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RESEARCH ARTICLE

Mean residual life order among largest order statistics arising from resilience-scale models with reduced scale parameters

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Abstract
In this paper, we identify some conditions to compare the largest order statistics from resilience-scale models with reduced scale parameters in the sense of mean residual life order. As an example of the established result, the exponentiated generalized gamma distribution is examined. Also, for the special case of the scale model, power-generalized Weibull and half-normal distributions are investigated.

1. Introduction

If operating of a system comprising n components depends to at least k active components, then it is called a k-out-of-n system. The lifetimes of k-out-of-n systems thus can be described by order statistics arising from the lifetimes of their components. Due to this intimate relation, order statistics play a fundamental rule in the context of reliability theory. In this regard, extreme order statistic, among others, corresponding to the lifetimes of series and parallel systems have received more attentions because of their importance in analyzing of the lifetimes of complex systems; see Barlow and Proschan [7]. For elaborate discussions on order statistics and their applications, one may refer to Balakrishnan and Rao [4,5] and David and Nagaraja [11].

Consider a nonnegative random variable T with distribution function F_T and assume that $F_T(x) = G^{\theta}(\delta x)$ for all $x \in \mathbb{R}_+ (= [0, \infty))$ wherein G is an absolutely continuous distribution function (centered on $\mathbb{R}_+$) with corresponding reversed hazard rate function $\tilde{r}$. Then, it is said that T follows the resilience-scale (RS) model with baseline distribution G, resilience parameter $\theta \in \mathbb{R}^+$ and scale parameter $\delta \in \mathbb{R}^+$, written as $T \sim RS(G; \theta, \delta)$. Note that the RS model becomes the scale model when $\theta = 1$. If $\tilde{r}_T$ denotes the reversed hazard rate function of T, then one can see that $r_T(x) = \theta\delta\tilde{r}(\delta x)$ for all $x \in \mathbb{R}_+$. Therefore, the RS model can be viewed as the scaled version of the proportional hazard rate model. On the other hand, the RS model can be achieved by using the exponentiation method on the scale model, which for this reason, it is also called as the exponentition scale model in the literature. For a comprehensive discussion on the exponentiation method and its applications, we refer the readers to AL-Hussaini and Ahsanullah [1]. It should be mentioned that the RS model contains some well-known life distributions such as generalized exponential distribution, exponentiated Weibull distribution, exponentiated Lomax distribution, generalized Rayleigh distribution, exponentiated gamma distribution and exponentiated generalized gamma distribution.

In the recent years, some studies have been focused on stochastic comparisons between the largest order statistics for several specific cases of the RS model. For example, one can see Balakrishnan *et al.* [3] and Kundu *et al.* [21] for the case of generalized exponential distribution; Fang and Zhang [14], Kundu and Chowdhury [20] and Barmalzan *et al.* [8] for the case of exponentiated Weibull distribution; Fang and Xu [13] for the case of exponentiated gamma distribution; Haidari *et al.* [16] and Haidari and Payandeh Najafabadi [15] for the case of the exponentiated generalized gamma distribution. However, stochastic ordering relations between the largest order statistics based on random variables following the general RS models have not received the attentions they deserve. Indeed, to the best of our knowledge, only few works in this direction are published so far including Zhang *et al.* [31], Haidari *et al.* [17] and Lu *et al.* [23].

Mean residual life function is a key tool in reliability, life testing and survival analysis. Provided that an item is of age $t \in \mathbb{R}_+$, the remaining lifetime after t is a random variable whose expected value is called the mean residual life (MRL) function at time t. As one can see, the MRL function sums up the entire residual life distribution of an item while the failure (hazard) rate function describes the effect of an immediate failure. From this point of view, the MRL function is likely to be more efficient than the failure (hazard) rate function. The MRL function has wide applications in other areas such as renewal theory, demography, social sciences and actuarial sciences; see Chapter 4 of Lai and Xie [22] and the references therein.

Before going into the background of the idea investigated in this paper, let us briefly recall some definitions. For two nonnegative random variables T_1 and T_2 with survival functions $\bar{F}_{T_1}$ and $\bar{F}_{T_2}$, and mean residual functions $m_{T_1}(t) = [\bar{F}_{T_1}(t)]^{-1}\int_t^\infty \bar{F}_{T_1}(u)\,du$ and $m_{T_2}(t) = [\bar{F}_{T_2}(t)]^{-1}\int_t^\infty \bar{F}_{T_2}(u)\,du$, it is said that T_1 is larger than T_2 with respect to the mean residual life order, denoted by $T_1 \geq_{\mathrm{mrl}} T_2$, if $m_{T_1}(t) \geq m_{T_2}(t)$ for all $t \in \mathbb{R}_+$; see Chapter 2 of Shaked and Shanthikumar [27] for more details on the mean residual life order and its properties. Let $u_{1:n} \leq \cdots \leq u_{n:n}$ and $v_{1:n} \leq \cdots \leq v_{n:n}$ denote the increasing arrangements of the components of nonnegative vectors $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$, respectively. Then, $\boldsymbol{u}$ is said to reciprocal majorize $\boldsymbol{v}$, written as $\boldsymbol{u} \overset{\mathrm{rm}}{\succ} \boldsymbol{v}$, if $\sum_{i=1}^k u_{i:n}^{-1} \geq \sum_{i=1}^k v_{i:n}^{-1}$ for all $k = 1, \ldots, n$. For additional details on the reciprocal mjorization order and its application, one may refer to Zhao and Balakrishnan [32].

Suppose $T_1, \ldots, T_n$ and $T_1^*, \ldots, T_n^*$ are two sets of independent exponential random variables with T_i, T_j, T_i^* and T_j^* having the respective hazard rates $\delta_1, \delta_2, \delta_1^*$ and δ_2^* for $i = 1, \ldots, l$ and $j = l+1, \ldots, n$ $(1 \leq l \leq n-1)$, and let $T_{n:n}$ (resp. $T_{n:n}^*$) denotes the largest order statistic based on $T_1, \ldots, T_n$ (resp. $T_1^*, \ldots, T_n^*$). In the sequel, a vector with all its elements being one is represented by $\mathbf{1}_k$. In Open Problem 2 of Balakrishnan and Zhao [6], the following idea concerning the mean residual life order between $T_{n:n}$ and $T_{n:n}^*$ is proposed:

Under the assumptions $\delta_1 \leq \delta_1^* \leq \delta_2^* \leq \delta_2$ *and* $(\delta_1\mathbf{1}_l, \delta_2\mathbf{1}_{n-l}) \overset{\mathrm{rm}}{\succ} (\delta_1^*\mathbf{1}_l, \delta_2^*\mathbf{1}_{n-l})$, *does the ordering* $T_{n:n} \geq_{\mathrm{mrl}} T_{n:n}^*$ *hold?*

For $\delta_1 \leq \delta_2$, set

$$\Omega(\delta_1, \delta_2) = \{(x, y) \in \mathbb{R}^{+2} : \delta_1 \leq x \leq y \leq \delta_2 \text{ and } l\delta_1^{-1} + (n-l)\delta_2^{-1} \geq lx^{-1} + (n-l)y^{-1}\}.$$

Now, the foresaid idea can be restated as follows:

If $(\delta_1^*, \delta_2^*) \in \Omega(\delta_1, \delta_2)$, *then does the ordering* $T_{n:n} \geq_{\mathrm{mrl}} T_{n:n}^*$ *hold?*

Zhao and Balakrishnan [33] showed that the answer of the above questions is positive for the special case when $n = 2$. However, the complete answer is provided by Wang and Cheng [29] with the aid of an effective method which is new in the context of stochastic orderings of order statistics; see Wang [28] and Wang and Cheng [30] for further details on this method and its applications.

