 0}, \ldots, \varphi_p|_{\mathbb{S}^1\times 0}$$

represent a set of generators of $\Pi_1(M)$.

Let

$$M_0 = M \setminus \bigcup_{i=1}^{p} \varphi_i\left(\mathbb{S}^1 \times \mathring{D}^3\right)$$

and

$$M' = M_0 \cup \bigcup_{i=1}^{p} (D^2 \times \mathbb{S}^2).$$

The manifold M' is a simply-connected closed 4-manifold obtained by surgeries killing the fundamental group $\Pi_1(M)$.

Moreover, the canonical inclusions $M_0 \to M$ and $M_0 \to M'$ induce isomorphisms

$$H_2(M) \xleftarrow[\cong]{} H_2(M_0) \xrightarrow[\cong]{} H_2(M'),$$

which respect the integral intersection forms (see for example [5]).

By Sullivan' s results (see [20]) there are natural isomorphisms

$$S : \mathcal{N}_4^{\mathrm{PL}}(M) \cong \{(a,b) \in H^2(M;\mathbb{Z}_2) \times H^4(M;\mathbb{Z}) : a^2 \equiv b \bmod 2\},$$
$$S' : \mathcal{N}_4^{\mathrm{PL}}(M') \cong \left\{(a',b') \in H^2(M';\mathbb{Z}_2) \times H^4(M';\mathbb{Z}) : (a')^2 \equiv b' \bmod 2\right\},$$
$$S_0 : \mathcal{N}_4^{\mathrm{PL}}(M_0,\partial M_0) \cong \{(a_0,b_0) \in H^2(M_0,\partial M_0;\mathbb{Z}_2) \times H^4(M_0,\partial M_0;\mathbb{Z}) : a_0^2 \equiv b_0 \bmod 2\}.$$

Recall that any homotopy equivalence of a manifold defines a normal invariant. This gives rise to maps

$$n : HE_{\mathrm{Id}}(M) \to \mathcal{N}_4^{\mathrm{PL}}(M),$$
$$n' : HE_{\mathrm{Id}}(M') \to \mathcal{N}_4^{\mathrm{PL}}(M'),$$
$$n_0 : HE_{\mathrm{Id}}(M_0,\partial M_0) \to \mathcal{N}_4^{\mathrm{PL}}(M_0,\partial M_0),$$

where $HE_{\mathrm{Id}}(M_0,\partial M_0) = \{f \in HE_{\mathrm{Id}}(M_0) : f|_{\partial M_0} = \mathrm{Id}\}$.

Furthermore, there are well-known maps

$$\Pi_2(M)\otimes\Pi_4(\mathbb{S}^2)\xrightarrow{N}\Pi_4(M)\xrightarrow{\tau}HE_{\mathrm{Id}}(M),$$
$$\Pi_2(M')\otimes\Pi_4(\mathbb{S}^2)\xrightarrow{N'}\Pi_4(M')\xrightarrow{\tau'}HE_{\mathrm{Id}}(M'),$$
$$\Pi_2(M_0)\otimes\Pi_4(\mathbb{S}^2)\xrightarrow{N_0}\Pi_4(M_0)\xrightarrow{\tau_0}HE_{\mathrm{Id}}(M_0,\partial M_0)$$

defined as follows. For any $[x]\in\Pi_2(M)$, $x:\mathbb{S}^2\to M$, let

$$N([x]\otimes 1)=[x\circ\eta\circ\Sigma\eta],$$

where $\eta:\mathbb{S}^3\to\mathbb{S}^2$ is the Hopf map and $\Sigma\eta:\mathbb{S}^4\to\mathbb{S}^3$ is its suspension. The map τ is obtained by "pinching off" a 4-sphere from a 4-cell B which we choose in Int M_0, that is, if $[y]\in\Pi_4(M)$, $y:\mathbb{S}^4\to M$, then

$$\tau([y]):M\longrightarrow M\vee\mathbb{S}^4\xrightarrow{\mathrm{Id}\vee y}M$$

(see [**8**] or [**18**]).

