THERMODYNAMIC TRANSPORT PROPERTIES IN DENSE STARS

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ABSTRACT

The thermodynamic transport properties of special relativistic imperfect fluids, as found in dense stars, are investigated. These properties, which include thermal and electrical conductivities, electrothermal coefficients, and bulk and shear viscosities may be formulated in terms of the momentum distribution functions obtained from the solution of the Boltzmann transport equation. Spherical harmonic solutions of the relaxation form of the relativistic magnetic Boltzmann transport equation have also been obtained which give the non-equilibrium momentum distribution function perturbation $f-f^{(0)} = \Delta f(p)$ in terms of electromagnetic and thermal fields.

I. INTRODUCTION

The thermodynamic transport properties (TTP) of imperfect fluids, initially developed by Marshak (1941), Mestel (1950), and Lee (1950), have been refined by Hubbard (1966), Lampe (1968a,b), Hubbard and Lampe (1969), Canuto (1970), Solinger (1970), Canuto and Solinger (1970), Kovetz and Shaviv (1973), and Flowers and Itoh (1976,1979). These works culminate in calculations which include complicating effects of inter-ionic correlations and strong degeneracy. Nevertheless, these calculations do not represent a piecewise complete formalism. In particular, they do not contain complete descriptions of all transport properties such as thermoelectric effects and both bulk and shear viscosities. Moreover, little is know about magnetic effects of the TTP. Finally, the existing TTP calculations are largely based on highly simplified or very complex solutions of the governing Boltzmann transport equation. It is obviously desirable to overcome each of these limitations.

In this work, we extend the formalism of relativistic fluid theory to study the TTP of imperfect fluids including thermal and electrical conductivities, electro-thermal coefficients, and bulk and shear viscosities. We also obtain spherical harmonic solutions of the relativistic

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Boltzmann transport equation which permit us to include magnetic effects into the TTP. Such effects are of importance in dense stellar magnetic plamsas such as in white dwarfs where surface magnetic fields of $B \simeq 10^6$ - 10^7 gauss are observed (cf. Liebert, Angel, and Landstreet 1975), pulsars, and neutron stars where $B \simeq 10^{12}$ gauss are expected (Ruderman 1974).

II. THE THERMODYNAMIC TRANSPORT PROPERTIES

For a relativistic fluid, the thermodynamic transport properties are obtained from the electrical current flow four-vector J^α and the energy-momentum flow rate (the stress-energy) four-tensor $T^{\alpha\beta}$ which are given by

$$J^\alpha = 2h^{-3} \int_0^\infty qU^\alpha f(x^\mu,p^\nu)\, d^4p = 2h^{-3} \int_0^\infty q\, (\gamma v^a,\gamma c)\, f\, (mc^2/W)\, d^3p$$

$$= 2h^{-3} \int_0^\infty q\, (v^a,c)\, f\, d^3p = (J^a,nqc)\ , \qquad (1)$$

$$T^{\alpha\beta} = 2h^{-3} \int_0^\infty p^\alpha U^\beta f(x^\mu,p^\nu) d^4p = 2h^{-3} \int_0^\infty (p^a,W/c)(\gamma v^b,\gamma c)\, f\, (mc^2/W) d^3p$$

$$= 2h^{-3} \int_0^\infty \begin{bmatrix} p^a v^b, & p^a c \\ Wv^b/c, & W \end{bmatrix} f\, d^3p = \begin{bmatrix} S^{ab}, & F^a/c \\ F^b/c, & w \end{bmatrix} . \qquad (2)$$

Here we have identified the (classical) three-forms of the particle number density n, the total energy density w, the electrical current density J^a, the energy flow rate (flux) F^a, and the classical stress-energy tensor S^{ab}. In obtaining these forms we have employed the four-vector momentum $p^\alpha = mU^\alpha = m\gamma(v^a,c) = (p^a,W/c)$ where W is the relativistic total energy of a fluid element.

