MARTINGALE CONVERGENCE THEOREMS FOR SEQUENCES OF STONE ALGEBRAS

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1. Introduction. A vector lattice W is *boundedly complete* when each subset $\{a_j : j \in J\}$ of W which is bounded above by an element of W has a least upper bound in W. The least upper bound of $\{a_j : j \in J\}$ is denoted by $\bigvee_{j \in J} a_j$ and the greatest lower bound by $\bigwedge_{j \in J} a_j$ whenever these exist.

Let $C(S)$ be the algebra of real valued continuous functions on a compact Hausdorff space S. Stone [4] shows that the vector lattice $C(S)$ is boundedly complete if and only if the closure of each open subset of S is open; in this event we call $C(S)$ a *Stone algebra*. For example, if $(X, \mathscr{B}, \mu)$ is a probability space, then $L^{\infty}(X, \mathscr{B}, \mu)$ is a Stone algebra satisfying the countable chain condition.

Let $\{a_n\}$ $(n = 1, 2, \ldots)$ be a bounded sequence in a Stone algebra $\mathscr{S}$; then

$$\bigvee_{n=1}^{\infty} \bigwedge_{r=n}^{\infty} a_r \leqq \bigwedge_{n=1}^{\infty} \bigvee_{r=n}^{\infty} a_r.$$

When these two terms are equal we define $\mathrm{LIM}\, a_n$ to be their common value and say the sequence is order convergent with order limit $\mathrm{LIM}\, a_n$. In the special case where $\mathscr{S}$ is of the form $L^{\infty}(X, \mathscr{B}, \mu)$ and μ is a probability measure, if a sequence $\{b_n\}$ $(n = 1, 2, \ldots)$ has order limit b, then the sequence $\{b_n\}$ $(n = 1, 2, \ldots)$ converges to b in the L^1-topology (L^{∞} is the dual of L^1). But Floyd [3] gives an example of a Stone algebra $\mathscr{S}$ such that there is no Hausdorff vector topology for $\mathscr{S}$ in which each bounded monotone increasing sequence converges to its least upper bound.

We shall postpone all further definitions till §2. In [7] we investigated Moy averaging operators on Stone algebras satisfying the countable chain condition. In this paper we consider a monotone increasing sequence $\{\mathscr{A}_n\}$ $(n = 1, 2, \ldots)$ of Stone subalgebras of a Stone algebra $\mathscr{A}_{\infty}$ such that the smallest Stone subalgebra containing $\bigcup_{n=1}^{\infty} \mathscr{A}_n$ is $\mathscr{A}_{\infty}$. Let $\mathscr{A}_{\infty}$ satisfy the countable chain condition and let $T_0 : \mathscr{A}_{\infty} \to \mathscr{A}_1$ be a Moy operator satisfying certain conditions. Then we show that there exists a sequence $\{T_n\}$ $(n = 1, 2, \ldots)$ of Moy operators on $\mathscr{A}_{\infty}$ such that:

(i) *T_n is a projection of $\mathscr{A}_{\infty}$ onto $\mathscr{A}_n$ for $n \geqq 1$.*

(ii) *$T_r T_n = T_r$ for $0 \leqq r < n$.*

(iii) *If b is a positive element of $\mathscr{A}_{\infty}$ and $T_n b = 0$ then $b = 0$.*

(iv) *For each $z \in \mathscr{A}_{\infty}$ the order limit* $\mathrm{LIM}\, T_n z$ *exists and* $\mathrm{LIM}\, T_n z = z$.

This result is a Corollary of Theorem 2.

Theorem 1 is a convergence theorem for a sequence of generalized conditional expectations with respect to a modular Stone algebra valued measure. For conditional expectations with respect to real valued measures such results are known in probability theory as martingale

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theorems; see Doob [2]. Theorem 1 was suggested by the classical work of Sparre Andersen and Jessen in [1]. The key step in generalizing their result to Stone algebra valued measures is Lemma 1.