The idea investigated in this paper is concerning the mean residual life order between the largest order statistics in the RS models. Consider two nonnegative random vectors $(T_1, \ldots, T_n)$ and $(T_1^*, \ldots, T_n^*)$

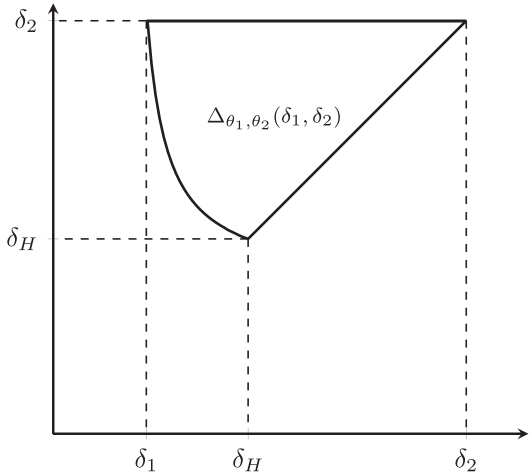

Figure 1. *Plot of the region* $\Delta_{\theta_1,\theta_2}(\delta_1,\delta_2)$.

with $T_i \sim RS(G;\theta_i,\delta_1), T_j \sim RS(G;\theta_j,\delta_2), T_i^* \sim RS(G;\theta_i,\delta_1^*)$ and $T_j^* \sim RS(G;\theta_j,\delta_2^*)$ for $i = 1,\ldots,l$ and $j = l+1,\ldots,n$. Assume that the region $\Delta_{\theta_1,\theta_2}(\delta_1,\delta_2)$ is formed by the lines $y = x$ and $y = \delta_2$, and the curve $\theta_1 x^{-1} + \theta_2 y^{-1} = \theta_1\delta_1^{-1} + \theta_2\delta_2^{-1}$ for $\delta_1 \le \delta_2$ and $\theta_i \ge 1$, $i = 1,2$. The graph of this region is plotted in Figure 1 in which $\delta_H = (\theta_1+\theta_2)/(\theta_1\delta_1^{-1}+\theta_2\delta_2^{-1})$ is the weighted harmonic mean of δ_1 and δ_2 with corresponding weights θ_1 and θ_2. Set $\xi_1 = \sum_{i=1}^{l}\theta_i$ and $\xi_2 = \sum_{j=l+1}^{n}\theta_j$. We will find some conditions on the baseline distribution G such that, for $(\delta_1^*,\delta_2^*) \in \Delta_{\xi_1,\xi_2}(\delta_1,\delta_2)$, the ordering $T_{n:n} \ge_{\mathrm{mrl}} T_{n:n}^*$ holds. We will also examine this result when the baseline distribution is the generalized gamma. When $\theta_i = 1$ for all $i = 1,\ldots,n$ (the scale model with reduced heterogeneity of scale parameters), the power-generalized Weibull and half-normal distributions are investigated as the examples. For many well-known distributions, a huge body of literature exists concerning comparisons of their largest order statistics with respect to magnitude orders such as the usual stochastic, hazard rate, and likelihood ratio orders. But, for the mean residual life order, attentions has been focused just on the exponential distribution whereas other distributions remain noticeably absent in the literature. The results established in this paper fill this gape by extension of the previous ones from the exponential framework to the exponentiated generalized gamma, power-generalized Weibull and half-normal frameworks.

The general structure of the paper can be summarized as follows: In Section 2, we present the main results. Several examples and illustrations are stated in Section 3. Finally, some discussions are made in Section 4. Throughout the paper, we write $P \overset{\mathrm{sgn}}{=} Q$ to mean that P and Q have the same sign. Furthermore, for any differentiable function $w(x)$, $w'(x)$ denotes the first derivative of $w(x)$ with respect to x while the notion $\partial_i b(x_1,x_2)$ is used for the partial derivative of any differentiable function $b(x_1,x_2)$ with respect to x_i, $i = 1,2$.

2. Main results

Here, we compare the largest order statistics arising from independent random variables following heterogeneous RS models with respect to the mean residual life order. In what follows, everywhere we use the notions $(T_1,\ldots,T_n) \sim RS(G;\boldsymbol{\theta},\boldsymbol{\delta}_l)$ and $(T_1^*,\ldots,T_n^*) \sim RS(G;\boldsymbol{\theta},\boldsymbol{\delta}_l^*)$ wherein $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_n)$, $\boldsymbol{\delta}_l = (\delta_1\mathbf{1}_l,\delta_2\mathbf{1}_{n-l})$ and $\boldsymbol{\delta}_l^* = (\delta_1^*\mathbf{1}_l,\delta_2^*\mathbf{1}_{n-l})$, it means that $T_i \sim RS(G;\theta_i,\delta_1)$, $T_j \sim RS(G;\theta_j,\delta_2)$, $T_i^* \sim RS(G;\theta_i,\delta_1^*)$ and $T_j^* \sim RS(G;\theta_j,\delta_2^*)$ for $i = 1,\ldots,l$ and $j = l+1,\ldots,n$. Furthermore, the survival, density, hazard rate and reversed hazard rate functions of the baseline distribution function G, centered on $\mathbb{R}_+$, are respectively denoted by $\bar{G}$, g, $r = g/\bar{G}$ and $\tilde{r} = g/G$. Set $\alpha(x) = x\tilde{r}(x)$, $\gamma(x) = x\tilde{r}'(x)/\tilde{r}(x)$, $F(x;u_1,u_2) = [G(u_1x)]^{\theta_1}[G(u_2x)]^{\theta_2}$ and $\bar{F}(x;u_1,u_2) = 1 - F(x;u_1,u_2)$ for all

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$x \in \mathbb{R}^+$ and $(u_1, u_2) \in \mathbb{R}^{+2}$. Under this setting, it can be easily seen that

$$\gamma(x) = \frac{xg'(x)}{g(x)} - \alpha(x), \quad x \in \mathbb{R}^+, \tag{1}$$

$$\frac{xr'(x)}{r(x)} = \frac{xg'(x)}{g(x)} + xr(x), \quad x \in \mathbb{R}^+. \tag{2}$$

To prove the main result, we need a series of lemmas which are presented in the sequel.

Lemma 1 (Mitrinović *et al.* [24, p. 71])**.** *Let $y_k \in (0,1)$ and $\nu_k \geq 1$ for all $k = 1, \ldots, n$. Then, we have*

$$1 - \prod_{k=1}^{n} (1 - y_k)^{\nu_k} \leq \sum_{k=1}^{n} \nu_k y_k.$$

Lemma 2 (Mitrinović *et al.* [24, p. 340])**.** *Consider three vectors $\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$, $\boldsymbol{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n$ and $\boldsymbol{d} = (d_1, \ldots, d_n) \in \mathbb{R}^{+n}$. Then, we have*

$$\min\left\{\frac{c_1}{d_1}, \ldots, \frac{c_n}{d_n}\right\} \leq \frac{\sum_{i=1}^n q_i c_i}{\sum_{i=1}^n q_i d_i} \leq \max\left\{\frac{c_1}{d_1}, \ldots, \frac{c_n}{d_n}\right\}.$$

Lemma 3. *Suppose that $\theta_i \geq 1$ for $i = 1, 2$. Let the function $L(\cdot; t) : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as:*

$$L(x;t) = \frac{\bar{F}(x;1,t)}{F(x;1,t)\alpha(x)},$$

wherein $t \geq 1$. Assume that the following conditions hold:

(a_1) $r(x)$ is increasing in $x \in \mathbb{R}^+$;
(a_2) $\alpha(x)/\bar{G}(x)$ is increasing in $x \in \mathbb{R}^+$.

