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Bowen–Walters expansiveness for semigroups of linear operators

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Abstract. We define *positively expansive semigroups of linear operators on Banach spaces*. We characterize these semigroups in terms of the point spectrum of the infinitesimal generator. In particular, we prove that a positively expansive semigroup is neither uniformly bounded nor equicontinuous. We apply our results to the Lasota equation.

Key words: expansive flow, positively expansive semigroup, Banach space, generator
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1. *Introduction*

To the best of our knowledge, there is no satisfactory definition of positively expansive semiflow on metric spaces (see [8] and references therein). In this paper, we will present one for the special cases of semigroups of bounded linear operators. We characterize this property in terms of the point spectrum of the infinitesimal generator. In particular, we prove that a positively expansive semigroup is neither uniformly bounded nor equicontinuous. We apply our results to the Lasota equation [3, 7]. Let us present our definition and results in a precise way.

A homeomorphism (respectively continuous map) of a metric space $f : Y \to Y$ is *expansive* (respectively *positively expansive*) [5, 10] if there is $\delta > 0$ such that if $y, y' \in Y$ and $d(f^n(y), f^n(y')) \leq \delta$ for every $n \in \mathbb{Z}$ (respectively $n \in \mathbb{N}$), then $x = y$.

A *flow* is a map $\psi : \mathbb{R} \times Y \to Y$ such that

$$\psi(0, y) = y \quad \text{and} \quad \psi(t, \psi(s, y)) = \psi(t + s, y) \quad \text{for all } y \in Y \text{ and } t, s \in \mathbb{R}.$$

Denote by $\psi(t) : Y \to Y$ the *time t-map* defined by $\psi(t)y = \psi(t, y)$ for $y \in Y$ and $t \in \mathbb{R}$. Then, a flow is just a family of maps $\{\psi(t) : Y \to Y\}_{t\in\mathbb{R}}$ such that $\psi(0) = I$ (the identity of Y) and $\psi(t + s) = \psi(t) \circ \phi(s)$ for every $t, s \in \mathbb{R}$.

The following is the classical definition of expansive flow by Bowen and Walters [2].

Definition 1.1. We say that a flow ψ of Y is *expansive* if for every $\epsilon > 0$, there is some $\delta > 0$ such that the following holds: If $x, y \in Y$ and a continuous function $s : \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$ satisfy

$$d(\psi(t)x, \psi(s(t))y) \leq \delta \quad \text{for all } t \in \mathbb{R},$$

then there is $t_* \in [-\epsilon, \epsilon]$ such that $y = \psi(t_*)x$.

However, a *semiflow* is a map $\phi : [0, \infty[\times Y \to Y$ such that

$$\phi(0, y) = y \quad \text{and} \quad \phi(t, \phi(s, y)) = \phi(t + s, y) \quad \text{for all } y \in Y \text{ and } t, s \geq 0.$$

Again $\phi(t) : Y \to Y$ denotes the *time t-map*, namely $\phi(t)y = \phi(t, y)$ for $y \in Y$ and $t \geq 0$. Then, a semiflow is just a family of maps $\{\phi(t) : Y \to Y\}_{t\geq 0}$ such that $\phi(0) = I$ and $\phi(t + s) = \phi(t) \circ \phi(s)$ for every $t, s \geq 0$. Every flow ϕ generates semiflow (still denoted by ϕ) equal to the restriction of ϕ to $[0, \infty)$.

Recall that a *semigroup of bounded linear operators* of a (complex) Banach space X is a semiflow T of X such that the time t-map $T(t) : X \to X$ is a bounded linear operator for every $t \geq 0$.

Our definition of a positively expansive semigroup will be motivated by the next characterization of positively expansive linear operators.

THEOREM 1.1. *A bounded linear operator $L : X \to X$ is positively expansive if and only if there is $\delta > 0$ such that if $x, y \in X$ and*

$$\|L^n(x) - L^n(y)\| \leq \delta\|x\| \quad \textit{for all } n \geq 0,$$

then $x = y$.

Note the role of the rescaling term $\delta\|x\|$ in the conclusion of this result. Rescaling terms were introduced in the theory of smooth expansive systems by Wen and Wen [11]. By plugging these into Bowen–Walters's definition, we obtain the main definition of this work.

Definition 1.2. We say that a semigroup of bounded linear operators T of a Banach space X is *positively expansive* if for every $\epsilon > 0$, there is some $\delta > 0$ such that the following holds: If $x, y \in X$ and a continuous function $s : [0, \infty[\to [0, \infty[$ with $s(0) = 0$ satisfy

$$\|T(t)x - T(s(t))y\| < \delta\|x\| \quad \text{for all } t \geq 0,$$

then there is $t_* \in [0, \epsilon]$ such that $y = T(t_*)x$ or $x = T(t_*)y$.