In a later publication I intend to discuss applications of the results of this paper to Boolean algebras.

The work of this paper depends essentially on that of [6]. This is because in [7] we used the results of [6] to establish the existence, under certain conditions, of generalized conditional expectations.

2. Convergence theorems. Throughout this paper $(X, \mathscr{B})$ is a measurable space and $C(S)$ is a Stone algebra. Stone algebra valued measures were defined in [5]. We require ρ to be a finite $C(S)$-valued measure on $(X, \mathscr{B})$; that is, ρ is to be a map of $\mathscr{B}$ into $C(S)$ such that

(i) $\rho E \geqq 0$ for each $E \in \mathscr{B}$;

(ii) if $\{E_j\}$ $(j = 1, 2, \ldots)$ is a pairwise disjoint family of sets in $\mathscr{B}$ then

$$\rho \bigcup_{j=1}^{\infty} E_j = \bigvee_{n=1}^{\infty} \sum_{j=1}^{n} \rho E_j.$$

A Stone algebra $\mathscr{S}$ satisfies the *countable chain condition* when each bounded subset of $\mathscr{S}$ contains a countable subset such that the two sets have the same least upper bound. This condition on $\mathscr{S}$ is equivalent (see Proposition 3.2 of [6]) to the Boolean algebra of idempotent elements of $\mathscr{S}$ satisfying the countable chain condition. From now onward we require $C(S)$ to satisfy the countable chain condition.

We defined L^p-spaces with respect to Stone algebras in [6], and it follows from Proposition 3.3 that $L^\infty(X, \mathscr{B}, \rho)$ is a Stone algebra satisfying the countable chain condition because $C(S)$ satisfies this condition.

We require the existence of an algebra homomorphism $\pi: C(S) \to L^\infty(X, \mathscr{B}, \rho)$ such that

$$\int_X \pi(a) f \, d\rho = a \int_X f \, d\rho \quad \text{for each} \quad f \in \mathscr{L}^1(X, \mathscr{B}, \rho).$$

Then ρ is a modular measure with respect to π, as defined in [6]. Close connections between modular measures and averaging operators were exhibited in [7].

Let $\mathscr{S}$ be a Stone algebra and $\mathscr{U}$ a subalgebra. $\mathscr{U}$ is a *Stone subalgebra* of $\mathscr{S}$, if the least upper bound, in $\mathscr{S}$, of each upper bounded subset of $\mathscr{U}$ is in $\mathscr{U}$; i.e. $\mathscr{U}$ is a Stone algebra and a bounded subset of $\mathscr{U}$ has the same least upper bound in $\mathscr{S}$ and $\mathscr{U}$.

Let T be a linear operator on a Stone algebra $\mathscr{S}$. T is an *averaging operator* if $T(fTg) = (Tf)(Tg)$ for each f and g in $\mathscr{S}$. T is a *Moy averaging operator* when T is a positive averaging operator and, if $\{f_n\}$ $(n = 1, 2, \ldots)$ is a monotone increasing sequence in $\mathscr{S}$ which is bounded above, then $T \bigvee_{n=1}^{\infty} f_n = \bigvee_{n=1}^{\infty} Tf_n$. For any operator T on $\mathscr{S}$ let

$$\mathscr{E}(T) = \{a \in \mathscr{S} : aTb = Tab \quad \text{for all} \quad b \in \mathscr{S}\}.$$

When T is an averaging operator the range of T is a subset of $\mathscr{E}(T)$. It is shown in [7] that when T is a Moy operator and $\mathscr{S}$ satisfies the countable chain condition then $\mathscr{E}(T)$ is a Stone subalgebra of $\mathscr{S}$.

