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GEOMETRIC PROGRAMMING WITH PROBABILISTIC DECISION VARIABLES

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Abstract

Here we consider a particular class of stochastic geometric programs in which the randomness occurs in the decision variables. Specifically we analyse a program in which we specify a joint normal probability for the decision variables and require the constraint set to be satisfied in the chance constrained mode. A numerical example is given to illustrate the approach.

1. Introduction

Uncertainty is a common feature in problems to which mathematical programming is applied. Indeed it is often assumed that the parameters are known exactly even though it is known that they are a result of statistical estimation. By this means, one eliminates the very significant technical problems that arise if the stochastic element is treated explicitly. However there are numerous applications in areas such as finance, engineering design and reliability [11, 12] where the stochastic element cannot be ignored. In order to cope with such problems the field of stochastic programming has evolved and considerable success has been obtained with the chance constrained approach of Charnes and Cooper [2] and recourse programming [11]. Typically these approaches allow some or all of the parameters of the mathematical program to be random variables derived from known probability density functions. In chance constrained programming, one permits constraint violations up to specified probability limits. For such problems Prekopa [8] has derived necessary and sufficient conditions for optimality for very general distributions including the normal distribution. In addition, he converts the chance constrained program into a solvable certainty equivalent.

Here we follow the chance-constrained philosophy but introduce randomness via the decision variables. We term this a mathematical program with probabilistic decision variables. Specifically we consider a geometric program [4] in which the decision variables are normally distributed and the linear constraint set is of the chance-constrained variety. By approximating the normal distribution we transform this into an equivalent deterministic posynomial program which may be readily solved by conventional techniques.

By a posynomial program, we mean a mathematical program of the form:

$$(P) \qquad \textit{minimize} \quad g_0(t) \tag{1}$$

subject to constraints

$$g_k(t) \leqslant 1, \quad k = 1, 2, \ldots, p, \tag{2}$$

$$g_k(t) \geqslant 1, \quad k = p+1, \ldots, r, \tag{3}$$

and positivity conditions

$$t_j > 0, \quad j = 1, \ldots, m. \tag{4}$$

Here

$$g_k(t) = \sum_{i \in [k]} c_i \prod_{i=1}^{m} t_j^{a_{ij}}, \quad k = 0, \ldots, r,$$

and $[k]$, $k = 0, 1, \ldots, r$, is a collection of disjoint index sets which form a sequential partition of the integers 1 to n (n is the total number of terms in the objective function plus all the constraints). The a_{ij} are arbitrary real exponents and the coefficients c_i are positive. The name *posynomial* is derived from the *polynomial* form of the functions and the *positivity* of the coefficients and variables.

Posynomials arise naturally in the disciplines of economics and engineering. Moreover as polynomials may be converted into posynomials, posynomials are of wide applicability [7]. Posynomial programs form the original class of programs to be analysed by geometric programming duality which provides the basis for efficient computer codes [3, 9, 10].

As a result of the versatility of the polynomial form for modelling and the basis for algorithms provided by geometric programming it is valuable if geometric programs with probabilistic decision variables can be converted into deterministic posynomial form. In Section 2, we carry this calculation through and give a computational example in Section 3.

2. Certainty equivalent program

We consider a linearly constrained posynomial program of the form:

$$(L) \qquad \textit{minimize} \quad g_0(t) \tag{5}$$

$$\textit{subject to} \quad \sum_{i \in [k]} c_i \, t_{j(i)} \leqslant 1, \quad k = 1, \ldots, p. \tag{6}$$

Here $j(i)$ determines the variable j appearing in the ith term. Although the linear constraints may appear to be a considerable restriction on the posynomial form, in many design and scheduling problems all the nonlinearity appears in the objective function. Let the variables t_j be independent normally distributed random variables with mean μ_j (a variable) and variance σ_j^2 (a constant) for all j that is $N(\mu_j, \sigma_j^2)$. Hence the left hand side of each constraint is a normal variate with mean $\sum_{i\in[k]} c_i \mu_{j(i)}$ and variance $\sum_{i\in[k]} c_i \sigma_{j(i)}^2$.

