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GROUPS ADMITTING ONLY FINITELY MANY NILPOTENT RING STRUCTURES

BY

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ABSTRACT. The abelian groups which are the additive groups of only finitely many non-isomorphic (associative) nilpotent rings are studied. Progress is made toward a complete classification of these groups. In the torsion free case, the actual number of non-isomorphic nilpotent rings these groups support is obtained.

i. All groups considered here are abelian with addition the group operation. The additive group of a ring R will be denoted by R^+. Terminology and notation follow [5].

Szele [7] initiated the study of groups allowing only finitely many non-isomorphic ring structures, Fuchs [4] made much progress towards classifying them, and Borho [2] concluded the classification of all such groups which are not nil. The object of this paper is to study the groups allowing only finitely many non-isomorphic nilpotent ring structures. For G a torsion free group, the number of non-isomorphic rings R satisfying $R^+ = G$ will be obtained.

ii. DEFINITION. *A group G is (associative) nilpotent nil if the only (associative) nilpotent ring R satisfying $R^+ = G$ is the zeroring, i.e., $R^2 = 0$. If G admits only finitely many non-isomorphic (associative) nilpotent ring structures, then G is said to be (associative) quasi-nilpotent nil.*

THEOREM 1. *Let G be a torsion group. The following are equivalent:*

1) *G is nilpotent nil.*

2) *G is associative nilpotent nil.*

3) *$G = D \oplus \bigoplus_{p \in P} Z(p)$ with D a divisible torsion group, P a set of distinct primes, and $D_p = 0$ for all $p \in P$.*

PROOF. Clearly 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 3): Suppose that G is associative nilpotent nil. $G = D \oplus H$, with D the maximal divisible subgroup of G. Suppose that $H_p \neq 0$, and $H_p \neq Z(p)$ for some prime p. Then H, and hence G, has a direct summand $K = Z(p^k)$, $1 < k < \infty$, or $K = Z(p) \oplus Z(p)$, [5, Corollaries 27.2 and 27.3]. It is readily seen that there exists an associative nilpotent

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ring S with $S^+ = K$, and $S^2 \neq 0$. Since a direct summand of an associative nilpotent nil group must clearly be associative nilpotent nil, we have a contradiction. Therefore $H = \bigoplus_{p \in P} Z(p)$, P a set of distinct primes. Suppose that $D_p \neq 0$ for some $p \in P$. Then G has a direct summand $K \oplus L$ with $K = Z(p)$, and $L = Z(p^\infty)$. Let a be a generator for K, and let $b \in L$ with $|b| = p$. The products $a^2 = b$, $ax = xa = xy = 0$ for all $x, y \in L$ induce an associative nilpotent ring structure S satisfying $S^+ = K \oplus L$, and $S^2 \neq 0$, a contradiction.

The implication 3) $\Rightarrow$ 1) is easily obtained.

The argument used above in proving the implication 2) $\Rightarrow$ 3) yields:

COROLLARY 1.1. *Let G be an associative nilpotent nil group. Then the torsion part of G, $G_t = D \oplus \bigoplus_{p \in P} Z(p)$, with D a divisible torsion group, P a set of distinct primes, and $D_p = 0$ for all $p \in P$.*

THEOREM 2. *Let G be a torsion group. The following are equivalent:*

1) *G is quasi-nilpotent nil.*

2) *G is associative quasi-nilpotent nil.*

3) *$G = D \oplus F \oplus \bigoplus_{p \in P} Z(p)$, with D a divisible torsion group, F a finite group, P a set of distinct primes, $D_p = 0$ for all but finitely many $p \in P$,* and *$F_p = 0$ for all $p \in P$.*

PROOF. Clearly 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 3): Suppose that G is associative quasi-nilpotent nil. $G = D \oplus H$, with D the maximal divisible subgroup of G. Suppose there exists an infinite set of primes P' such that $H_p \neq 0$, and $H_p \neq Z(p)$ for all $p \in P'$. Then for $p \in P'$ there exists an associative nilpotent ring S_p with $S_p^+ = H_p$, and $S_p^2 \neq 0$. Now $G = H_p \oplus K(p)$. Let $R(p)$ be the ring direct sum $R(p) = S_p \oplus T(p)$, with $T(p)^+ = K(p)$ and $T(p)^2 = 0$. Clearly $R(p)$ is an associative nilpotent ring, $R(p)^+ = G$, and for distinct primes $p, q \in P'$, $R(p) \not\simeq R(q)$, a contradiction. It therefore suffices to show that H_p is finite for every prime p. Suppose there exists a prime p such that H_p is an infinite group. Repeated applications of [5, Corollaries 27.2 and 27.3] yield that for every positive integer n, H_p, and hence G, has a direct summand $H_p(n') = \bigoplus_{i=1}^n Z(p^{k_i})$, with k_i a positive integer, $1 \leq i \leq n$, and $k_1 \leq k_2 \leq \ldots \leq k_n$. Let a_i be a generator for $Z(p^{k_i})$, $i = 1, \ldots, n$. The products