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When $\mathscr{B}_1$ is a Boolean σ-subalgebra of $\mathscr{B}$ and ρ_1 is the restriction of ρ to $\mathscr{B}_1$ then $L^\infty(X, \mathscr{B}_1, \rho_1)$ can be identified with a Stone subalgebra of $L^\infty(X, \mathscr{B}, \rho)$. If $\pi[C(S)]$ is a subalgebra of $L^\infty(X, \mathscr{B}_1, \rho_1)$, then for each $f \in \mathscr{L}^1(X, \mathscr{B}, \rho)$ we can find a $\mathscr{B}_1$-measureable function $f_1 \in \mathscr{L}^1(X, \mathscr{B}_1, \rho_1)$ such that

$$\int_E f\, d\rho = \int_E f_1\, d\rho_1 \quad \text{for each } E \in \mathscr{B}_1.$$

This is Lemma 2.1 of [7].

Definition 1. *Let $\mathscr{B}_1$ be a σ-subalgebra of $\mathscr{B}$ such that $\pi[C(S)] \subset L^\infty(X, \mathscr{B}_1, \rho_1)$. The conditional expectation of $\mathscr{B}_1$ with respect to ρ is the map $C: L^1(X, \mathscr{B}, \rho) \to L^1(X, \mathscr{B}_1, \rho_1)$ such that for each $[f]_\rho \in L^1(X, \mathscr{B}, \rho)$ we have $C[f]_\rho = [f_1]_\rho$, where*

$$\int_E f\, d\rho = \int_E f_1\, d\rho \quad \textit{for each } E \in \mathscr{B}_1.$$

The generalized conditional expectation operator C, defined above, is a positive linear map of $L^1(X, \mathscr{B}, \rho)$ onto $L^1(X, \mathscr{B}_1, \rho_1)$ such that $C^2 = C$. The restriction of C to $L^\infty(X, \mathscr{B}, \rho)$ is a Moy averaging operator whose range is $L^\infty(X, \mathscr{B}_1, \rho_1)$.

Lemma 1. *Let $\mathscr{W}$ be a Boolean subalgebra of $\mathscr{B}$ such that $\mathscr{B}$ is the smallest σ-algebra of subsets of X containing $\mathscr{W}$. Let $f \in \mathscr{L}^1(X, \mathscr{B}, \rho)$ be such that $\int_E f\, d\rho \geqq 0$ for each $E \in \mathscr{W}$. Then $\int_E f\, d\rho \geqq 0$ for each $E \in \mathscr{B}$.*

Proof. Let $\mathscr{U} = \left\{E \in \mathscr{B} : \int_E f\, d\rho \geqq 0\right\}$; then by hypothesis $\mathscr{W} \subset \mathscr{U}$. An argument using Zorn's lemma shows that there is a maximal Boolean algebra $\mathscr{M}$ such that $\mathscr{W} \subset \mathscr{M} \subset \mathscr{U}$.

Let $\mathscr{M}^* = \{E \subset X : \chi_E = \lim \chi_{E_n}$, where each $E_n \in \mathscr{M}\}$, so that $\mathscr{M} \subset \mathscr{M}^*$. If $A \in \mathscr{M}^*$ and $B \in \mathscr{M}^*$ then $A \cap B$ and $X - A$ are in $\mathscr{M}^*$. Hence $\mathscr{M}^*$ is a Boolean algebra containing $\mathscr{M}$.

Let $E \in \mathscr{M}^*$; then $\chi_E = \lim \chi_{E_n}$, where $E_n \in \mathscr{M}$ for each n. Then, by the analogue for Stone algebra valued measures of the Dominated Convergence Theorem established in [5], we have

$$\int_E f\, d\rho = \int_X f \chi_E\, d\rho = \mathrm{LIM} \int_X f \chi_{E_n}\, d\rho = \mathrm{LIM} \int_{E_n} f\, d\rho.$$

Thus $\int_E f\, d\rho \geqq 0$ and so $\mathscr{M}^* \subset \mathscr{U}$. It now follows from the maximality of $\mathscr{M}$ that $\mathscr{M} = \mathscr{M}^*$. Thus $\mathscr{M}$ is a Boolean σ-algebra containing $\mathscr{W}$ and thus $\mathscr{M} = \mathscr{U} = \mathscr{B}$.