We may anticipate a solution form of the Boltzmann transport equation (see section III below) such that

$$f\text{-}f^{(0)} = \tau(p)\, (\partial f^{(0)}/\partial\varepsilon)\, \left[(\partial\varepsilon/\partial v^b)(\partial v^b/\partial x^a) + L(\varepsilon,\mu,T)(\partial T/\partial x^a) - qE_a\right] \cdot$$

$$\cdot \left[\Gamma_0\, v^a + \Gamma_1\, (\tau e/m)\, E^a + \cdots\right] , \qquad (3)$$

where $f^{(0)}$ is the equilibrium distribution function given by

$$f^{(0)} = \left[\exp(\varepsilon-\mu)/kT + \Theta\right]^{-1}, \qquad \Theta = (+1,0,-1), \qquad (4)$$

where $L(\varepsilon,\mu,T) = \left[(\varepsilon-\mu)/T + \partial\mu/\partial T\right]$, and where Γ_0 and Γ_1 are geometrical coefficients which depend on $\underline{\nabla}T$, $\underline{E}$, and $\underline{B}$.

This solution of the relativistic Boltzmann transport equation, when inserted into equations (1) and (2) yields the following forms for the transport properties

$$w = 2h^{-3} \int W\, f\, d^3p = \rho_0 c^2 + U + (K_1)^a\, (\partial T/\partial x^a) + (K_2)^a\, E_a + \cdots , \qquad (5)$$

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$$J^a = 2h^{-3} \int qV^a f\, d^3p = (L_{11})^{ab} (\partial T/\partial x^b) + (L_{12})^{ab} E_b + \cdots \quad , \tag{6}$$

$$F^a = 2h^{-3} \int Wv^a f\, d^3p = (L_{21})^{ab} (\partial T/\partial x^b) + (L_{22})^{ab} E_b + \cdots \quad , \tag{7}$$

$$S^{ab} = 2h^{-3} \int p^a V^b f\, d^3p = S_o^{ab} + (M_1)^{abc} (\partial T/\partial x^c) + (M_2)^{abc} E_c + \cdots \quad , \tag{8}$$

$$S_o^{ab} = \left[-P + (\zeta - 2\eta/3)\,(\partial v^c/\partial x^c)\right] \delta^{ab} + 2\eta\, e^{ab} + 2\,\psi\,\omega^{ab} \ , \tag{9}$$

where η, ζ, and ψ are, respectively, the coefficients of shear viscosity, bulk viscosity, and vorticity, and where $e^{ab} = \frac{1}{2}\left[(\partial v^a/\partial x^b) + (\partial v^b/\partial x^a)\right]$ and $\omega^{ab} = \frac{1}{2}\left[(\partial v^a/\partial x^b) - (\partial v^b/\partial x^a)\right]$ are, respectively, the rate of deformation and vorticity tensors. Also in equations (5) to (8), K_i, L_{ij} and M_i (i,j=1,2) are coupling coefficient tensors to be determined. The classical thermo-electric transport coefficients may be obtained from equations (6) and (7) according to

$$E_a = (\sigma^{-1})_{ab}\, J^b - \varepsilon_{ab}\, (\partial T/\partial x^b) \quad , \tag{10}$$

$$F_a = -\kappa_{ab}\, (\partial T/\partial x^b) - \pi_{ab}\, J^b \quad , \tag{11}$$

where σ_{ab}, κ_{ab}, ε_{ab}, and π_{ab} are, respectively, the tensor coefficients of electrical conductivity, thermal conductivity, thermo-electromotive force, and Peltier heat transfer.

III. THE RELATIVISTIC BOLTZMANN TRANSPORT EQUATION

The relativistic Boltzmann transport (RBT) equation may be obtained from the four-dimensional formalism developed by Synge (1957), Stewart (1971), and exploited by Kovetz and Shaviv (1973). In that formalism, Liouville's theorem becomes

$$L(f) \equiv p^\alpha\, (\partial f/\partial x^\alpha) + (q/c)\, F^{\alpha\beta}\, p_\beta\, (\partial f/\partial p^\alpha) = -\,(p^\gamma \lambda_\gamma/c)\, Q(f), \tag{12}$$