Then, $L(x;t)$ is decreasing in $x \in \mathbb{R}^+$.

Proof. Since $F'(x;1,t) = (\theta_1 \tilde{r}(x) + \theta_2 t \tilde{r}(tx)) F(x;1,t)$ for all $x \in \mathbb{R}^+$, then we find

$$\begin{aligned}
(xL(x;t))' &\overset{\text{sgn}}{=} -F'(x;1,t)F(x;1,t)\tilde{r}(x) - (F'(x;1,t)\tilde{r}(x) + F(x;1,t)\tilde{r}'(x))\bar{F}(x;1,t) \\
&= -F'(x;1,t)F(x;1,t)\tilde{r}(x) - F'(x;1,t)\bar{F}(x;1,t)\tilde{r}(x) - F(x;1,t)\bar{F}(x;1,t)\tilde{r}'(x) \\
&= -F'(x;1,t)\tilde{r}(x)(F(x;1,t) + \bar{F}(x;1,t)) - F(x;1,t)\bar{F}(x;1,t)\tilde{r}'(x) \\
&= -F'(x;1,t)\tilde{r}(x) - F(x;1,t)\bar{F}(x;1,t)\tilde{r}'(x) \\
&= -(\theta_1 \tilde{r}(x) + \theta_2 t \tilde{r}(tx))F(x;1,t)\tilde{r}(x) - F(x;1,t)\bar{F}(x;1,t)\tilde{r}'(x) \\
&= -\frac{F(x;1,t)\bar{F}(x;1,t)\tilde{r}(x)}{x}\left(\frac{\theta_1 x \tilde{r}(x) + \theta_2 t x \tilde{r}(tx)}{\bar{F}(x;1,t)} + \gamma(x)\right) \\
&\overset{\text{sgn}}{=} -\left(\frac{\theta_1 \alpha(x) + \theta_2 \alpha(tx)}{\bar{F}(x;1,t)} + \gamma(x)\right), \quad x \in \mathbb{R}^+.
\end{aligned} \tag{3}$$

We also have

$$\begin{aligned}
\frac{\theta_1 \alpha(x) + \theta_2 \alpha(tx)}{\bar{F}(x;1,t)} &\geq \frac{\theta_1 \alpha(x) + \theta_2 \alpha(tx)}{\theta_1 \bar{G}(x) + \theta_2 \bar{G}(tx)} \\
&\geq \min\left\{\frac{\alpha(x)}{\bar{G}(x)}, \frac{\alpha(tx)}{\bar{G}(tx)}\right\} \\
&= \frac{\alpha(x)}{\bar{G}(x)}, \quad x \in \mathbb{R}^+,
\end{aligned} \tag{4}$$

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wherein the first inequality is obtained from Lemma 1, the second inequality is derived from Lemma 2 and finally the last identity is established based on Condition (a_2) and the restriction $t \geq 1$. Now, upon combining Eqs. (1), (2) and (4), it follows from Condition (a_1) that

$$\begin{aligned}\frac{\theta_1\alpha(x)+\theta_2\alpha(tx)}{\bar{F}(x;1,t)}+\gamma(x) &\geq \frac{\alpha(x)}{\bar{G}(x)}+\frac{xg'(x)}{g(x)}-\alpha(x)\\ &= \frac{xg'(x)}{g(x)}+xr(x)\\ &= \frac{xr'(x)}{r(x)}\\ &\geq 0, \quad x\in\mathbb{R}^+,\end{aligned}$$

which confirms the right-hand side of (3) is non-positive. Thus, $xL(x;t)$ is decreasing in $x \in \mathbb{R}^+$ and so, $L(x;t)$ is also decreasing in $x \in \mathbb{R}^+$, as required. □

Lemma 4. *For $t \in \mathbb{R}^+$ and $u_2 \geq u_1$, let the function $\Xi(\cdot;t) : \mathbb{R}^+ \to \mathbb{R}$ be defined as:*

$$\Xi(x;t)=\alpha(t)F(1;u_1,u_2)\bar{F}\left(\frac{x}{t};u_1,u_2\right)-\alpha(x)F\left(\frac{x}{t};u_1,u_2\right)\bar{F}(1;u_1,u_2).$$

Assume that the following conditions hold:

(a_1) $r(x)$ is increasing in $x \in \mathbb{R}^+$;
(a_2) $\alpha(x)/\bar{G}(x)$ is increasing in $x \in \mathbb{R}^+$;
(a_3) $\tilde{r}(c_1x)/\tilde{r}(c_2x)$ is increasing in $x \in \mathbb{R}^+$ for $0 < c_1 \leq c_2$.

Then, we have

(i) $\Xi(x;u_1) \leq 0$ for all $x \geq u_1$;
(ii) $\Xi(x;u_1) - \Xi(x;u_2) \leq 0$ for all $x \geq u_2$.

Proof. (i) The function $\Xi(x;u_1)$ can be rewritten as

$$\Xi(x;u_1)=\alpha(u_1)\alpha(x)F(1;u_1,u_2)F\left(\frac{x}{u_1};u_1,u_2\right)\left(\frac{\bar{F}(\frac{x}{u_1};u_1,u_2)}{F(\frac{x}{u_1};u_1,u_2)\alpha(x)}-\frac{\bar{F}(1;u_1,u_2)}{F(1;u_1,u_2)\alpha(u_1)}\right),\quad x\geq u_1.$$

On the other hand, we have

$$\begin{aligned}F\left(\frac{x}{u_1};u_1,u_2\right) &= \left[G\left(u_1\frac{x}{u_1}\right)\right]^{\theta_1}\left[G\left(u_2\frac{x}{u_1}\right)\right]^{\theta_2}\\ &= [G(x)]^{\theta_1}\left[G\left(\frac{u_2}{u_1}x\right)\right]^{\theta_2}\\ &= F\left(x;1,\frac{u_2}{u_1}\right),\quad x\in\mathbb{R}^+.\end{aligned}$$

Now, from the above observation and the notion of Lemma 3, one can easily find that

$$\begin{aligned}\Xi(x;u_1) &= \alpha(u_1)\alpha(x)F(1;u_1,u_2)F\left(\frac{x}{u_1};u_1,u_2\right)\left(\frac{\bar{F}(x;1,\frac{u_2}{u_1})}{F(x;1,\frac{u_2}{u_1})\alpha(x)}-\frac{\bar{F}(u_1;1,\frac{u_2}{u_1})}{F(u_1;1,\frac{u_2}{u_1})\alpha(u_1)}\right)\\ &= \alpha(u_1)\alpha(x)F(1;u_1,u_2)F\left(\frac{x}{u_1};u_1,u_2\right)\left(L\left(x;\frac{u_2}{u_1}\right)-L\left(u_1;\frac{u_2}{u_1}\right)\right),\quad x\geq u_1.\end{aligned}$$

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By Lemma 3, we know $L(x; u_2/u_1)$ is decreasing in $x \in \mathbb{R}^+$ which results in $\Xi(x; u_1) \le 0$ for all $x \ge u_1$.