$$a_i \circ a_j = \begin{cases} a_{\min(i,j)-1} & \text{for } i \neq 1 \text{ and } j \neq 1 \\ 0 & \text{for } i = 1 \text{ or } j = 1 \end{cases}$$

induce an associative nilpotent ring structure $S(n)$ with $S(n)^+ = H_p(n)$. Now $G = H_p(n) \oplus K(n)$. Let $T(n)$ be the zeroring with $T(n)^+ = K(n)$. Then the ring direct sum $R(n) = S(n) \oplus T(n)$ is associative, nilpotent, and $R(n)^+ = G$. It is readily seen that for distinct positive integers n, m, $R(n) \not\simeq R(m)$, a contradiction.

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Let p be a prime for which $D_p \neq 0$, and $H_p \neq 0$. Then $G = Z(p^\infty) \oplus Z(p^n) \oplus K$, with $Z(p^\infty)$ a direct summand of D, and $Z(p^n)$ a direct summand of H. Let a be a generator of $Z(p^n)$, and let $b \in Z(p^\infty)$ satisfying $|b| = p^n$. For $x_i = c_i + k_i a + y_i$ with $c_i \in Z(p^\infty)$, $y_i \in K$, and k_i an integer, $i = 1, 2$, define $x_1 \cdot x_2 = k_1 k_2 b$. These products induce an associative nilpotent ring structure $R(p)$ on G. For distinct primes p, q, $R(p) \not\cong R(q)$. Therefore $D_p = 0$ for all but finitely many primes p for which $H_p \neq 0$. We now have that $G = D \oplus F' \oplus \bigoplus_{p \in P'} Z(p)$, with F' a finite group, P' a set of distinct primes, and $D_p = 0$ for all but finitely many $p \in P'$. Let $\{p_1, \ldots, p_k\}$ be the set of primes $p \in P'$ for which $F'_p \neq 0$. Put $F = F' \oplus \bigoplus_{i=1}^{k} Z(p_i)$, and let $P = P' - \{p_1, \ldots, p_k\}$. Then $G = D \oplus F \oplus \bigoplus_{p \in P} Z(p)$ with $F_p = 0$ for all $p \in P$.

3) $\Rightarrow$ 1): Let $G = D \oplus F \oplus \bigoplus_{p \in P} Z(p)$, with D a divisible torsion group, F a finite group, P a set of distinct primes, and $D_p = 0$ for all but finitely many $p \in P$. It may be assumed that $F_p = 0$ for all $p \in P$. The group of ring multiplications on G, Mult G, is isomorphic to Hom $(G \otimes G, G)$ [5, Theorem 118.1], and [3, 1.2.1]. Let $\{p_1, \ldots, p_n\}$ be the set of all primes in P such that $D_{p_i} \neq 0$, $i = 1, \ldots, n$. It follows from [5, Theorem 431.1 and vol. 1, p. 255(D)] that Mult $G \simeq$ Hom $(F \otimes F, D) \oplus \bigoplus_{i=1}^{n} \operatorname{Hom} Z(p_i), D) \oplus \operatorname{Hom} (F \otimes F, F) \oplus \Pi_{p \in P} \operatorname{Hom} Z(p) \otimes Z(p), Z(p))$. Clearly Hom $(F \otimes F, D)$, $\bigoplus_{i=1}^{n}$ Hom $Z(p_i), D)$ and Hom $(F \otimes F, F)$ are finite groups. A multiplication on G corresponding to a nonzero homomorphism $Z(p) \otimes Z(p) \rightarrow Z(p)$ induces a field structure on a direct summand $Z(p)$ of G. Therefore G is quasi-nilpotent nil.

Counting the number of non-isomorphic (associative) ring structures which can be defined on an (associative) quasi-nilpotent nil torsion group is a difficult problem. An asymptotic expression for the number of associative nilpotent rings of order p^n, p a prime, may be found in [6, Corollary 5.2.12].