where $f(x^\alpha, p^\beta)$ coincides with the classical (phase space) momentum distribution function $f(t,\underline{r},\underline{p})$. Also, $p^\alpha = mU^\alpha = m\gamma(v^a, c) = (p^a, W/c)$ and $p_\alpha = \eta_{\alpha\beta}\, p^\beta = (p^a, -W/c)$ are the four-vector momenta of the scattered particles, such that $p^\alpha p_\alpha = p^2 - W^2/c^2 = m^2c^2$; λ^γ is the unit world velocity of the scattering particles, $Q(f)$ is the scattering integral, and $F^{\alpha\beta}$ is the Maxwell electromagnetic field tensor (cf. Panofsky and Phillips 1955). Straightforward calculations, noting that $p^\gamma \lambda_\gamma = -\,W/c^2$, leads to the form

$$(c^2/W)\, L(f) = (\partial f/\partial t) + \underline{v} \cdot (\partial f/\partial \underline{r}) + q(\underline{E} + \underline{v}\times\underline{B}/c) \cdot (\partial f/\partial \underline{p}) = Q(f), \tag{13}$$

which is equivalent to the "standard" or pre-covariant form of the RBT equation.

To simplify equation (13) we may introduce the relaxation time $\tau(p)$ in place of the scattering integral $Q(f)$ such that

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$$(f^{(0)}-f)/\tau(p) = Q(f) = 2h^{-3} \int d^3p_s \int d\Omega' \,|\underline{v}-\underline{v}_s|\, (d\sigma/d\Omega') \cdot \left[f'f'_s\,(1-\Theta f)\,(1-\Theta_s f_s) - ff_s\,(1-\Theta f')\,(1-\Theta_s f'_s)\right] . \tag{14}$$

Here f and f' refer to the distribution functions before and after collisions, f and f_s refer to the distribution functions for the test and scattering particles, $d\sigma/d\Omega'$ is the differential scattering cross-section, and Θ, Θ_s = (+1, 0, -1) for Fermions, Maxwellons, and Bosons.

To solve equation (13) in the steady state limit, where $\partial f/\partial t = 0$, we may first suppose that the distribution function $f^{(0)}(\varepsilon,\mu,T)$, given by equation (4), and $\mu(T)$ are specified, and that $\partial f/\partial \underline{r} \simeq \partial f^{(0)}/\partial \underline{r}$ but $\partial f/\partial \underline{p} \neq \partial f^{(0)}/\partial \underline{p}$. We may then cast equation (13) into the dimensionless form

$$\nu(x)\ (f-f^{(0)}) + \left[\underline{a} + \underline{x} \times \underline{\omega}(1+x^2)^{-\frac{1}{2}}\right]\cdot \partial(f-f^{(0)})/\partial \underline{p} = \underline{K}(x) \cdot \underline{x}(1+x^2)^{-\frac{1}{2}}, \tag{15}$$

where $$\underline{x} = \underline{p}/mc, \quad \underline{a} = q\underline{E}/mc, \quad \underline{\omega} = q\underline{B}/mc, \tag{16a}$$

$$\nu(x) = \tau(p)^{-1}, \ \underline{K}(x) = c(\partial f^{(0)}/\partial\varepsilon)\left[L(x)\ (\partial T/\partial \underline{r}) - q\ \underline{E}\right] , \tag{16b}$$

and $$L(x) = L(\varepsilon,\mu,T) = \left[(\varepsilon-\mu)/T + \partial\mu/\partial T\right] . \tag{16c}$$

To obtain a spherical harmonic solution of the RBT equation (15), we may now suppose that

$$f-f^{(0)} \equiv U(\underline{x}) = \Sigma_{\ell,m}\, U_{\ell m}(x)\ Y_{\ell m}\,(\theta,\phi). \tag{17}$$