Initially we consider the problem with one constraint of the form (6). We require the probability of the left-hand side to be less than 1 with at least probability q. Hence

$$P(x \leqslant 1) \geqslant q \tag{7}$$

where x is

$$N(\sum_{i\in[k]} c_i \mu_{j(i)}, \quad \sum_{i\in[k]} c_i \sigma_{j(i)}^2).$$

Thus we seek $\mu_{j(i)}$, $\sigma_{j(i)}$ which minimize the objective function and such that a realization from the distribution will satisfy the constraint with probability q. In order to convert inequality (7) into a posynomial form we make use of a numerical approximation due to Hastings [5] who approximates the integral

$$\frac{2}{\sqrt{\pi}} \int_0^z \exp(-u^2)\, du = \Phi(z) \tag{8}$$

by

$$\Phi_1(z) = 1 - 1/(1 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4)^4 \tag{9}$$

to an accuracy of 0.5×10^{-3} where $b_1 = 0.278393$, $b_2 = 0.230389$, $b_3 = 0.000972$, $b_4 = 0.078108$. A closer approximation is

$$\Phi_2(z) = 1 - 1/(1 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + d_5 z^5 + d_6 z^6)^{16} \tag{10}$$

to an accuracy of 0.3×10^{-6} where $d_1 = 0.0705230784$, $d_2 = 0.0422820123$, $d_3 = 0.0092705272$, $d_4 = 0.0001520143$, $d_5 = 0.0002765672$, $d_6 = 0.0000430638$.

For the purposes of this paper, $\Phi_1(z)$ will be employed. The use of $\Phi_2(z)$ is analogous.

Assuming that q is close to one we have that

$$P(x \leqslant 1) = \frac{1}{(2\Pi \sum_{i\in[k]} c_i \sigma_{j(i)}^2)^{\frac{1}{2}}} \int_{-\infty}^{1} \{\exp - (x - \sum_{i\in[k]} c_i \mu_{j(i)})^2 / (2(\sum_{i\in[k]} c_i \sigma_{j(i)}^2))\}\, dx.$$

Setting

$$z = (x - \sum_{i\in[k]} c_i \mu_{j(i)})/(2 \sum_{i\in[k]} c_i \sigma_{j(i)}^2)^{\frac{1}{2}}, \quad P(x \leqslant 1) \geqslant q$$

can be approximated by

$$z \leqslant (1 - \sum_{i \in [k]} c_i \mu_{j(i)})/(2 \sum_{i \in [k]} c_i \sigma_{j(i)}^2)^{\frac{1}{2}} \tag{11}$$

and

$$0.5 + 0.5\Phi_1(z) \geqslant q. \tag{12}$$

Substituting equation (9) into (12), it may be rewritten in the form

$$((1-q)/(0.5))^{\frac{1}{2}}(1 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4) \geqslant 1 \tag{13}$$

We are now able to develop the certainty equivalent posynomial program for programs of the form

(*S1*) $$\textit{minimize} \quad g_0(E(t)) \tag{14}$$

$$\textit{subject to} \quad P(\sum_{i \in [1]} c_i t_{j(i)} \leqslant 1) \geqslant q. \tag{15}$$

The equivalent deterministic program to ($S1$) is

($C1$) $$\textit{minimize} \quad g_0(\mu) \tag{16}$$

$$\textit{subject to} \quad \sum_{i \in [k]} c_i \mu_{j(i)} + (2 \sum_{i \in [k]} c_i \sigma_{j(i)}^2)^{\frac{1}{2}} z \leqslant 1, \tag{17}$$

$$((1-q)/0.5)^{\frac{1}{2}}(1 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4) \geqslant 1 \tag{18}$$

where the variables are the scalar z and vector μ.