Theorem 1. *Suppose that ρ is a finite $C(S)$-valued measure on the measurable space $(X, \mathscr{B})$ and suppose that ρ is modular with respect to π. Let $\{\mathscr{B}_n\}$ $(n = 1, 2, \ldots)$ be a monotone increasing sequence of σ-subalgebras of $\mathscr{B}$ such that $\mathscr{B}$ is the smallest σ-subalgebra of $\mathscr{B}$ containing $\bigcup_{n=1}^\infty \mathscr{B}_n$. Further, let $\pi[C(S)]$ be a subalgebra of $L^\infty(X, \mathscr{B}_1, \rho)$. For each n let T_n be the*

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generalized conditional expectation of $\mathscr{B}_n$ *with respect to* ρ. *Let* $f \in \mathscr{L}^1(X, \mathscr{B}, \rho)$ *and let* $f_n \in L^1(X, \mathscr{B}_n, \rho)$ *be such that* $[f_n]_\rho = T_n[f]_\rho$ *for each n. Then* $\lim f_n(x) = f(x)$ *almost everywhere with respect to* ρ.

Proof. The set $F = \{x \in X : \underline{\lim} f_n(x) < f(x)\}$ is the countable union of all sets of the form

$$F_{\alpha,\beta} = \{x \in X : \underline{\lim} f_n(x) \leqq \alpha < \beta \leqq f(x)\},$$

where α and β are rational and $\alpha < \beta$. Assume that $\rho F \neq 0$; then $\rho F_{\lambda,\mu} \neq 0$ for some rational numbers λ and μ, $\lambda < \mu$.

Let $L_\lambda = \{x \in X : \underline{\lim} f_n(x) \leqq \lambda\}$ and, for each natural number n, let

$$H_n = \left\{x \in X \;:\; \inf_{r > n} f_r(x) < \lambda + \frac{1}{n}\right\}.$$

Let

$$H_{n,1} = \left\{x \in X \;:\; f_{n+1}(x) < \lambda + \frac{1}{n}\right\}$$

and, for $q \geqq 2$,

$$H_{n,q} = \left\{x \in X : \min\{f_r(x) : n < r < n+q\} \geqq \lambda + \frac{1}{n} \quad \text{and} \quad f_{n+q}(x) < \lambda + \frac{1}{n}\right\}.$$

Since f_{n+q} is $\mathscr{B}_{n+q}$-measurable, $H_{n,q} \in \mathscr{B}_{n+q}$. Also $\{H_{n,q}\}$ $(q = 1, 2, \ldots)$ is a pairwise disjoint family such that $H_n = \bigcup_{q=1}^{\infty} H_{n,q}$. We also have $L_\lambda = \bigcap_{n=1}^{\infty} H_n$.

Choose $A \in \bigcup_1^\infty \mathscr{B}_n$, so that $A \in \mathscr{B}_N$ for some N. Then $H_{n,q} \cap A \in \mathscr{B}_{n+q}$ for $n \geqq N$ and $q \geqq 1$. By Proposition 3.3 of [6]

$$\int_A f\chi_{H_n} d\rho = \mathrm{LIM} \int_A \sum_{q=1}^{r} f\chi_{H_{n,q}} d\rho.$$

From the definition of T_{n+q} and f_{n+q} we have, for $n \geqq N$,

$$\int_A f\chi_{H_{n,q}} d\rho = \int_{A \cap H_{n,q}} f_{n+q}\, d\rho$$

$$\leqq \left(\lambda + \frac{1}{n}\right) \int_A \chi_{H_{n,q}} d\rho.$$

So

$$\int_A f\chi_{H_n} d\rho \leqq \left(\lambda + \frac{1}{n}\right) \int_A \chi_{H_n} d\rho \quad \text{for} \quad n \geqq N.$$