Here we take the vector $\underline{a} = q\underline{E}/mc$ along the principal axis $(\theta,\phi) = (0,0)$, $\underline{K}(\gamma,0)$ in the $\phi = 0$ plane, but keep the vectors $\underline{x}(\theta,\phi)$ and $\underline{\omega}\ (\beta,\lambda)$ arbitrary. In terms of the momentum ($\underline{x}$-) space unit vectors $(\hat{u}_x, \hat{u}_\theta, \hat{u}_\phi)$, we have

$$\underline{a} = a\ \left[\cos\theta\ \hat{u}_x - \sin\theta\ \hat{u}_\theta\right] , \tag{18a}$$

$$\underline{x} \times \underline{\omega} = x\omega\left\{-\sin\beta\ \sin(\lambda-\phi)\ \hat{u}_\theta + \left[\cos\beta\ \sin\theta - \sin\beta\ \cos\theta\ \cos(\lambda-\phi)\right]\hat{u}_\phi\right\} \tag{18b}$$

Inserting these relations (17) and (18) into the RBT equation (15) and using the recurrence, combinatorial, and orthogonality properties of the spherical harmonics, we obtain a set of inhomogeneous coupled ordinary differential equations for the radial momentum functions $U_{\ell m}(x)$. These equations are

$$\begin{aligned} &ax^{-(j+2)}\frac{d}{dx}\left[x^{(j+2)}U_{j+1,n}\right]\alpha_{j+1,n} + ax^{(j+1)}\frac{d}{dx}\left[x^{-(j+1)}U_{j-1,n}\right]\alpha_{j,n} + \\ &+ U_{j,n-1}\left[\mu(x)\ k_{j,n}(\beta,\lambda)\right] + U_{j,n}\left[\nu(x) + \mu(x)h_{j,n}(\beta)\right] + \\ &+ U_{j,n+1}\left[\mu(x)\,\ell_{j,n}(\beta,\lambda)\right] - K(x)\ x\ (1+x^2)^{-\frac{1}{2}}\ g_{j,n}(\gamma) = 0 , \end{aligned} \tag{19}$$

where $j = 0,1,2,\ldots;\ n = -j,\ldots 0,\ldots +j$, and where

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$$\alpha_{j,n} = \left[(j^2 - n^2)\ (4j^2 - 1)\right]^{\frac{1}{2}}, \tag{20a}$$

$$g_{j,n} = (4\pi/3)^{\frac{1}{2}}\ \delta_{j,1}\ \left[\cos\gamma\ \delta_{o,n} + \sin\gamma\ (\delta_{1,n} - \delta_{-1,n})/2^{\frac{1}{2}}\right], \tag{20b}$$

$$\mu(x) = \omega(1 + x^2)^{-\frac{1}{2}}, \tag{20c}$$

$$k_{j,n}(\beta,\lambda) = \left[(j-n+1)(j+n)\right]^{\frac{1}{2}} \cdot \frac{1}{2} \sin\beta\ e^{i(\pi/2-\lambda)}, \tag{20d}$$

$$h_{j,n}(\beta) = n\ \cos\beta\ e^{i\pi/2}, \tag{20e}$$

$$\ell_{j,n}(\beta,\lambda) = \left[(j+n+1)(j-n)\right]^{\frac{1}{2}} \cdot \frac{1}{2} \sin\beta\ e^{-i(\pi/2-\lambda)}. \tag{20f}$$

In the weak electric field case, equations (19) may be solved by supposing a series expansion of the form

$$U_{jn}(x) = U_{jn}^{(0)} + aU_{jn}^{(1)} + a^2U_{jn}^{(2)} + \cdots . \tag{21}$$

Inserting this expansion into equations (19) and separating terms according to powers a^s leads to a set of recurrsive differential equations for the function $U_{jn}^{(s)}$. The solutions, for s = 0,1, are

$$U_{1n}^{(0)} = \sum_{n'=-1}^{+1} M_{nn'}^{(1)} \left[K(x)\ x\ (1 + x^2)^{-\frac{1}{2}}\right] g_{1n'}(\gamma), \tag{22a}$$

$$U_{on}^{(1)} = M_{on}^{(0)} \left\{ -x^2 \frac{d}{dx} \left[x^2\ U_{1n}^{(0)}\right] \alpha_{1n} \right\}, \tag{22b}$$