As was the case in Theorem 1, the argument used to prove the implication 2) $\Rightarrow$ 3) remains true for G an arbitrary associative quasi-nilpotent nil group. Hence:

COROLLARY 2.1. *Let G be an associative quasi-nilpotent nil group. Then $G_t = D \oplus F \oplus \bigoplus_{p \in P} Z(p)$, with D a divisible torsion group, F a finite group, P a set of distinct primes, $D_p = 0$ for all but finitely many $p \in P$, and $F_p = 0$ for all $p \in P$.*

COROLLARY 2.2. *Let G be an associative quasi-nilpotent nil group with $G_t = D \oplus F \oplus \bigoplus_{p \in P} Z(p)$ as in Corollary 2.1. Then for all but finitely many $p \in P$, $G = H(p) \oplus Z(p)$ with $H(p)$ a p-divisible subgroup of G.*

PROOF. Since $G_p = Z(p)$ for all but finitely many $p \in P$, there exists a subgroup $H(p)$ of G such that $G = H(p) \oplus Z(p)$ for all but finitely many $p \in P$, [5, Theorem 27.5] (in fact it is not difficult to show that $Z(p)$ is a direct summand of G for all $p \in P$). Suppose that $H(p)$ is not p-divisible. Then the canonical homomorphism $H(p) \rightarrow H(p)/pH(p)$ followed by a projection yield an epimorphism $\varphi: H(p) \rightarrow (a)$, where (a) is a cyclic group of order p generated by a. The product $a \cdot a = a$ induces a field structure on (a). Let b generate the direct summand $Z(p)$ of G, and let $\psi: (a) \rightarrow (b)$ be the epimorphism induced by the map $a \rightarrow b$. Every element in G is

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of the form $x = h + nb$ with $h \in H(p)$, and n an integer. Let $x_i = h_i + n_i b$, $i = 1, 2$, be elements in G written in this form. Then the products $x_1 x_2 = \psi[\varphi(h_1)\varphi(h_2)]$, with the product in the square brackets being the field multiplication on (a), yield an associative nilpotent ring $R(p)$ with $R(p)^+ = G$, and $(R(p)^2)^+ = Z(p)$. Therefore $H(p)$ is p-divisible for all but finitely many $p \in P$.

EXAMPLE 2.3. *A group admitting only finitely many non-isomorphic (associative) ring structures, decomposes into the direct sum of a torsion group, and a torsion free group both satisfying the same property*, [3, Theorem 2.2.7].

This is not the case for an (associative) quasi-nilpotent nil group G. Although by Corollary 2.1, G_t is (associative) quasi-nilpotent nil, G/G_t may not be, and G need not split into the direct sum of a torsion and torsion free group. Let P be an infinite set of distinct primes. It is well known that $G = \Pi_{p \in P} Z(p)$ does not split into the direct sum of a torsion and torsion free group. $G/G_t \simeq \bigoplus_c Q^+$, with Q the field of rationals, and c the powers of the continuum, is clearly not associative quasi-nilpotent nil. Let R be a nilpotent ring with $R^+ = G$. Since $G_t = \bigoplus_{p \in P} Z(p)$ is nilpotent nil, $G_t^2 = 0$. Let $a \in G_t$, $a \neq 0$, $x \in R$, with $|a| = n$. Since G/G_t is divisible, there exist $y \in R$, and $b \in G_t$ such that $x = ny + b$. Hence $xa = nya = 0$. Similarly $ax = 0$, and so G_t annihilates R. For $z \in G$, let z_p denote the p-component of z for every $p \in P$. Let $x, y \in R$. As above there exist $x_1 \in G$, and $a \in G_t$ such that $x = px_1 + a$. Therefore $xy = px_1 y$, and so $(xy)_p = 0$ for all $p \in P$, i.e., $R^2 = 0$, and G is nilpotent nil.