Thus

$$\int_A \left(\lambda + \frac{1}{n}\right) \chi_{H_n} - f\chi_{H_n} d\rho \geqq 0 \quad \text{for} \quad n \geqq N.$$

But $\lim_n \chi_{H_n} = \chi_{L_\lambda}$ and so

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$$\lim_n \left(\lambda+\frac{1}{n}\right)\chi_{H_n} - f\chi_{H_n} = (\lambda - f)\chi_{L_\lambda}.$$

So, by Proposition 3.5 of [6],

$$\int_A (\lambda - f)\chi_{L_\lambda}\, d\rho \geqq 0 \quad \text{for each} \quad A \in \bigcup_{n=1}^{\infty} \mathscr{B}_n.$$

We observe that $\bigcup_{n=1}^{\infty} \mathscr{B}_n$ is a Boolean subalgebra of $\mathscr{B}$ and, by hypothesis, $\mathscr{B}$ is the σ-algebra generated by $\bigcup_{n=1}^{\infty} \mathscr{B}_n$. It now follows from Lemma 1 that

$$\int_A (\lambda - f)\chi_{L_\lambda}\, d\rho \geqq 0 \quad \text{for each} \quad A \in \mathscr{B}.$$

We replace A by $F_{\lambda,\mu}$ in the above inequality and since $F_{\lambda,\mu} \subset L_\lambda$, obtain

$$\lambda\rho F_{\lambda,\mu} \geqq \int_{F_{\lambda,\mu}} f\, d\rho \geqq \mu\rho F_{\lambda,\mu}.$$

Since $\mu > \lambda$ this implies that $F_{\lambda,\mu} = 0$. This is a contradiction; so the assumption $\rho F \neq 0$ must be false. Thus $f(x) \leqq \underline{\lim}\, f_n(x)$ for almost all x.

Applying this result to $-f$ we obtain $f(x) \geqq \overline{\lim}\, f_n(x)$ for almost all x.

So $\lim f_n$ exists and equals f almost everywhere with respect to ρ.

We now strip away the measure theory of Theorem 1 and obtain the following abstract martingale theorem.

THEOREM 2. Let $\{\mathscr{A}_n\}$ $(n = 1, 2, \ldots)$ *be an increasing sequence of Stone subalgebras of a Stone algebra* $\mathscr{A}_\infty$ *such that the smallest Stone subalgebra containing* $\bigcup_{n=1}^{\infty} \mathscr{A}_n$ *is the whole of* $\mathscr{A}_\infty$. *Let* $\mathscr{A}_0$ *be a Stone algebra satisfying the countable chain condition and* $\pi: \mathscr{A}_0 \to \mathscr{A}_1$ *an algebra homomorphism. Let* $T_0: \mathscr{A}_\infty \to \mathscr{A}_0$ *be a positive linear map such that*:

(i) *If* $b \geqq 0$ *and* $T_0 b = 0$ *then* $b = 0$.

(ii) $T_0(\pi(a)z) = aT_0 z$ *for each* $z \in \mathscr{A}_\infty$ *and each* $a \in \mathscr{A}_0$.

(iii) *If* $\{z_n\}$ $(n = 1, 2, \ldots)$ *is a bounded monotone increasing sequence of positive elements of* $\mathscr{A}_\infty$ *then*

$$T_0\left(\bigvee_{n=1}^{\infty} z_n\right) = \bigvee_{n=1}^{\infty} T_0 z_n.$$

Then there exists a sequence of Moy operators $\{T_n\}$ $(n = 1, 2, \ldots)$ *such that*:

(i) T_n *is a projection of* $\mathscr{A}_\infty$ *onto* $\mathscr{A}_n$ *for each* $n \geqq 1$.

(ii) *If* $b \geqq 0$ *and* $T_n b = 0$ *then* $b = 0$.

(iii) $T_r T_n = T_r$ *for* $0 \leqq r < n$.

(iv) *For each* $z \in \mathscr{A}_\infty$ *the order limit* LIM $T_n z$ *exists and* LIM $T_n z = z$.