(ii) Let us define $\mathcal{D} = \{y \ge u_2 : \Xi(y; u_2) \le 0\}$. For $x \in \mathcal{D}^c$ (the complement of $\mathcal{D}$), we have from Part (i) that $\Xi(x; u_1) - \Xi(x; u_2) \le 0$. In the sequel, it is assume that $x \in \mathcal{D}$. Because $F(x; u_1, u_2)$ is increasing in $x \in \mathbb{R}^+$, it follows that $F(x/u_2; u_1, u_2) \le F(x/u_1; u_1, u_2)$ for all $x \in \mathbb{R}^+$ and $u_2 \ge u_1$. Now, using this observation, we have

$$\begin{aligned}
&\Xi(x; u_1) - \Xi(x; u_2)\\
&\quad = F\left(\frac{x}{u_1}; u_1, u_2\right) F(1; u_1, u_2)\alpha(x)\alpha(u_1)\left(\frac{\bar{F}(\frac{x}{u_1}; u_1, u_2)}{\alpha(x)F(\frac{x}{u_1}; u_1, u_2)} - \frac{\bar{F}(1; u_1, u_2)}{\alpha(u_1)F(1; u_1, u_2)}\right)\\
&\qquad - F(\frac{x}{u_2}; u_1, u_2)F(1; u_1, u_2)\alpha(x)\alpha(u_2)\left(\frac{\bar{F}(\frac{x}{u_2}; u_1, u_2)}{\alpha(x)F(\frac{x}{u_2}; u_1, u_2)} - \frac{\bar{F}(1; u_1, u_2)}{\alpha(u_2)F(1; u_1, u_2)}\right)\\
&\quad \le F\left(\frac{x}{u_1}; u_1, u_2\right) F(1; u_1, u_2)\alpha(x)\alpha(u_1)\left(\frac{\bar{F}(\frac{x}{u_1}; u_1, u_2)}{\alpha(x)F(\frac{x}{u_1}; u_1, u_2)} - \frac{\bar{F}(1; u_1, u_2)}{\alpha(u_1)F(1; u_1, u_2)}\right)\\
&\qquad - F\left(\frac{x}{u_1}; u_1, u_2\right) F(1; u_1, u_2)\alpha(x)\alpha(u_2)\left(\frac{\bar{F}(\frac{x}{u_2}; u_1, u_2)}{\alpha(x)F(\frac{x}{u_2}; u_1, u_2)} - \frac{\bar{F}(1; u_1, u_2)}{\alpha(u_2)F(1; u_1, u_2)}\right)\\
&\quad = F\left(\frac{x}{u_1}; u_1, u_2\right) F(1; u_1, u_2)\alpha(x)\left(\frac{\bar{F}(\frac{x}{u_1}; u_1, u_2)}{\alpha(x)F(\frac{x}{u_1}; u_1, u_2)}\alpha(u_1) - \frac{\bar{F}(\frac{x}{u_2}; u_1, u_2)}{\alpha(x)F(\frac{x}{u_2}; u_1, u_2)}\alpha(u_2)\right)\\
&\quad = \Upsilon(x), \quad \text{say.}
\end{aligned}$$

Setting $c_1 = u_1/u_2$ and $c_2 = 1$ in Condition (a_3), it follows that $\tilde{r}(u_1/u_2 x)/\tilde{r}(x) \ge \tilde{r}(u_1)/\tilde{r}(u_2)$ or equivalently $\alpha(u_1/u_2 x)/\alpha(x) \ge \alpha(u_1)/\alpha(u_2)$ for all $x \ge u_2$. From this observation and Lemma 3, we obtain

$$\begin{aligned}
\Upsilon(x) &\overset{\text{sgn}}{=} \frac{\bar{F}(\frac{x}{u_2}; u_1, u_2)}{\alpha(x)F(\frac{x}{u_1}; u_1, u_2)}\alpha(u_1) - \frac{\bar{F}(\frac{x}{u_2}; u_1, u_2)}{\alpha(\frac{u_1}{u_2}x)F(\frac{x}{u_2}; u_1, u_2)}\frac{\alpha(\frac{u_1}{u_2}x)\alpha(u_2)}{\alpha(x)}\\
&\le \frac{\alpha(\frac{u_1}{u_2}x)\alpha(u_2)}{\alpha(x)}\left(\frac{\bar{F}(\frac{x}{u_1}; u_1, u_2)}{\alpha(x)F(\frac{x}{u_1}; u_1, u_2)} - \frac{\bar{F}(\frac{x}{u_2}; u_1, u_2)}{\alpha(\frac{u_1}{u_2}x)F(\frac{x}{u_2}; u_1, u_2)}\right)\\
&= \frac{\alpha(\frac{u_1}{u_2}x)\alpha(u_2)}{\alpha(x)}\left(L\left(x; \frac{u_2}{u_1}\right) - L\left(\frac{u_1}{u_2}x; \frac{u_2}{u_1}\right)\right)\\
&\le 0, \quad x \ge u_2.
\end{aligned}$$

Thus, we can conclude that $\Xi(x; u_1) - \Xi(x; u_2) \le 0$ for all $x \ge u_2$, as desired. □

For a given point (δ_1, δ_2) with $0 < \delta_1 \le \delta_2$ and $\theta_i \ge 1$, $i = 1, 2$, we can redefine the region $\Delta_{\theta_1,\theta_2}(\delta_1, \delta_2)$, proposed in Introduction, as follows:

$$\Delta_{\theta_1,\theta_2}(\delta_1, \delta_2) = \{(x, y) \in \mathbb{R}^{+2} : \delta_1 \le x \le y \le \delta_2 \text{ and } \theta_1\delta_1^{-1} + \theta_2\delta_2^{-1} \ge \theta_1 x^{-1} + \theta_2 y^{-1}\}.$$

A question arises here: what condition the function $\varphi : \mathbb{R}^{+2} \to \mathbb{R}$ must have to satisfy the inequality $\varphi(\delta_1, \delta_2) \ge \varphi(\delta_1^*, \delta_2^*)$ for $(\delta_1^*, \delta_2^*) \in \Delta_{\theta_1,\theta_2}(\delta_1, \delta_2)$? If $\theta_1{\delta^*}_1^{-1} + \theta_2{\delta^*}_2^{-1} = \theta_1\delta_1^{-1} + \theta_2\delta_2^{-1}$, then the point (δ_1^*, δ_2^*) lies on the curve $\theta_1 x^{-1} + \theta_2 y^{-1} = \theta_1\delta_1^{-1} + \theta_2\delta_2^{-1}$. Because the vector field of this curve is $(1, -\theta_1\theta_2^{-1}x^{-2}y^2)$, then the inequality $\varphi(\delta_1, \delta_2) \ge \varphi(\delta_1^*, \delta_2^*)$ holds if $\varphi(x, y)$ is decreasing along the vector $(1, -\theta_1\theta_2^{-1}x^{-2}y^2)$. If (δ_1^*, δ_2^*) lies inside the region $\Delta_{\theta_1,\theta_2}(\delta_1, \delta_2)$, that is, $\theta_1{\delta^*}_1^{-1} + \theta_2{\delta^*}_2^{-1} < \theta_1\delta_1^{-1} + \theta_2\delta_2^{-1}$, then there exists a point (δ_1', δ_2^*) on the curve $\theta_1 x^{-1} + \theta_2 y^{-1} = \theta_1\delta_1^{-1} + \theta_2\delta_2^{-1}$ such that $\delta_1 < \delta_1'$. In this case, if $\varphi(x, y)$ is decreasing along the vectors $(1, 0)$ and $(1, -\theta_1\theta_2^{-1}x^{-2}y^2)$, then we have $\varphi(\delta_1, \delta_2) \ge \varphi(\delta_1', \delta_2^*) \ge \varphi(\delta_1^*, \delta_2^*)$. Consequently, we can state the following lemma.