We are now in a position to consider the more interesting case of p constraints where multivariate correlations play an important role. We consider the following program:

(Sp) $$\textit{minimize} \quad g_0(E(t)) \tag{19}$$

$$\textit{subject to} \quad P(\sum_{i \in [k]} c_i t_{j(i)} \leqslant 1,\ k = 1, \ldots, p) \geqslant q. \tag{20}$$

In this generalization, the probability distribution is multivariate normal with mean vector (of dimension p) with components $M_k = \sum_{i \in [k]} c_i \mu_{j(i)}$, $k = 1, \ldots, p$, and variance covariance matrix $\Sigma = (\sigma_{kl})$ where

$$\sigma_{kl} = \sum_{i \in [k]} \sum_{i' \in [l]} c_i c_{i'} \sigma_{j(i)}^2 \delta_{j(i), j(i')} \tag{21}$$

where σ_j^2 is the variance of the j variable and $\delta_{j(i), j(i')}$ is a Kronecker delta.

If we define Δ to be the triangular square root of Σ^{-1}, then

$$y = \Delta^{-1}(x - M) \tag{22}$$

is $N(0, I)$ for x, $N(M, \Sigma)$, that is, the transformation removes the correlation. In order to employ the approximation $\Phi_1(z)$, we require the transformation $z = y/\sqrt{2}$ so that $x = M + \sqrt{(2\Delta)}\, z$.

Program (Sp) can now be written as

$$(Cp) \qquad \textit{minimize} \quad g_0(\mu) \tag{23}$$

$$\textit{subject to} \quad \sum_{i\in[k]} c_i\,\mu_{j(i)} + \sqrt{2}\sum_{l=1}^{p} \Delta_{kl}\, z_l \leqslant 1, \quad k = 1, \ldots, p, \tag{24}$$

$$(2v_k)^{\frac{1}{4}}(1 + b_1 z_k + b_2 z_k^2 + b_3 z_k^3 + b_4 z_k)^4 \geqslant 1, \quad k = 1, \ldots, p, \tag{25}$$

$$v_k + w_k \leqslant 1, \quad k = 1, \ldots, p, \tag{26}$$

$$q \prod_{k=1}^{p} w_k^{-1} \leqslant 1 \tag{27}$$

over the variables μ, v, w and z. Here inequality (24) arises analogously to (17), and inequality (25) is the generalization of (18) which has been combined with (26) such that (25) is in posynomial form. Inequality (27) is the joint probability constraint which is factorizable since the correlations have been removed. If any of the Δ_{kl} are negative, equation (24) can be converted into posynomial form using the transformations found in [7].

Program (Cp) will always have reversed constraints (constraints of the form $g_k(t) \geqslant 1$) which means that the geometric program is not necessarily convex and hence the Kuhn–Tucker conditions are not necessary and sufficient. Prekopa [8] has shown that (Sp) is at least quasi-convex and the Kuhn–Tucker conditions are necessary and sufficient. The key transformation for geometric programming is the natural logarithm. Since $\partial \ln t_j / \partial t_j = 1/t_j > 0$ for $t_j > 0$ the transformation is non-singular. Thus, appealing to the following theorem, Kuhn–Tucker conditions are necessary and sufficient for (Cp).

THEOREM. *Assuming a constraint qualification holds, the Kuhn–Tucker conditions are necessary and sufficient for mathematical programs if there exists a transformation of the variable with non-zero Jacobian which transforms the program into a program for which the Kuhn–Tucker conditions are both necessary and sufficient.*

PROOF. Consider the mathematical program

$$(F) \qquad \textit{min} \quad f_0(x)$$

$$\textit{subject to} \quad f_i(x) \leqslant 0, \quad i = 1, m$$

for which the Kuhn–Tucker conditions are necessary and sufficient. Thus $\hat{x}$ is optimal for (F) if and only if

$$f_i(\hat{x}) \leqslant 0, \quad i = 1, \ldots, m$$

and there exists $\lambda_i \geqslant 0$ such that

$$\lambda_i\, f_i(\hat{x}) = 0, \quad i$$

and

$$\nabla f_0(\hat{x}) + \sum_{i=1}^{m} \lambda_i \nabla f_i(\hat{x}) = 0.$$

Let T be a transformation with non-zero Jacobian of y into x:

$$x = T(y).$$

Then (F) is equivalent to

$$\begin{aligned} &\textit{min} \quad f_0(T(y)) \\ &\textit{subject to} \quad f_i(T(y)) \leqslant 0, \quad i = 1, \ldots, m. \end{aligned}$$