Proof. Let $\mathscr{A}_\infty \cong C(E)$, the ring of continuous functions on an extremally disconnected compact Hausdorff space E. For each Borel set A in E there is a unique idempotent $k(A)$

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in $C(E)$ which differs from χ_A only on a meagre Borel set. We recall from [5] that k is a $C(E)$-valued measure, the map $f \to \int_E f dk$ is an algebra homomorphism of $B^\infty(E)$ (the bounded Borel functions on E) onto $C(E)$ and the kernel of this homomorphism is the set of Borel functions vanishing outside a meagre Borel set.

Let m be defined on the Borel sets of E by $mB = T_0(kB)$. Then m is a (finite) $\mathscr{A}_0$-valued measure on the Borel sets of E and for each $f \in B^\infty(E)$ we have

$$\int_E f dm = T_0\left(\int_E f dk\right).$$

Let B be any Borel set of E; then $mB = 0$ if and only if $kB = 0$, that is, if and only if B is meagre. Thus

$$L^\infty(E, m) \cong C(E) \cong \mathscr{A}_\infty.$$

For each $a \in \mathscr{A}_0$ and $f \in B^\infty(E)$ we have

$$\int_E \pi(a) f dm = T_0\left(\int_E \pi(a) f dk\right) = T_0\left(\pi(a) \int_E f dk\right).$$

But, by hypothesis,

$$T_0\left(\pi(a) \int_E f dk\right) = a T_0 \int_E f dk = a \int_E f dm.$$

Thus m is a modular $\mathscr{A}_0$-valued measure with respect to π.

Let $\mathscr{B}_n$ be the collection of all Borel sets B of E such that $kB \in \mathscr{A}_n$. Then $L^\infty(E, \mathscr{B}_n, m) \cong \mathscr{A}_n$ for each $n \geqq 1$. Let $\mathscr{B}_\infty$ be the smallest σ-subalgebra of the Borel sets of E which contains $\bigcup_{n=1}^{\infty} \mathscr{B}_n$. Thus $L^\infty(E, \mathscr{B}_\infty, m)$ is a Stone subalgebra of $L^\infty(E, m) \cong \mathscr{A}_\infty$ and contains each of the algebras $\mathscr{A}_n$ $(n = 1, 2, \ldots)$. Thus $L^\infty(E, \mathscr{B}_\infty, m) \cong \mathscr{A}_\infty \cong L^\infty(E, m)$, although $\mathscr{B}_\infty$ may not contain all the Borel sets of E.

Since $\pi[\mathscr{A}_0] \subset \mathscr{A}_n$ for $n \geqq 1$ and m is an $\mathscr{A}_0$-valued measure, which is modular with respect to π, there exists a generalized conditional expectation operator T_n mapping $\mathscr{A}_\infty$ onto $\mathscr{A}_n$. Thus T_n is a projection of $\mathscr{A}_\infty$ onto $\mathscr{A}_n$; if b is a positive element of $\mathscr{A}_\infty$ and $T_n b = 0$, then $b = 0$; T_n is the unique linear operator from $\mathscr{A}_\infty$ into $\mathscr{A}_n$ such that for each idempotent $e \in \mathscr{A}_n$ and each $z \in \mathscr{A}_\infty$ we have $T_0(eT_n z) = T_0(ez)$. Let $1 \leqq r < n$ and let e be an idempotent of $\mathscr{A}_r$ and $z \in \mathscr{A}_\infty$; then $T_0(eT_r T_n z) = T_0(eT_n z) = T_0(ez)$, and so $T_r T_n = T_r$.