Lemma 5. *Consider the function* $\varphi : \mathbb{R}^{+2} \to \mathbb{R}$. *If* $\varphi(u_1, u_2)$ *is decreasing along the vectors* $\boldsymbol{v}_1 = (1, 0)$ *and* $\boldsymbol{v}_2 = (1, -\theta_1\theta_2^{-1}u_1^{-2}u_2^2)$, *then we have*

$$(\delta_1^*, \delta_2^*) \in \Delta_{\theta_1,\theta_2}(\delta_1, \delta_2) \Rightarrow \varphi(\delta_1, \delta_2) \geq \varphi(\delta_1^*, \delta_2^*).$$

Next theorem deals with the mean residual life order between the largest order statistics arising from the RS models.

Theorem 1. *Suppose* $(T_1, \ldots, T_n) \sim RS(G; \boldsymbol{\theta}, \boldsymbol{\delta}_l)$ *and* $(T_1^*, \ldots, T_n^*) \sim RS(G; \boldsymbol{\theta}, \boldsymbol{\delta}_l^*)$. *Set* $\xi_1 = \sum_{i=1}^{l} \theta_i$ *and* $\xi_2 = \sum_{i=l+1}^{n} \theta_i$. *Assume that the following conditions hold:*

(a_1) $r(x)$ *is increasing in* $x \in \mathbb{R}^+$;
(a_2) $\alpha(x)/\bar{G}(x)$ *is increasing in* $x \in \mathbb{R}^+$;
(a_3) $\tilde{r}(c_1x)/\tilde{r}(c_2x)$ *is increasing in* $x \in \mathbb{R}^+$ *for* $0 < c_1 \leq c_2$.

If $(\delta_1^*, \delta_2^*) \in \Delta_{\xi_1,\xi_2}(\delta_1, \delta_2)$, *then we have* $T_{n:n} \geq_{\mathrm{mrl}} T_{n:n}^*$.

Proof. The distribution functions of $T_{n:n}$ and $T_{n:n}^*$ are respectively as

$$F_{T_{n:n}}(x) = [G(\delta_1 x)]^{\xi_1}[G(\delta_2 x)]^{\xi_2}, \quad F_{T_{n:n}^*}(x) = [G(\delta_1^* x)]^{\xi_1}[G(\delta_2^* x)]^{\xi_2}, \quad x \in \mathbb{R}^+.$$

Evidently, the distribution function of $T_{n:n}$ (resp. $T_{n:n}^*$) is the same as that of $T_{2:2}$ (resp. $T_{2:2}^*$) by replacing θ_i by ξ_i for $i = 1, 2$. Therefore, it is enough to prove the required result for the special case of $n = 2$. The mean residual life functions of $T_{2:2}$ and $T_{2:2}^*$ can be rewritten respectively as

$$m_{T_{2:2}}(x) = x\varphi(\delta_1 x, \delta_2 x), \quad m_{T_{2:2}^*}(x) = x\varphi(\delta_1^* x, \delta_2^* x), \quad x \in \mathbb{R}^+,$$

wherein

$$\varphi(u_1, u_2) = \frac{\int_1^\infty \bar{F}(x; u_1, u_2)\, dx}{\bar{F}(1; u_1, u_2)}, \quad 0 < u_1 \leq u_2.$$

According to Lemma 5, the desired result follows if we could show that $\varphi(u_1, u_2)$ is decreasing at the directions $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. The partial derivative of $\varphi(u_1, u_2)$ with respect to u_1 can be expressed as

$$\begin{aligned}\partial_1\varphi(u_1, u_2) &= [\bar{F}(1; u_1, u_2)]^{-2}\left\{-\theta_1\bar{F}(1; u_1, u_2)\int_1^\infty x\tilde{r}(u_1x)F(x; u_1, u_2)\, dx\right. \\ &\quad \left. +\theta_1\tilde{r}(u_1)F(1; u_1, u_2)\int_1^\infty \bar{F}(x; u_1, u_2)\, dx\right\} \\ &= \theta_1 u_1^{-2}[\bar{F}(1; u_1, u_2)]^{-2}\int_{u_1}^\infty \Xi(x; u_1)\, dx,\end{aligned}$$

wherein the function $\Xi(.; u_1)$ is defined in Lemma 4. Similarly, the partial derivative of $\varphi(u_1, u_2)$ with respect to u_2 is

$$\partial_2\varphi(u_1, u_2) = \theta_2 u_2^{-2}[\bar{F}(1; u_1, u_2)]^{-2}\int_{u_2}^\infty \Xi(x; u_2)\, dx.$$

Using Part (i) of Lemma 4, one can easily observe that $\Xi(x; u_1) \leq 0$ for all $x \geq u_1$. Hence, we find that

$$\nabla_{\boldsymbol{v}_1}\varphi = \partial_1\varphi(u_1, u_2) \leq 0,$$

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and so, φ is decreasing at the direction $\boldsymbol{v}_1$. The gradient of φ along the vector $\boldsymbol{v}_2$ is

$$\begin{aligned}\nabla_{\boldsymbol{v}_2}\varphi &= \partial_1\varphi(u_1,u_2) - \theta_1\theta_2^{-1}u_2^2u_1^{-2}\partial_2\varphi(u_1,u_2)\\ &\overset{\text{sgn}}{=} \int_{u_1}^{\infty}\Xi(x;u_1)\,dx - \int_{u_2}^{\infty}\Xi(x;u_2)\,dx\\ &= \int_{u_1}^{u_2}\Xi(x;u_1)\,dx + \int_{u_2}^{\infty}(\Xi(x;u_1)-\Xi(x;u_2))\,dx\end{aligned}$$

According to Lemma 4, it readily follows $\nabla_{\boldsymbol{v}_2}\varphi \leq 0$ which results in φ is also decreasing at the direction $\boldsymbol{v}_2$, completing the proof of the theorem. □

In Theorem 1, the inference is focused on the scale parameters while both involved vectors of random variables have the common resilience parameters. It is worthwhile to point that the effect of resilience parameters on the mean residual life function of $T_{n:n}$ is also an interesting problem. To see more information in this direction, we refer the readers to Haidari *et al.* [17]. It should be mentioned that the result of Theorem 1 extends those of Zhao and Balakrishnan [33] and Wang and Cheng [29] which are established when the baseline distribution is exponential.

Now, let us give a reliability explanation of Theorem 1. Consider a factory that produces some specific units with parallel structures made up n components. Suppose the components used in building the units come from a supplier, say Supplier I, which has two production lines. Due to production policies, the factory selects l components from one of the production line and the remaining $n-l$ components from the another one. Supplier I asserts the lifetimes of its produced components in each line follow the RS models with same scale parameters but with possibly different resilience parameters. For some reasons such as high price or unavailability of the components in a specific period of time, the factory decides to purchase its required components from a new supplier, say Supplier II. The produced components by Supplier II, like Supplier I, are built in two production lines with their lifetimes following the RS models with same scale parameters but with possibly different resilience parameters. In such a case, changing the components may impress the quality of the units of the factory. Therefore, to avoid the quality loss of the units, the factory must investigate the effect of these changes. In this situation, Theorem 1 gives some sufficient conditions to compare the mean residual life functions of the units comprising the components of Suppliers I and II.