The Kuhn–Tucker conditions for this program are

$$f_i(T(y)) \leqslant 0, \quad i = 1, \ldots, m,$$

$$\nabla f_0(T(y))\, T'(y) + \sum_{i=1}^{m} \lambda_i \nabla f_i(T(y))\, T'(y) = 0 \tag{28}$$

or

$$\left[\nabla f_0(T(y)) + \sum_{i=1}^{m} \lambda_i \nabla f_i(T(y))\right] T'(y) = 0. \tag{29}$$

Since $T'(y)$ is non-singular (29) holds if and only if

$$\nabla f_0(T(y)) + \sum_{i=1}^{m} \lambda_i \nabla f_i(T(y)) = 0$$

which proves the theorem.

The example in the next section demonstrates this approach.

3. Example

Consider the process of making a box of length t_1, width t_2, depth t_3 subject to the constraints on $t_1 + t_2 + t_3$ being less than 140 cm and $t_2 + t_3$ less than 80 cm. The box which maximizes the volume is found by solving the geometric program

$(V1)$

$$\begin{aligned} &\textit{minimize} \quad t_1^{-1} t_2^{-1} t_3^{-1} \\ &\textit{subject to} \quad \frac{t_2}{80} + \frac{t_3}{80} \leqslant 1, \\ &\frac{t_1}{140} + \frac{t_2}{140} + \frac{t_3}{140} \leqslant 1. \end{aligned}$$

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The solution to $(V1)$ is

$$t_1 = 60, \quad t_2 = 40, \quad t_3 = 40$$

Suppose now that there is some variation in the production process so that the final dimensions are normally distributed with means μ_1, μ_2, μ_3 and variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$. For the example let the variances all be $\frac{1}{50}$. If the boxes exceed the specifications of 140 cm and 80 cm they must be thrown away. Suppose we do not wish to reject boxes more than 5% of the time. This problem gives the following stochastic program:

$$(Se) \qquad \text{minimize} \quad (E(t_1 t_2 t_3))^{-1}$$

$$\text{subject to} \quad P\left(\frac{t_2}{80}+\frac{t_3}{80} \leqslant 1, \frac{t_1}{140}+\frac{t_2}{140}+\frac{t_3}{140} \leqslant 1\right) \geqslant 0.95$$

For this problem $M_1 = \mu_1+\mu_2+\mu_3$ and $M_2 = \mu_1+\mu_2$, and we have

$$\Sigma = \frac{1}{50}\begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix},$$

$$\Delta = \sqrt{50}\begin{pmatrix} \sqrt{3/2} & -\sqrt{2/3} \\ 0 & \sqrt{1/3} \end{pmatrix}.$$

The equivalent deterministic program is

$$(Ce) \qquad \text{minimize} \quad \mu_1^{-1}\mu_2^{-1}\mu_3^{-1}$$

$$\text{subject to} \quad 10\sqrt{3/2}\, z_1 \mu_1^{-1} + \tfrac{1}{2}\mu_2 \mu_1^{-1} + \tfrac{1}{2}\mu_3 \mu_1^{-1} + 20\mu_1^{-1} \leqslant 1,$$

$$\frac{\sqrt{3}}{14} z_2 + \frac{\mu_2}{140} + \frac{\mu_3}{140} + \frac{\mu_1}{140} \leqslant 1,$$

$$\frac{v_k^{\frac{1}{4}}}{0.5}(1+b_1 z_k + b_2 z_k^2 + b_3 z_k^3 + b_4 z_k^4) \geqslant 1, \quad k = 1, 2,$$

$$v_k + w_k \leqslant 1, \quad k = 1, 2,$$

$$0.95 w_1^{-1} w_2^{-1} \leqslant 1.$$

The solution to (Ce) is

$$\mu_1 = 60.29, \quad \mu_2 = 39.47, \quad \mu_3 = 39.47,$$

$$w_1 = 0.989, \quad w_2 = 0.957.$$

Of interest is the fact that the setting for width and depth is reduced to 39.47 whereas the length is increased to 60.29.

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