We have the following remark.

Remark 1.2. The domain (and range) of s in the above definition were changed to $[0, \infty)$ since a semigroup is not defined for negative times. Likewise, the conclusion

'$t_* \in [-\epsilon, \epsilon]$ *such that* $y = \psi(t_*)x$' in Bowen–Walters's definition was changed to '*there is* $t_* \in [0, \epsilon]$ *such that* $y = T(t_*)x$ *or* $x = T(t_*)y$'.

To state our main result, we will use the following standard definitions. The *infinitesimal generator* [9] of a semigroup of bounded linear operators T of a Banach space X is the linear operator $A : D(A) \subset X \to X$ with domain

$$D(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

defined by

$$A(x) = \lim_{t\to 0^+} \frac{T(t)x - x}{t} \quad \text{for all } x \in D(A).$$

The *point spectrum* and *resolvent* of A will be denoted by $\sigma_p(A)$ and $\rho(A)$ respectively (see [4] for the corresponding definitions).

We say that T is a C_0-*semigroup* (or *strongly continuous*) if

$$\lim_{t\to 0^+} T(t)x = x \quad \text{for all } x \in X.$$

In such a case, A is densely defined, closed [9], and there are constants $M \geq 1$ and $\omega \geq 0$ such that $\|T(t)\| \leq Me^{\omega t}$ for $t \geq 0$. If $\omega = 0$, the semigroup is called *uniformly bounded* and if additionally $M = 1$, then T is called *semigroup of contractions*.

A semigroup of linear operators T of a Banach space X is:

- *equicontinuous* if for every $\epsilon > 0$, there is $\delta > 0$ such that if $x, y \in X$ satisfies $\|x - y\| \leq \delta$, then $\|T(t)x - T(t)y\| \leq \epsilon$ for $t \geq 0$;
- *hypercyclic* if there is $x \in X$ such that $\{T(t)x : t \geq 0\}$ is dense in X;
- *chaotic* if it is hypercyclic and the periodic points are dense in X.

(Recall that $x \in X$ is a periodic point if there is $t > 0$ such that $T(t)x = x$.) Examples of chaotic semigroups of linear operators can be found in [6]. We also say that T is *injective* if $T(t)$ is an injective linear operator for every $t \geq 0$.

With these definitions, we can state our result.

THEOREM 1.3. *Let T be a C_0-semigroup of linear operators with infinitesimal generator A of a Banach space X.*

(1) *If T is positively expansive, then $\sigma_p(A) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$. Moreover, T is not uniformly bounded, equicontinuous, or chaotic.*

(2) *If there is $\Delta > 0$ such that $(-\infty, \Delta) \subset \rho(A)$ and*

$$\|(\gamma I - A)^{-1}\| \leq \frac{1}{\Delta - \gamma} \quad \textit{for all } \gamma \in (-\infty, \Delta),$$

then T is positively expansive.

Let us present two short applications of this theorem.

Example 1.4. Consider the semigroup $T(t)x = 2^t x$ for $t \geq 0$ in a Banach space X. It follows that $A(x) = (\ln 2)x$, so $(-\infty, \ln 2) \subset \rho(A)$ and $\|(\gamma I - A)^{-1}\| = 1/(\gamma - \ln 2)$ for all $\gamma < \ln 2$. Then, T is positively expansive by item (2) of Theorem 1.3.

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Example 1.5. Let X be the Banach space of bounded uniformly continuous functions on $]-\infty, \infty[$ with the supremum norm (see [9, Example 2.9]). Then, the semigroup T of X defined by $T(t)f(x) = f(x+t)$ for $f \in X$, $-\infty < x < \infty$, and $t \geq 0$ is uniformly bounded and so it is not positively expansive by item (1) of Theorem 1.3.

A third application will be given in the last section. In §2, we present some preliminary results and in §3, we prove the theorems.

2. *Preliminary results*

To prove Theorem 1.1, we will use the following characterization.