Next proposition is an immediate consequence of Theorem 1 because the scale model can be obtained from the RS model when all the resilience parameters are equal to 1.

Proposition 1. *Consider two sets of independent nonnegative random variables $T_1,\ldots,T_n$ and $T_1^*,\ldots,T_n^*$ following the multiple-outlier scale models with common baseline distribution function G and respective vectors of scale parameters $(\delta_1\mathbf{1}_l,\delta_2\mathbf{1}_{n-l})$ and $(\delta_1^*\mathbf{1}_l,\delta_2^*\mathbf{1}_{n-l})$. Assume that the following conditions hold:*

(a_1) $r(x)$ is increasing in $x\in\mathbb{R}^+$;
(a_2) $\alpha(x)/\bar{G}(x)$ is increasing in $x\in\mathbb{R}^+$;
(a_3) $\tilde{r}(c_1x)/\tilde{r}(c_2x)$ is increasing in $x\in\mathbb{R}^+$ for $0<c_1\leq c_2$.

If $(\delta_1^*,\delta_2^*)\in\Delta_{l,n-l}(\delta_1,\delta_2)$, then we have $T_{n:n}\geq_{\text{mrl}}T_{n:n}^*$.

Remark 1. It should be noted that Condition (a_2) in Theorem 1 (Proposition 1) satisfies if and only if

$$1+\gamma(x)+xr(x)\geq 0,\quad \text{for all } x\in\mathbb{R}^+,$$

which, by Eqs. (1) and (2), can be rewritten as

$$1+\frac{xr'(x)}{r(x)}-\alpha(x)\geq 0,\quad \text{for all } x\in\mathbb{R}^+.$$

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Furthermore, Condition (a_3) in Theorem 1 (Proposition 1) is equivalent to say that $\gamma(x)$ is decreasing in $x \in \mathbb{R}^+$.

3. Illustration with examples

In this section, we present some examples of well-known distributions verifying the conditions of the results given in the previous section. Recall that $\bar{G}$, r and $\tilde{r}$ are respectively the survival, hazard rate and reversed hazard rate functions of the baseline distribution in both RS and scale models. Also, the functions α and γ are defined as $\alpha(x) = x\tilde{r}(x)$ and $\gamma(x) = x\tilde{r}'(x)/\tilde{r}(x)$ for $x \in \mathbb{R}^+$.

3.1. Exponentiated generalized gamma distribution

If a random variable Y admits the following distribution function

$$F(x;\tau,\beta,\theta,\delta) = \left[\int_0^x \frac{\tau\delta^\beta}{\Gamma(\frac{\beta}{\tau})} u^{\beta-1} e^{-(\delta u)^\tau}\, du\right]^\theta, \quad x \in \mathbb{R}^+,\ (\theta,\tau,\beta,\delta) \in \mathbb{R}^{+4},$$

wherein $\Gamma(\cdot)$ is the incomplete gamma function, then it is said that Y has the exponentiated generalized gamma (EGG) distribution with shape parameters (θ,τ,β) and scale parameter δ, denoted by $Y \sim \mathrm{EGG}(\theta,\tau,\beta,\delta)$. This distribution is introduced and investigated comprehensively by Cordeiro *et al.* [10]. The EGG distribution contains some known distributions such as Weibull, generalized gamma (GG), generalized exponential, exponentiated Weibull and exponentiated gamma as special cases. Note that, the EGG distribution belongs to the RS model when the baseline distribution is the GG distribution with shape parameters (τ,β) and scale parameter 1 (denoted by $\mathrm{GG}(\tau,\beta)$ and called as the GG distribution with shape parameters (τ,β)); see Kleiber and Kotz [19] for more details on the GG distribution and its applications.

To show Theorem 1 can be applied for the EGG distribution, we need the following lemma.

Lemma 6. *The GG distribution with shape parameters (τ,β) satisfies all conditions of Theorem 1 when $\tau \geq \beta \geq 1$.*

Proof. Khaledi *et al.* [18] proved the followings for $\mathrm{GG}(\tau,\beta)$:

$$\alpha(x) \leq \beta, \quad x \in \mathbb{R}^+; \tag{5}$$

$$\beta - 1 < x\frac{r'(x)}{r(x)} < \tau - 1, \quad x \in \mathbb{R}^+,\ \tau > \beta. \tag{6}$$

When $\tau \geq \beta \geq 1$, it is well-known that the GG distribution has an increasing hazard rate function, and so, Condition (a_1) of Theorem 1 is fulfilled. Also, upon combining Eqs. (5) and (6), we find that $1 + (xr'(x))/r(x) - \alpha(x) \geq 0$ for all $x \in \mathbb{R}^+$ and $\tau > \beta$. Using this observation and Remark 1, we see that $\alpha(x)/\bar{G}(x)$ is increasing in $x \in \mathbb{R}^+$ when $\tau > \beta$. Furthermore, if $\tau = \beta$, then the GG distribution is reduced to Weibull distribution. In this case, we have $\alpha(x)/\bar{G}(x) = \beta x^\beta e^{x^\beta}$ which clearly is increasing in $x \in \mathbb{R}^+$ for all $\beta \in \mathbb{R}^+$. Hence, Condition (a_2) of Theorem 1 is satisfied for $\tau \geq \beta$. As Ding *et al.* [12] have shown, $\gamma(x)$ is decreasing in $x \in \mathbb{R}^+$ for all $(\tau,\beta) \in \mathbb{R}^{+2}$ and so, by Remark 1, we can conclude that Condition (a_3) of Theorem 1 is held. The proof is now completed. □

Next corollary is a direct consequence of Theorem 1 and Lemma 6.

Corollary 1. *Let $T_1,\ldots,T_n$ and $T_1^*,\ldots,T_n^*$ be two sets of independent random variables with $T_i \sim \mathrm{EGG}(\theta_i,\tau,\beta,\delta_1)$, $T_j \sim \mathrm{EGG}(\theta_j,\tau,\beta,\delta_2)$, $T_i^* \sim \mathrm{EGG}(\theta_i,\tau,\beta,\delta_1^*)$ and $T_j^* \sim \mathrm{EGG}(\theta_j,\tau,\beta,\delta_2^*)$ for $i = 1,\ldots,l$ and $j = l+1,\ldots,n$. If $\tau \geq \beta \geq 1$ and $(\delta_1^*,\delta_2^*) \in \Delta_{\xi_1,\xi_2}(\delta_1,\delta_2)$, then we have $T_{n:n} \geq_{\mathrm{mrl}} T^*{}_{n:n}$.*

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Several well-known lifetime distributions satisfy in Corollary 1 as listed below:

(i) Set $\theta_1 = \cdots = \theta_n = 1$ and $\tau = \beta = 1$. In this case, we have the multiple-outlier exponential models. This statement is proved by Zhao and Balakrishnan [33] for $n = 2$, while the general case is established by Wang and Cheng [29].
(ii) Set $\theta_1 = \cdots = \theta_n = 1$ and $\tau = \beta$. This case deals with Weibull distributed random variables with common shape parameter τ and reduced scale parameters. Note that, the mean residual life order holds under the restriction $\tau \geq 1$. For the special case of $\tau = 2$, Rayleigh distribution is involved.
(iii) Set $\tau = \beta = 1$. In this statement, we have the generalized exponential distributed random variables with heterogeneous shape parameters and reduced scale parameters.
(iv) Set $\tau = \beta$. This case concerns the exponentiated Weibull distributed random variables. Like Case (ii), a restriction $\tau \geq 1$ is appeared for the mean residual life order to be hold.