It remains to show that, if $z \in \mathscr{A}_\infty$, then the order limit LIM $T_n z$ exists and equals z. Let us identify $\mathscr{A}_\infty$ with $C(E)$ so that z and each $T_n z$ $(n \geqq 1)$ are continuous functions in $C(E)$. We have from Theorem 1 that there exists a Borel set B such that $mB = 0$ and $\lim (T_n z)(t)$ exists and equals $z(t)$ for each $t \in E - B$. The sequence $\{T_n z\}$ $(n = 1, 2, \ldots)$ is uniformly bounded because each T_n is a positive operator and $T_n 1 = 1$. Since $mB = 0$ only if $kB = 0$, we have, by the analogue of the Dominated Convergence Theorem proved in [5], that

$$\text{LIM} \int_E T_n z\, dk \quad \text{exists and equals} \quad \int_E z\, dk.$$

Thus LIM $T_n z$ exists and equals z.

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COROLLARY. *Let $\{\mathscr{A}_n\}$ $(n = 1, 2, \ldots)$ be an increasing sequence of Stone subalgebras of a Stone algebra $\mathscr{A}_\infty$ such that the smallest Stone subalgebra containing $\bigcup_{n=1}^{\infty} \mathscr{A}_n$ is the whole of $\mathscr{A}_\infty$. Let $\mathscr{A}_\infty$ satisfy the countable chain condition. Let T_0 be a Moy operator on $\mathscr{A}_\infty$ whose range is a subset of $\mathscr{A}_1$, and is such that if b is a positive element of $\mathscr{A}_\infty$ and $T_0 b = 0$ then $b = 0$. Then there exists a sequence of Moy operators $\{T_n\}$, $n = 1, 2, \ldots$, such that*:

(i) *T_n is a projection of $\mathscr{A}_\infty$ onto $\mathscr{A}_n$ for $n \geqq 1$.*

(ii) *If b is a positive element of $\mathscr{A}_\infty$ and $T_n b = 0$ then $b = 0$.*

(iii) *$T_r = T_r T_n$ for $0 \leqq r < n$.*

(iv) *For each $z \in \mathscr{A}_\infty$ the order limit* LIM $T_n z$ *exists and* LIM $T_n z = z$.

Proof. Since $\mathscr{A}_\infty$ satisfies the countable chain condition, we have that

$$\mathscr{E}(T_0) = \{a \in \mathscr{A}_\infty : aTb = Tab \quad \text{for all } b \in \mathscr{A}_\infty\}$$

is a Stone subalgebra of $\mathscr{A}_\infty$. Let $\mathscr{A}_0$ be the smallest Stone subalgebra of $\mathscr{A}_\infty$ containing the range of T_0. Thus $\mathscr{A}_0 \subset \mathscr{A}_1$ and $\mathscr{A}_0 \subset \mathscr{E}(T_0)$. Let $\pi : \mathscr{A}_0 \to \mathscr{A}_\infty$ be the natural embedding. Then $T_0 \pi(a) z = aT_0 z$ for each $a \in \mathscr{A}_0$ and each $z \in \mathscr{A}_\infty$. The corollary now follows from Theorem 2.

These methods can be adapted to prove analogous convergence theorems, where instead of $\{\mathscr{A}_n\}$ $(n = 1, 2, \ldots)$ being monotone increasing it is monotone decreasing and

$$\pi[\mathscr{A}_0] \subset \bigcap_{n=1}^{\infty} \mathscr{A}_n = \mathscr{A}_\infty.$$

In Theorem 1 we required the measure ρ to be modular so as to ensure the existence of the generalized conditional expectations T_n. We observe that we can dispense with the hypothesis that ρ is modular if we know that the conditional expectation T_1 of $\mathscr{B}_1$ with respect to ρ exists. This is because T_1 may be regarded as an $L^\infty(X, \mathscr{B}_1, \rho)$-valued modular measure and so there is a conditional expectation T_n of $\mathscr{B}_n$ with respect to T_1 for each n. A straightforward computation shows that T_n is the conditional expectation of $\mathscr{B}_n$ with respect to ρ. The proof, in Theorem 1, that $\{T_n f\}$ $(n = 1, 2, \ldots)$ converges almost everywhere to f depends only on the existence of the conditional expectations T_n and not on the modularity of ρ.

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