LEMMA 3. *Let G be an (associative) quasi-nilpotent nil group, and let R be an (associative) nilpotent ring with $R^+ = G$. Then $(R^2)^+ = E \oplus F$ with E a divisible group, and F a finite group.*

PROOF. Let R be an (associative) nilpotent ring with $R^+ = G$. For every positive integer n, the products $a x_n b = n(ab)$ for all $a, b \in G$, with products on the righthand side of the equality being multiplication in R, induce an (associative) nilpotent ring structure $R(n)$ on G. Since G is (associative) quasi-nilpotent nil, there exist integers $m_1, \ldots, m_k$ such that for every positive integer n, there exists $1 \leq i \leq k$ for which $R(n) \simeq R(m_i)$. Put $m = \Pi_{i=1}^k m_i$. Then $m(R^2)^+$ is divisible. Hence $(R^2)^+ = E \oplus B$ with E divisible, and $mB = 0$. Since $B \leq G_t = D \oplus F \oplus \bigoplus_{p \in P} Z(p)$, with decomposition of G_t as in Corollary 2.1, it suffices to show that $B' = \pi_D(B)$ is finite, where π_D is the natural projection of G_t onto D. Let C be a direct summand of B', and let π_C be a projection of B' onto C. Let $x, y \in R$. Then $x \cdot y = e + b$ with $e \in E$, $b \in B$. Define $x *_C y = \pi_C \cdot \pi_D(b)$. These products induce a ring structure R_C on G. Let $x, y, z \in G$. Since $x *_C y \in D$, there exists $d \in D$ such that $x *_C y = md$. Hence $(x *_C y) *_C z = m(d *_C z) = 0$.Similarly, $x *_C (y *_C z) = 0$. Therefore $R_C^3 = 0$, and so R_C is an associative nilpotent ring with $(R_C^2)^+ = C$. Since G is (associative) quasi-nilpotent nil, B' is a bounded group possessing only finitely many pairwise non-isomorphic direct summands. This clearly implies that B' is finite.

COROLLARY 3.1. *A reduced torsion free group is (associative) quasi-nilpotent nil if and only if it is (associative) nilpotent nil.*

If G is a quasi-nilpotent nil, then an argument similar to that used in proving Lemma 3 shows that $(R^2)^+$ has only finitely many pairwise non-isomorphic direct summands. This together with Lemma 3 yields:

COROLLARY 3.2. *Let G be a quasi-nilpotent group, and let R be a nilpotent ring with $R^+ = G$. Then $(R^2)^+ = \bigoplus_{\text{finite}} Q \oplus \bigoplus_{p \in P} \bigoplus_{\alpha_p} Z(p^\infty) \oplus F$, with P a finite set of primes, α_p a finite cardinal for each $p \in P$, and F a finite group.*

THEOREM 4. *Let G be a torsion free (associative) quasi-nilpotent nil group. Then either G is a reduced (associative) nilpotent nil group, or the rank of G, $r(G) \leq 2$. Conversely, every non-reduced torsion free group of rank ≤ 2 is associative quasi-nilpotent nil.*

PROOF. Let G be a torsion free (associative) quasi-nilpotent nil group. By Corollary 3.1 it may be assumed that G is not reduced, i.e., $G \simeq Q^+ \oplus H$. Suppose that $r(H) > 1$. Then choose $b_0 \in Q$, $b_0 \neq 0$, and independent elements $b_1, b_2 \in H$. Let $A = (\alpha_{ij})$, $1 \leq i, j \leq 2$ be a 2×2 matrix with components in Q. Since $Q^+ \otimes G \simeq Qb_0 \oplus Qb_1 \oplus Qb_2 \oplus K$, after identifying elements with their isomorphic images, every element of $Q^+ \otimes G$ can be uniquely written in the form $r_0b_0 + r_1b_1 + r_2b_2 + c$ with $r_0, r_1, r_2 \in Q$, and $c \in K$. Let $x = r_0b_0 + r_1b_1 + r_2b_2 + c$, and $y = r_0'b_0 + r_1'b_1 + r_2'b_2 + c'$ be elements in $Q^+ \otimes G$ written in the above form. The products $xy = \Sigma_{i,j=1}^2 r_ir_j'\alpha_{ij}b_0$ induce an associative nilpotent Q-algebra structure on $Q^+ \otimes G$. Identifying elements $g \in G$ with $1 \otimes g \in Q^+ \otimes G$, and restricting the above multiplication to G, yields an associative nilpotent ring R_A with $R_A^+ = G$. Let $A = (\alpha_{ij})$, $B = (\beta_{ij})$ be two nonzero 2×2 matrices over Q, and let $\varphi: R_A \to R_B$ be an isomorphism. Since φ extends to an algebra isomorphism $\varphi: Q \otimes R_A \to Q \otimes R_B$, $\varphi(b_i) = \Sigma_{k=0}^2 p_{ik}b_k$, with $p_{ik} \in Q$ for $i, k = 0, 1, 2$. Choose $i, j \in \{1, 2\}$ such that $\alpha_{ij} \neq 0$. Then $\varphi(b_{1i}b_j) = \alpha_{ij}\varphi(b_0) = \Sigma_{k=0}^2 \alpha_{ij}p_{0k}b_k$. However $\varphi(b_i)\varphi(b_j) = (\Sigma_{k=1}^2 \Sigma_{\ell=1}^2 p_{ik}p_{j\ell}\beta_{k\ell})b_0$. Hence $p_{0k} = 0$ for $k = 1, 2$, and $p_{00}\alpha_{ij} = \Sigma_{k=1}^2 \Sigma_{\ell=1}^2 p_{ik}p_{j\ell}\beta_{k\ell}$. The same argument used in proving [3, Theorem 2.2.4] shows that there are infinitely many nonisomorphic rings R_A, A a 2×2 matrix over Q.