3.2. Power-generalized Weibull distribution

A random variable Y is said to have the power-generalized Weibull (PGW) distribution with shape parameters (ρ, γ) and scale parameter δ, denoted by PGW(ρ, γ, δ), if its distribution function is as follows:

$$F(x; \rho, \gamma, \delta) = 1 - e^{1-(1+(\delta x)^{\rho})^{1/\gamma}}, \quad x \in \mathbb{R}^+, (\rho, \gamma, \delta) \in \mathbb{R}^{+3}.$$

This distribution is introduced by Bagdonavicius and Nikulin [2] in the context of accelerated failure time models. It contains exponential, Rayleigh and Weibull distributions as special cases. The PGW distribution is a suitable model to analysis the lifetime data sets due to its flexible hazard rate function which admits monotone and non-monotone shapes; see Nikulin and Haghighi [26] and Nadaraja and Haghighi [25]. It is clear that the PGW distribution belongs to the scale model with the baseline distribution as PGW$(\rho, \gamma, 1)$ (called as the PGW distribution with shape parameters (ρ, γ)).

Before presenting an application of Proposition 1 for the case of PGW distribution, we state the next lemma.

Lemma 7. *The PGW distribution with shape parameters (ρ, γ) satisfies all conditions of Proposition 1 when $\rho \geq 1$ and $\gamma \leq 1$.*

Proof. With the restrictions $\rho \geq 1$ and $\gamma \leq 1$, Condition (a_1) of Proposition 1 is satisfied because the PGW distribution admits an increasing hazard rate function under the mentioned restrictions. Furthermore, as shown by Ding *et al.* [12], we have

$$x\frac{r'(x)}{r(x)} = \rho - 1 + \rho\left(\frac{1}{\gamma} - 1\right)\frac{x^{\rho}}{1 + x^{\rho}}, \quad x \in \mathbb{R}^+.$$

It is obvious that $(xr'(x))/r(x)$ is increasing in $x \in \mathbb{R}^+$ for $\gamma \leq 1$. Also, one can easily find that

$$\lim_{x\to 0} x\frac{r'(x)}{r(x)} = \rho - 1, \quad \lim_{x\to\infty} x\frac{r'(x)}{r(x)} = \frac{\rho}{\gamma} - 1.$$

Upon combining the above observations, we obtain

$$\rho - 1 \leq x\frac{r'(x)}{r(x)} \leq \frac{\rho}{\gamma} - 1, \quad x \in \mathbb{R}^+, \ \gamma \leq 1. \tag{7}$$

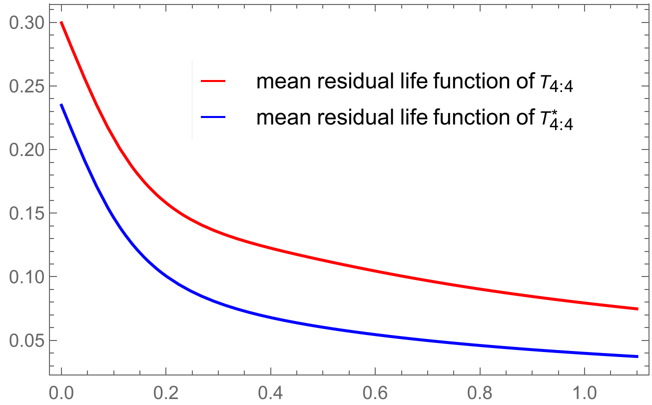

Figure 2. *Plot of the mean residual functions of $T_{4:4}$ and $T^*_{4:4}$ when $l = 2$, $\rho = 1$, $\gamma = 0.5$, $(\delta_1, \delta_2) = (2, 3.5)$ and $(\delta^*_1, \delta^*_2) = (3, 3.2)$ for random variables with PGW distributions.*

Furthermore, using the L'Hopital's rule, it follows that

$$\begin{aligned}\lim_{x\to 0} \alpha(x) &= \frac{\rho}{\gamma} \lim_{x\to 0} \frac{x^\rho (1 + x^\rho)^{1/\gamma - 1}}{e^{(1+x^\rho)^{1/\gamma}-1} - 1} \\ &= \lim_{x\to 0} \frac{\rho(1 + x^\rho) + \rho(\frac{1}{\gamma} - 1)x^\rho}{(1 + x^\rho)e^{(1+x^\rho)^{1/\gamma}-1}} \\ &= \rho. \end{aligned} \tag{8}$$

According to Lemma A.6 of Khaledi *et al.* [18], we know that $\alpha(x)$ is decreasing in $x \in \mathbb{R}^+$ which along with Eq. (8) result in

$$\alpha(x) \le \rho, \quad x \in \mathbb{R}^+. \tag{9}$$

Now, using Eqs. (7) and (9), we readily find that $1 + (xr'(x))/r(x) - \alpha(x) \ge 0$ for all $x \in \mathbb{R}^+$ and $\gamma \le 1$ and so, based on Remark 1, one can observe that $\alpha(x)/\bar{G}(x)$ is increasing in $x \in \mathbb{R}^+$ when $\gamma \le 1$. Hence, Condition (a_2) of Proposition 1 is fulfilled under the restriction $\gamma \le 1$. Furthermore, from Remark 1 once again and Lemma 4.11 of Ding *et al.* [12], we see that Condition (a_3) of Proposition 1 is satisfied when $\gamma \le 1$, completing the proof of the lemma. □

From Proposition 1 and Lemma 7, the next corollary readily follows.

Corollary 2. *Let $T_1, \ldots, T_n$ and $T^*_1, \ldots, T^*_n$ be two sets of independent random variables with $T_i \sim \mathrm{PGW}(\rho, \gamma, \delta_1)$, $T_j \sim \mathrm{PGW}(\rho, \gamma, \delta_2)$, $T^*_i \sim \mathrm{PGW}(\rho, \gamma, \delta^*_1)$ and $T^*_j \sim \mathrm{PGW}(\rho, \gamma, \delta^*_2)$ for $i = 1, \ldots, l$ and $j = l + 1, \ldots, n$. If $\rho \ge 1$, $\gamma \le 1$ and $(\delta^*_1, \delta^*_2) \in \Delta_{l,n-l}(\delta_1, \delta_2)$, then we have $T_{n:n} \ge_{\mathrm{mrl}} T^*_{n:n}$.*

Next example illustrates the result given in Corollary 2.

Example 1. Set $n = 4$, $l = 2$, $\rho = 1$, $\gamma = 0.5$, $(\delta_1, \delta_2) = (2, 3.5)$ and $(\delta^*_1, \delta^*_2) = (3, 3.2)$. In this case, we have

$$\Delta_{2,2}(2, 3.5) = \left\{(x, y) \in \mathbb{R}^{+2} : 2 \le x \le y \le 3.5 \text{ and } x^{-1} + y^{-1} \le \frac{11}{14}\right\}.$$

It can be readily seen that $(\delta^*_1, \delta^*_2) \in \Delta_{2,2}(2, 3.5)$ which, according to Corollary 2, results in $T_{4:4} \ge_{\mathrm{mrl}} T^*_{4:4}$. In Figure 2, the mean residual life functions of $T_{4:4}$ and $T^*_{4:4}$ is plotted over the interval $(0, 1.1]$.