$$U_{2n}^{(1)} = \sum_{n''=-2}^{+2} M_{nn''}^{(2)} \left\{ -x^2 \frac{d}{dx} \left[x^{-1}\ U_{1n}^{(0)}\right] \alpha_{2n''} \right\}, \tag{22c}$$

where the coefficients $M_{p,q}^{(j)}$ are elements of the (inverse) matrix

$$M_{p,q}^{(j)} = \left[(\nu+\mu h_{j,p})\ \delta_{p,q} + \mu(k_{j,p}\ \delta_{p,q+1} + \ell_{j,p}\ \delta_{p,q-1})\right]^{-1}. \tag{23}$$

To extract the geometrical nature of the solutions (22), it is convenient to define the projected functions $U_{jn;n'}^{(s)}$ via

$$U_{jn}^{(s)} = \sum_{n'=-1}^{+1} U_{jn;n'}^{(s)}\ g_{1n'}(\gamma). \tag{24}$$

The weak electric field magnetic solution thus becomes

$$f-f^{(0)} = U(\underline{x}) = \sum_{j=0}^{\infty} \sum_{n=-j}^{+j} \sum_{s=0}^{\infty} \sum_{n'=-1}^{+1} U_{jn;n'}^{(s)}\ a^s\ g_{1n'}\ N_{jn}\ P_{jn}(\cos\theta)e^{in\phi}. \tag{25}$$

Expanding in terms of the projected functions $U_{jn;n'}^{(s)}$, introducing the explicit forms of $g_{1n'}$, N_{jn}, P_{jn}, and $U_{jn;n'}^{(s)}$ as determined from equations (22) to (24), leads to the result

$$f-f^{(0)} = \left[\frac{K(x)}{\nu(x)}\ (1+x^2)^{-\frac{1}{2}}\right] \left\{\cos\gamma \left[xC_o(x) - \frac{a}{\nu(x)}\ (1 + x^2)^{\frac{1}{2}}\ C_1(x) + \cdots\right] + \right.$$

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$$+ \sin\gamma \left[x\, S_0(x) - \frac{a}{\nu(x)} (1+x^2)^{\frac{1}{2}} S_1(x) + \cdots \right] \Bigg\} , \quad (26)$$

where the coefficients $C_0(x)$, $C_1(x)$, $S_0(x)$, and $S_1(x)$ are given in full detail by Edwards and Merlian (1980).

Returning to physical variables via equations (20), we find the distribution function perturbation to be given by

$$f-f^{(0)} = \tau(v)\ (\delta f^{(0)}/\delta\varepsilon)\ \ | L(\varepsilon,\mu,T)\ (\delta T/\delta \underline{r}) - q\ \underline{E} | \ . \quad (27)$$
$$\cdot \Big\{ v \left[C_0 \cos\gamma + S_0 \sin\beta \right] - (\tau q E/m) \left[C_1 \cos\gamma + S_1 \sin\gamma \right] + \cdots \Big\},$$

which agress in form with that supposed by equation (3).

IV. SUMMARY

It is expected that the solution of the relativistic Boltzmann transport (RBT) equation obtained here will be important in the calculation of realistic hydrodynamics models of highly collapsed magnetic stars such as white dwarfs and neutron stars. Further effort, including the investigation of the relevant collisional cross-sections and relaxation rates, as well as numerical implementation of these solutions necessary to specify the magnetic thermodynamic transport properties, is in progress.

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DISCUSSION

J. COX: Do you have any numbers yet?

EDWARDS: No, that's next year's work.

KEELEY: Will you be able to handle fields strong enough with your weak electric field approximation to be relevant to hydrodynamic cases?

EDWARDS: I can find a theoretical solution of the arbitrary electric field problem. I did not present that here, but it is not too much work.

KELLEY: Is it possible to extend this to include the Schwarzchild criterion?

EDWARDS: I think so. It may be more appropriate to do it in general, however. The parametric reduction is even harder. There has been some work done on the relativistic problem, but not in this detail and only in the case where the collision integral functions are not functions of momentum. We know that in a real fluid, this is not true.