Conversely, let G be a non-reduced group with $r(G) \leq 2$. If $r(G) = 1$, then the condition "G non-reduced" is superfluous, because a ring with rank one torsion free additive group is either isomorphic to a subring of Q, or is the zeroring, [5, Theorem 121.1]. Therefore every rank one torsion free group is nilpotent nil. If $r(G) = 2$, then either (A) $G \simeq Q^+ \oplus H$ with H a reduced rank one torsion free group, or (B) $G \simeq Q^+ \oplus Q^+$. Let R be an associative nilpotent ring with $R^+ = G$. If G is of form (A), then by Lemma 3, $R^2 \subseteq Q^+$. Choose $a_1 \in Q^+$, $a_2 \in H$, $a_i \neq 0$, $i = 1, 2$. Then $a_ia_j = r_{ij}a_1$, with $r_{ij} \in Q$, $i, j = 1, 2$. For every positive integer n, $a_1^n = r_{11}^{n-1}a_1$, $a_1a_2^n = r_{12}^na_1$, and $a_2 \cdot a_1^n = r_{21}^na_1$. Therefore $r_{11} = r_{12} = r_{21} = 0$, and so every associative nilpotent ring with additive group G is obtained by defining

$$a_ia_j = \begin{cases} ra_1 \text{ for } i = j = 2 \\ 0 \quad \text{otherwise} \end{cases}$$

with r an arbitrary rational number. Let R_r be the ring obtained in this manner. Let $r \neq 0$, $s \neq 0$ be rational numbers. Every element in R_r is of the form $r_1a_1 + r_2a_2$ with $r_1r_2 \in Q$. The map $\varphi: R_r \rightarrow R_s$ via $\varphi(r_1a_1 + r_2a_2) = r_1r^{-1}sa_1 + r_2a_2$ is an isomorphism.

Suppose that $G \simeq Q^+ \oplus Q^+$, and let R be an associative nilpotent ring with $R^+ = G$, and $R^2 \neq 0$. Then there exists $a \in R$ such that a and a^2 are independent in G, [1, Lemmas 1 and 2], and $a^3 = 0$, [3, Theorem 3.1.3]. If S is any associative nilpotent ring, with $S^+ = G$, $S^2 \neq Q$, and $b \in S$ such that b abd b^2 are independent in G, then the map $\varphi: R \rightarrow S$ via $\varphi(ra + sa^2) = rb + sb^2$ for all $r, s \in Q$, is an isomorphism.

The following result is implicit in the proof of Theorem 4:

COROLLARY 4.1. *Let G be a torsion free group. If $r(G) = 1$ then G is the additive group of only one nilpotent ring. If $r(G) = 2$, and G is not reduced then G is the additive group of precisely two non-isomorphic associative nilpotent rings. If $r(G) > 2$, and G is not reduced, then G is the additive group of infinitely many non-isomorphic associative nilpotent rings. If G is a reduced group, then either G is the additive group of only one (associative) nilpotent ring, or of infinitely many non-isomorphic (associative) nilpotent rings.*

The following results shed some light on the mixed case.