Let us denote by

$$\sigma:\mathcal{N}_4^{\mathrm{PL}}(M)\to L_4(\Pi_1(M)),$$
$$\sigma':\mathcal{N}_4^{\mathrm{PL}}(M')\to L_4(1),$$
$$\sigma_0:\mathcal{N}_4^{\mathrm{PL}}(M_0,\partial M_0)\to L_4(\Pi_1(M_0))$$

the surgery obstruction maps. By [**4**] we have

$$L_4(\Pi_1(M))\cong L_4(1)\cong L_4(\Pi_1(M_0))\cong\mathbb{Z}.$$

The isomorphisms are induced by either $1\to *_p\mathbb{Z}$ or $*_p\mathbb{Z}\to 1$. Since σ,σ' and σ_0 detect the "top-splitting invariant", one obtains

$$\operatorname{Ker}\sigma=\{a\in H^2(M;\mathbb{Z}_2):a^2=0\}=w_2(M)^{\perp},$$
$$\operatorname{Ker}\sigma'=\left\{a'\in H^2(M';\mathbb{Z}_2):(a')^2=0\right\}=w_2(M')^{\perp},$$
$$\operatorname{Ker}\sigma_0=\left\{a_0\in H^2(M_0,\partial M_0;\mathbb{Z}_2):(a_0)^2=0\right\}=w_2(M_0)^{\perp},$$

as shown in the proof of Theorem 10.

Let us consider the following diagram

$$\begin{array}{ccccc}
H_2(M';\mathbb{Z}_2)=\Pi_2(M')\otimes\Pi_4(\mathbb{S}^2) & \xrightarrow{\tau'\circ N'} & HE_{\mathrm{Id}}(M') & \xrightarrow{S'\circ n'} & \operatorname{Ker}\sigma' \\
\cong\uparrow & & & & \alpha'\downarrow\cong \\
H_2(M_0;\mathbb{Z}_2)\subset\Pi_2(M_0)\otimes\Pi_4(\mathbb{S}^2) & \xrightarrow{\tau_0\circ N_0} & HE_{\mathrm{Id}}(M_0,\partial M_0) & \xrightarrow{S_0\circ n_0} & \operatorname{Ker}\sigma_0 \\
\cong\downarrow & & & & \alpha\uparrow\cong \\
H_2(M;\mathbb{Z}_2)\subset\Pi_2(M)\otimes\Pi_4(\mathbb{S}^2) & \xrightarrow{\tau\circ N} & HE_{\mathrm{Id}}(M) & \xrightarrow{S\circ n} & \operatorname{Ker}\sigma
\end{array}$$

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where $\alpha : \ker\sigma \to \operatorname{Ker}\sigma_0$ and $\alpha' : \operatorname{Ker}\sigma' \to \operatorname{Ker}\sigma_0$ are the isomorphisms induced by

$$H^2(M_0, \partial M_0; \mathbb{Z}_2) \xrightarrow[\cong]{} H^2(M_0; \mathbb{Z}_2)$$

and

$$H^2(M; \mathbb{Z}_2) \xrightarrow[\cong]{} H^2(M_0; \mathbb{Z}_2) \xleftarrow[\cong]{} H^2(M'; \mathbb{Z}_2)$$

as $w_2(\cdot)$ is natural. It can be easily seen that the above diagram commutes. One of the main result of [8] is that the maps $S' \circ n'$ and $\tau' \circ N'$ are bijective (see Theorem 5.2, Section 5).

Proof of Theorem 1:

Let $h: N \to M$ be a simple homotopy equivalence, hence $n(h) \in \operatorname{Ker}\sigma$. By the above diagram and [8] there exists a homotopy equivalence $f : M \to M$ such that $n(f) = n(h)$.

Let

$$(W^5, N, M) \xrightarrow{\xi} (M \times I, M \times \partial I)$$

be a normal TOP cobordism between f and h $(I = [0,1])$. The proof now follows from the following lemma.

Lemma 12. *If $x \in L_5(\Pi_1(M \times I)) \cong \oplus_p \mathbb{Z}$ (see [4]), then there exists a normal TOP cobordism*

$$\eta : (V^5, \partial V = M \cup M) \to (M \times I, M \times \partial I)$$

such that $\eta|_{\partial V} = f \cup f$ and $\sigma(\eta) = x$.