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3.3. Half-normal distribution

Consider a random variable Y with its distribution function has the following form:

$$F(x;\lambda) = \int_0^x \frac{\delta\sqrt{2}}{\sqrt{\pi}} e^{-(\delta y)^2/2}\,dy, \quad x \in \mathbb{R}^+, \delta \in \mathbb{R}^+.$$

Then, it is said that Y follows the half-normal (HN) distribution with scale parameter δ, denoted by $Y \sim HN(\delta)$. Clearly, the HN distribution belongs to the scale model and the baseline distribution in this case obtains by choosing $\delta = 1$ (called as the standard HN).

Lemma 8. *The standard HN satisfies all conditions of Proposition 1.*

Proof. The hazard function of the standard HN is as follows:

$$r(x) = \frac{e^{-x^2/2}}{\int_x^\infty e^{-y^2/2}\,dy}, \quad x \in \mathbb{R}^+.$$

Taking derivative of $r(x)$ with respect to $x \in \mathbb{R}^+$, we find that

$$\begin{aligned} r'(x) &\overset{\text{sgn}}{=} e^{-x^2/2} - x\int_x^\infty e^{-y^2/2}\,dy \\ &= s(x), \quad \text{say}. \end{aligned}$$

It is easy to see that $s'(x) \leq 0$ for all $x \in \mathbb{R}^+$. From this observation and the fact that $\lim_{x\to\infty} s(x) = 0$, we can readily conclude that $r(x)$ is increasing in $x \in \mathbb{R}^+$. Thus, Condition (a_1) of Proposition 1 is held. Moreover, from Ding *et al.* [12], we have

$$\gamma(x) = -x^2 - \alpha(x), \quad x \in \mathbb{R}^+,$$

which, for all $x \in \mathbb{R}^+$, results in

$$\begin{aligned} 1 + \gamma(x) + xr(x) &= (1 - \alpha(x)) + x(r(x) - x) \\ &= D_1 + D_2, \quad \text{say}. \end{aligned}$$

As pointed out by Ding *et al.* [12], we know $\alpha(x) \leq 1$ for all $x \in \mathbb{R}^+$ and hence $D_1 \geq 0$. Furthermore, it is shown in the above that $s(x) \geq 0$ or equivalently $r(x) \geq x$ for all $x \in \mathbb{R}^+$ which results in $D_2 \geq 0$. Therefore, $1 + \gamma(x) + xr(x) \geq 0$ for all $x \in \mathbb{R}^+$ and so, according to Remark 1, Condition (a_2) of Proposition 1 is also satisfied. Finally, from Ding *et al.* [12] and Remark 1, we can see that Condition (a_3) of Proposition 1 is obtained. The proof is now completed. □

From Proposition 1 and Lemma 8, the next corollary immediately follows.

Corollary 3. *Let $T_1, \ldots, T_n$ and $T_1^*, \ldots, T_n^*$ be two sets of independent random variables with $T_i \sim HN(\delta_1)$, $T_j \sim HN(\delta_2)$, $T_i^* \sim HN(\delta_1^*)$ and $T_j^* \sim HN(\delta_2^*)$ for $i = 1, \ldots, l$ and $j = l+1, \ldots, n$. If $(\delta_1^*, \delta_2^*) \in \Delta_{l,n-l}(\delta_1, \delta_2)$, then we have $T_{n:n} \geq_{\text{mrl}} T_{n:n}^*$.*

In the following example, the result given in Corollary 3 is investigated numerically.

Example 2. Set $n = 2$, $l = 1$, and $(\delta_1, \delta_2) = (0.5, 1)$, $(\delta_1^*, \delta_2^*) = (0.75, 0.8)$. Then, it is easy to see that $(\delta_1^*, \delta_2^*) \in \Delta_{1,1}(0.5, 1)$ wherein

$$\Delta_{1,1}(0.5, 1) = \{(x, y) \in \mathbb{R}^{+2} : 0.5 \leq x \leq y \leq 1 \text{ and } x^{-1} + y^{-1} \leq 3\}.$$

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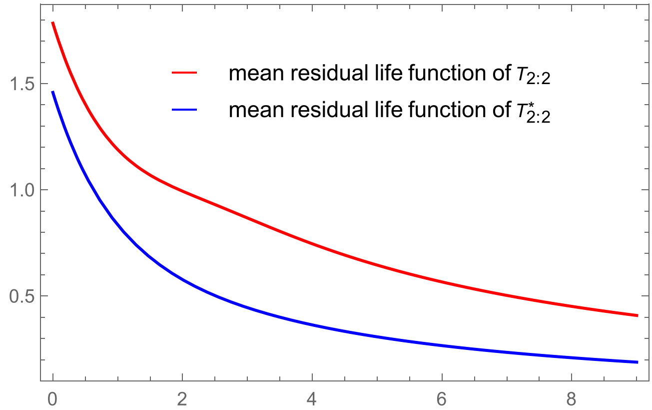

Figure 3. *Plot of the mean residual functions of* $T_{2:2}$ *and* $T^*_{2:2}$ *when* $l = 1$, $(\delta_1, \delta_2) = (0.5, 1)$ *and* $(\delta_1^*, \delta_2^*) = (0.75, 0.8)$ *for random variables with HN distributions.*

Now, by Corollary 3, we can conclude $T_{2:2} \geq_{\text{mrl}} T^*_{2:2}$. In Figure 3, the mean residual life functions of $T_{2:2}$ and $T^*_{2:2}$ is plotted over the interval $(0, 9]$.

4. Discussion

In this paper, we have established some conditions for the mean residual life order between the largest order statistics in the RS models to be hold. We have also studied an application of this result for the case of exponentiated generalized gamma, power-generalized Weibull and half-normal distributions. The results established here extend and reinforce those of Zhao and Balakrishnan [33] and Wang and Cheng [29]. Following the definitions of weighted majorization and related orders presented in Cheng [9], let us define the weighted version of reciprocal majorization order. Set $\mathcal{E}_n^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^{+n} : 0 < x_1 \leq \cdots \leq x_n\}$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$. For two vectors $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ in $\mathcal{E}_n^+$, we say that $(u_1, \ldots, u_n)$ is greater than $(v_1, \ldots, v_n)$ on $\mathcal{E}_n^+$ with respect to $\boldsymbol{\theta}$-reciprocal majorization order, denoted by $(u_1, \ldots, u_n) \overset{\text{rm}}{\succ}_{\boldsymbol{\theta}} (v_1, \ldots, v_n)$ on $\mathcal{E}_n^+$, if $\sum_{k=1}^{i} \theta_k u_k^{-1} \geq \sum_{k=1}^{i} \theta_k v_k^{-1}$ for all $i = 1, \ldots, n$. In the definition of the region $\Delta_{\theta_1,\theta_2}(\delta_1, \delta_2)$, the restrictions $\theta_1 \geq 1$ and $\theta_2 \geq 1$ are appeared because of utilizing the inequality in Lemma 1. Now, if $(\delta_1^*, \delta_2^*) \in \Delta_{\theta_1,\theta_2}(\delta_1, \delta_2)$ without any restriction on θ_i's, then we can see that $\delta_1 \leq \delta_1^* \leq \delta_2^* \leq \delta_2$ and $(\delta_1, \delta_2) \overset{\text{rm}}{\succ}_{(\theta_1,\theta_2)} (\delta_1^*, \delta_2^*)$ on $\mathcal{E}_2^+$. Thus, if the restrictions are dropped, then the weighted version of reciprocal majorization order can be used in comparison of the largest order statistics in the RS models. With the help of some numerical examples, we conjecture that the mean residual life order given in Theorem 1 may hold without any restrictions on θ_i's. We are currently working on this problem and hope to report the findings in the future works.

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