LEMMA 5. *Let G be an associative quasi-nilpotent nil group, with $G_t = D \oplus F \oplus \bigoplus_{p \in R} Z(p)$ as in Corollary 2.1. Then for every prime p, there exists a subgroup $K(p)$ of G such that $G = G_p \oplus K(p)$, p-divisible for all primes p for which $D_p \neq 0$, and for all but finitely many primes $p \in P$.*

PROOF. G_p is a direct summand of G for every prime $p \in P$ by Corollary 2.1 and [5, Theorems 21.2 and 27.5], i.e., $G = G_p \oplus K(p)$. For all primes p for which $F_p = 0$, $K(p) \simeq H(p) \oplus D_p$, with $H(p)$ the group in the proof of Corollary 2.2. Since D_p is p-divisible for all primes p, and $H(p)$ is p-divisible for all but finitely many primes $p \in P$, it follows that $K(p)$ is p-divisible for all but finitely many primes $p \in P$. Let p be a prime for which $D_p \neq 0$. Then $K = Z(p^\infty) \oplus K(p)$ is a direct summand of G, and so K is associative quasi-nilpotent nil. If $K(p)$ is not p-divisible, then for every positive integer n, there exists an epimorphism $\varphi: [K(p)/P^nK(p)] \otimes [K(p)/p^nK(p)] \rightarrow Z(p^n)$, where $Z(p^n)$ is a subgroup of $Z(p^\infty)$. Let $d_i \in Z(p^\infty)$, $a_i \in K(p)$, $i = 1, 2$. The products $(d_1 + a_1)(d_2 + a_2) = \varphi(a_1 \otimes a_2)$ induce an associative nilpotent ring structure R_n on K. Since $(R_n^2) = Z(p^n)$, $R_n \not\simeq R_m$ for positive integers $n \neq m$, a contradiction.

COROLLARY 5.1. *Let G and G_t be as in Lemma 5. Then G/G_t is p-divisible for every prime p such that $D_p \neq 0$, and for all but finitely many $p \in P$. If G/G_t is p-divisible for only finitely many primes p, then G_t is a direct summand of G.*

PROOF. Since G/G_t is a homomorphic image of $G/G_p \simeq K(p)$, it follows from Lemma 5, that G/G_t is p-divisible for all primes p such that $D_p \neq 0$, and for all but finitely many $p \in P$. Therefore if G/G_t is p-divisible for only finitely many primes p, then P is a finite set of primes and so G_t is the direct sum of a bounded (in fact finite) group and a divisible group. By [5, Theorem 100.1], G_t is a direct summand of G.

COROLLARY 5.2. *Let G be an associative quasi-nilpotent nil group, and let H be a torsion free direct summand of G. If $r(H) > 1$, then G_t is reduced.*

PROOF. Suppose that $r(H) > 1$, and that G_t is not reduced. Then $K = Z(p^\infty) \oplus K(p)$ is a direct summand of G for some prime p, with $K(p)$ as in Lemma 5. Now K is associative quasi-nilpotent nil, and $K(p)$ is p-divisible by Lemma 5. Let $Z(p^\infty) = \bigcup_{i=1}^{\infty} (a_i)$ with (a_i) a cyclic group of order p^i generated by a_i, $i = 1, 2, \ldots$. The p-adic integers are the endomorphisms of $Z(p^\infty)$, [5, vol. 1, p. 181, ex 3]. Let u, v be independent elements in H. Every element $x \in K$ is of the form $x = d + ru + sv + w$ with $d \in Z(p^\infty)$; $r, s \in Q$, $w \in K(p)$. Let $x_i = d_i + r_i u + s_i v + w_i$, $i = 1, 2$, be elements in K written in the above form, and let π be a p-adic integer. The products $x_1 x_2 = (r_1 r_2 + s_1 s_2 \pi) a_1$ induce an associative nilpotent ring structure R on K. The same argument used in proving [3, Theorem 2.2.7] shows that there are infinitely many non-isomorphic rings R_π, a contradiction.

COROLLARY 5.3. *Let G be an associative quasi-nilpotent nil group, and let D be the maximal divisible subgroup of G. Then either (A) D is a torsion group, (B) $D \simeq Q^+ \oplus Q^+$, or (C) $D \simeq D_t \oplus Q^+$, and $D_p = 0$ for all but finitely many primes p.*

PROOF. D must have form (A), (B) or $D \simeq D_t \oplus Q^+$ by Corollary 5.2, and [5, Theorem 23.1]. Suppose that $D \simeq D_t \oplus Q^+$, p is a prime for which $D_p \neq 0$, and $\varphi: Q \otimes Q \to Z(p^\infty)$ is a nonzero homomorphism into a direct summand $Z(p^\infty)$ of D. Let $d_i \in D$, $a_i \in Q$, $i = 1, 2$. The products $(d_1 + a_1)(d_2 + a_2) = \varphi(a_1 \otimes a_2)$ induce an associative nilpotent ring structure R_p on D. Since $R_p \not\simeq R_q$ for primes $p \neq q$, $D_p = 0$ for all but finitely many primes p.

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