If we take $y = -x = -\sigma(\eta)$, then the surgery obstruction of the composed normal map

$$\xi \cup \eta : (W \cup V, N \cup M) \to (M \times [0,2], M \times \partial[0,2])$$

is zero, hence by surgery one obtains an s-cobordism between f and h in $S_4^{\mathrm{TOP}}(M)$. Thus the manifolds N and M are TOP s-cobordant as claimed. □

Proof of Lemma 12:

The proof proceeds in the usual way (see also Theorem 8). We start with the homotopy equivalence

$$f \times \mathrm{Id} : (M \times I, M \times \partial I) \to (M \times I, M \times \partial I).$$

Since $f \in HE_{\mathrm{Id}}(M)$, we can assume that f is the identity on each $\varphi_i(\mathbb{S}^1 \times D^4)$, $i = 1, 2, \ldots, p$, which represent the generators of $\Pi_1(M)$.

Let N_i, $i = 1, 2, \ldots, p$, be the topological 5-manifold obtained by deleting $\varphi_i\left(\mathbb{S}^1 \times \overset{\circ}{D}{}^4\right)$ from $M \times I$ and substituting $(\mathbb{S}^1 \times \|E_8\|) \setminus \left(\mathbb{S}^1 \times \overset{\circ}{D}{}^4\right)$ by an obvious identification of their boundaries (compare also with the proof of Theorem 8).

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Because f is the identity on each $\varphi_i(\mathbb{S}^1 \times D^4)$, using an appropriate normal map $\mathbb{S}^1 \times \|E_8\| \to \mathbb{S}^1 \times \mathbb{S}^4$, we obtain a normal TOP cobordism

$$\xi_i : N_i \to M \times I = (M \times I) \setminus \varphi_i\left(\mathbb{S}^1 \times \mathring{D}^4\right) \bigcup_{\mathbb{S}^1 \times \mathbb{S}^3} \left(\mathbb{S}^1 \times \mathbb{S}^4 \setminus \mathbb{S}^1 \times \mathring{D}^4\right).$$

such that $\xi|_{\partial N_i} = f \cup f$ and the surgery obstruction $\sigma(\xi_i)$ is exactly the i-th generator of $L_5(\Pi_1) = \oplus_p Z$. This proves the lemma. □

EXAMPLE. Suppose $p = 1$, that is, $\Pi_1(M) \cong \mathbb{Z}$, λ is an intersection form on a finitely generated free $\mathbb{Z}[\mathbb{Z}]$-module, $k \in \mathbb{Z}_2$ and if λ is even, then we assume $k \equiv (\text{signature}\,\lambda)/8 \pmod 2$. Then there is an oriented closed manifold M_λ with $\Pi_1 \cong \mathbb{Z}$, intersection form λ and Kirby-Siebenmann invariant k (see [**15**, p.113]). Hence the connected sum

$$M = M_{\lambda_1} \# \dots \# M_{\lambda_p} \# M'$$

(where M' is a simply connected manifold) has $\Pi_1 \simeq *_p\mathbb{Z}$ and intersection form

$$\lambda_M = \lambda_1 \oplus \cdots \oplus \lambda_p \oplus \lambda_{M'}$$

on the Λ-module $\Pi_2(M, \Lambda)$, $\Lambda = \mathbb{Z}[*_p\mathbb{Z}]$.

4. Surface groups

In this section we shall prove Theorem 5. The statement follows in the same way as in section 2 together with the following result.

THEOREM 13. *Let M be a closed connected orientable (smooth) 4-manifold with $\Pi_1(M) \cong \Pi_1(F)$, where F is a closed aspherical (orientable) surface. Then the sequence*

$$\mathrm{HE}_{\mathrm{Id}}(M) \xrightarrow{n} \mathcal{N}_4^{\mathrm{PL}}(M) \xrightarrow{\sigma_4^{\mathrm{PL}}} L_4(\Pi_1(M)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$$

is exact.

