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ON SOME COMBINATORIAL INTERPRETATIONS OF SLATER'S IDENTITIES

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ABSTRACT. Four combinatorial interpretations of identities due to L. J. Slater that have recently been published are slightly incorrect. I show how they may be corrected and also provide a new interpretation of one of these identities.

In [2] and [3], theorems are given that provide combinatorial interpretations of various of the identities found by Slater [1]. It seems to me that four of these theorems, namely 2.3 and 2.4 in [2] and 1.5 and 1.6 [3], are not quite correct as stated, though each becomes correct if an extra hypothesis is imposed. I have been in correspondence with M. V. Subbarao about this matter and it is at his suggestion that I write this note.

With the notation of [2] and [3], I propose that the following hypotheses be included in the statements of these theorems:

(1) $$[2]\ 2.3:\ a_{s-1} \geqq a_s + s (t = 2s-1) \textit{ and } a_s \geqq a_{s+1} + s - 1 (t = 2s).$$

$$[2]\ 2.4:\ a_s > a_{s+1} \geqq a_{s+2} + s - 1 (t = 2s+1) \textit{ and }$$

$$a_s \geqq a_{s+1} + s - 1 (t = 2s).$$

(2) $$[3]\ 1.5 \textit{ and } 1.6:\ b_{s+1} \geqq b_{s+2} + s - 1 (t = 2s+1)$$

(and, in 1.6, "...minimal difference 2").

Take, for example, [3], 1.5. This states that, for each positive natural number, n, $u(n) = v(n)$, where

$u(n) :=$ *the number of partitions of* n *with parts* $\equiv \pm 1,\ \pm 4,\ \pm 6$ *or* $\pm 7 \bmod 16$, $v(n) :=$ *the number of partitions of* n *into an odd number of parts, say* $n = b_1 + \ldots + b_{2s+1}$, *which also satisfy*

(3) $$\begin{cases} b_i \geqq b_{i+1} + 2 \ (\textit{for } 1 \leqq i < s), \\ b_s > b_{s+1} \geqq s \textit{ and} \\ b_i \geqq b_{i+1} \geqq 1 \ (\textit{otherwise}). \end{cases}$$

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The proof given in [3] calls on the identity

$$\sum_{s=0}^{\infty} \frac{q^{2s(s+1)}}{(q;\, q)_{2s+1}} = (q^8;\, q^8)_\infty (q^3;\, q^8)_\infty \times (q^5;\, q^8)_\infty (q^2;\, q^{16})_\infty (q^{14};\, q^{16})_\infty (q;\, q)_\infty^{-1} \tag{4}$$

([1], (86), p. 161). The coefficient of q^n on the right-hand side of (4) is $u(n)$. On the left-hand side, the coefficient of q^n is the number of representations of n as

$$n = 2s(s+1) + c_1 + \ldots + c_{2s+1}, \tag{5}$$

with

$$c_1 \geqq c_2 \geqq \ldots \geqq c_{2s+1} \geqq 0 \tag{6}$$

and the argument of [3] claims to build such a representation out of a partition $n = b_1 + \cdots + b_{2s+1}$ (satisfying (3)) by taking

$$\begin{cases} c_i = b_i - 3s + 2i - 1 \ (1 \leqq i \leqq s), \\ c_{s+1} = b_{s+1} - s, \\ c_i = b_i - 1 \text{ (otherwise)}. \end{cases} \tag{7}$$

However, if $b_{s+1} < b_{s+2} + s - 1$, then $c_{s+1} < c_{s+2}$ and (6) is violated. For example, I find that $u(13) = 14$, whereas $v(13) = 15$; the culprit is the partition $5 + 3 + 2 + 2 + 1$ of 13.

On the other hand, it is a simple matter to check that the inverse of the transformation (7) converts a representation (5) to a partition satisfying (2) as well as the conditions (3). So, if we include (2) among the conditions defining the partitions counted by $v(n)$, then it is true that $u(n) = v(n)$ for each positive natural number, n.

Theorem 2.3 in [2], augmented with (1), follows from the identity

$$(q;\, q)_\infty \sum_{s-0}^{\infty} \frac{q^{2s^2}}{(q;\, q)_{2s}} = (q^8;\, q^8)_\infty (q;\, q^8)_\infty (q^7;\, q^8)_\infty (q^6;\, q^{16})_\infty (q^{10};\, q^{16})_\infty, \tag{8}$$

which is (83) in [1]. Another interpretation of (8) is:

THEOREM. *For each natural number, n, the number of partitions of n into an even number, say* 2*s*, *of parts in which the s largest parts differ from each other by at least* 4 *is equal to the number of partitions of n into parts congruent to* ±2, ±3, ±4 *or* ±5 *modulo* 16.

I leave the proof to the diligent reader.

References

1. L. J. Slater, *Further identities of the Rogers – Ramanujan type*, Proc. London Math. Soc., Vol. 54 (1951–52), pp. 147–167.

2. M. V. Subbarao, *Some Rogers – Ramanujan type partition theorems*, Pacific J. Math., Vol. 120 (1985), pp. 431–435.

3. M. V. Subbarao and A. K. Agarwal, *Further theorems of the Rogers – Ramanujan type theorems*, Canad. Math. Bull., Vol. 31 (2) (1988), pp. 210–214.

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