PROOF: As proved in [**13**, p.279], we have

$$\begin{aligned}\Pi_2(M) \cong H_2(M;\Lambda) &\cong \mathrm{Ext}^2_\Lambda(H_0(M;\Lambda),\Lambda) \oplus \mathrm{Ext}^0_\Lambda(H_2(M;\Lambda),\Lambda)\\ &\cong H^2(F) \oplus \mathrm{Ext}^0_\Lambda(H_2(M;\Lambda),\Lambda),\end{aligned}$$

where $Q = \mathrm{Ext}^0_\Lambda(H_2(M;\Lambda),\Lambda)$ is stably Λ-free. Using the universal coefficient spectral sequence

$$\mathrm{Tor}^\Lambda_p(H_q(M;\Lambda),\mathbb{Z}) \Longrightarrow H_{p+q}(M;\mathbb{Z}),$$

we obtain

$$\begin{aligned} H_2(M;\mathbb{Z}) &\cong \mathrm{Tor}_0^{\Lambda}(H_2(M;\Lambda),\mathbb{Z}) \oplus \mathrm{Tor}_2^{\Lambda}(H_0(M;\Lambda),\mathbb{Z}) \\ &\cong H_2(M;\Lambda)\otimes_\Lambda \mathbb{Z} \oplus H_2(\Pi_1;\mathbb{Z}) \cong \Pi_2(M)\otimes_\Lambda \mathbb{Z}\oplus H_2(F;\mathbb{Z}), \end{aligned}$$

hence

$$H^2(M;\mathbb{Z}_2)\cong \Pi_2(M)\otimes_\Lambda \mathbb{Z}_2 \oplus H_2(F;\mathbb{Z}_2)\cong \Pi_2(M)\otimes_\Lambda \mathbb{Z}_2\oplus \mathbb{Z}_2.$$

It follows that

$$\mathcal{N}_4^{\mathrm{TOP}}(M)\cong [M, G/\mathrm{TOP}]\cong \Pi_2(M)\otimes_\Lambda \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}.$$

Since the surgery obstruction map $\sigma_4^{\mathrm{TOP}}\colon \mathcal{N}_4^{\mathrm{TOP}}(M)\to L_4(\Pi_1)\cong \mathbb{Z}_2\oplus\mathbb{Z}$ (see [4]) is onto (see [**13**, Lemma 8]), we have that $\mathrm{Ker}\,\sigma_4^{\mathrm{TOP}}\cong \Pi_2(M)\otimes_\Lambda\mathbb{Z}_2$, that is, any element of $\mathrm{Ker}\,\sigma_4^{\mathrm{TOP}}$ can be realised by a map $\mathbb{S}^2\to M$. Let j denote the injection $\mathcal{N}_4^{\mathrm{PL}}(M)\to\mathcal{N}_4^{\mathrm{TOP}}(M)$. Since $\sigma_4^{\mathrm{PL}}=\sigma_4^{\mathrm{TOP}}\circ j$, we have that

$$j\left(\mathrm{Ker}\,\sigma_4^{\mathrm{PL}}\right)\subset \mathrm{Ker}\,\sigma_4^{\mathrm{TOP}},$$

hence $j\left(\mathrm{Ker}\,\sigma_4^{\mathrm{PL}}\right)\subset \Pi_2(M)\otimes_\Lambda\mathbb{Z}_2$. Now the proof can be completed as shown in Theorem 10 since any element of $\mathrm{Ker}\,\sigma_4^{\mathrm{PL}}$ can be realised by spherical maps. Indeed, the exact sequence

$$\Pi_2(M)\to H_2(M;\mathbb{Z}_2)\cong \Pi_2(M)\otimes_\Lambda\mathbb{Z}_2\oplus H_2(F;\mathbb{Z}_2)\to H_2(\Pi_1;\mathbb{Z}_2)\cong H_2(F;\mathbb{Z}_2)\to 0$$

shows that $\Pi_2(M)$ covers $\Pi_2(M)\otimes_\Lambda\mathbb{Z}_2$, as requested. ∎

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