LEMMA 2.1. *The following properties are equivalent for bounded linear operators on Banach spaces $L : X \to X$.*

(1) *There is $\delta > 0$ such that if $x, y \in X$ and $\|L^n(x) - L^n(y)\| \leq \delta\|x\|$ for every $n \geq 0$, then $x = y$.*

(2) *There is $\delta > 0$ such that if $x, y \in X$, $\|x\| = 1$ and $\|L^n(x) - L^n(y)\| \leq \delta$ for every $n \geq 0$, then $x = y$.*

Proof. The only difference between items (1) and (2) above is that $\|x\| = 1$ in the latter but not in the former. Then, we only need to prove that item (2) implies item (1). For this, take $\delta > 0$, as in item (2), and suppose that $x, y \in X$ satisfy $\|L^n(x) - L^n(y)\| \leq \epsilon\|x\|$ for all $n \geq 0$. If $x = 0$, then $\|x - y\| = 0$ and hence $x = y$. Therefore, we can assume that $x \neq 0$. In such a case, $\|L^n(x/\|x\|) - L^n(y/\|x\|)\| \leq \epsilon$ for all $n \geq 0$. Since $\|x/\|x\|\| = 1$, we get $x/\|x\| = y/\|x\|$ and hence $x = y$. □

To prove Theorem 1.3, we need the following lemmas.

LEMMA 2.2. *Let T a semiflow of a metric space X and $x \in X$ be such that the map $t \in \mathbb{R}_0^+ \mapsto T(t)x$ is continuous. If such a map is not injective in $[0, \Delta]$ for every $\Delta > 0$, then $T(t)x = x$ for every $t \geq 0$.*

Proof. It follows from the hypothesis that there are sequences $0 \leq t_1^k < t_2^k \leq 1/k$ such that $T(t_1^k)x = T(t_2^k)x$ for every $k \in \mathbb{N}$. Then, $T(t_2^k - t_1^k)T(t_1^k)x = T(t_1^k)x$ and so

$$T(n(t_2^k - t_1^k))T(t_1^k)x = T(t_1^k)x \quad \text{for all } k, n \in \mathbb{N}.$$

Now take any $t \geq 0$. Since $t_2^k - t_1^k > 0$, there are sequences $n_k \in \mathbb{N} \cup \{0\}$ and $0 \leq r_k \leq t_2^k - t_1^k$ such that $t = n_k(t_2^k - t_1^k) + r_k$ for every $k \in \mathbb{N}$. It follows that

$$\begin{aligned} T(t + t_1^k)x &= T(t)T(t_1^k)x \\ &= T(n_k(t_2^k - t_1^k) + r_k)T(t_1^k)x \\ &= T(r_k)T(n_k(t_2^k - t_1^k))T(t_1^k)x \\ &= T(r_k + t_1^k)x \quad \text{for all } k \in \mathbb{N}. \end{aligned}$$

Since $t \mapsto T(t)x$ is continuous, we get $T(t)x = x$ by letting $k \to \infty$. □

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LEMMA 2.3. *The following properties are equivalent for every semigroup T of linear operators of a Banach space:*
(1) *T is equicontinuous;*
(2) *T is uniformly bounded.*

Proof. Suppose that T is equicontinuous. Take $\lambda > 0$ from this property for $\epsilon = 1$. If $x \in X$ and $\|x\| = 1$, then $\|\delta x\| = \delta$ so $\|T(t)\delta x\| \leq 1$ and thus $\|T(t)x\| \leq 1/\delta$ for all $t \geq 0$. Hence, $\|T(t)\| \leq 1/\delta$ for all $t \geq 0$ proving that T is uniformly bounded. Conversely, suppose that T is uniformly bounded. Then, there is $M \geq 1$ with $\|T(t)\| \leq M$ for $t \geq 0$. If $\epsilon > 0$ is given and we take $\|x\| \leq \delta$ with $\delta = \epsilon/M$, $\|T(t)x\| \leq M\|x\| \leq M\delta = \epsilon$ for $t \geq 0$ proving that T is equicontinuous. □

LEMMA 2.4. *A semigroup of linear operators T of a Banach space X is positively expansive if and only if for every $\epsilon > 0$, there is $\delta > 0$ such that if $x, y \in X$ with $\|x\| = 1$ satisfies*

$$\|T(t)x - T(s(t))y\| < \delta \quad \textit{for all } t \in \mathbb{R}_0^+,$$

for some continuous function $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $s(0) = 0$, then $y = T(t_*)x$ or $x = T(t_*)y$ for some $0 \leq t_* \leq \epsilon$.

To prove item (2) of Theorem 1.3, we will use the following lemma.

LEMMA 2.5. *Let T be a strongly continuous semigroup of linear operators with infinitesimal generator A of a Banach space X. If there is $\Delta > 0$ such that $\rho(A) \supset (-\infty, \Delta)$ and*

$$\|(\gamma I - A)^{-1}\| \leq \frac{1}{\Delta - \gamma} \quad \textit{for all } \gamma < \Delta,$$

then $(T(t))^{-1}$ exists and $\|(T(t))^{-1}\| \leq e^{-\Delta t}$ for all $t \geq 0$.

Proof. The linear operator $B = \Delta I - A$ satisfies that the domain $D(B) = D(A)$ dense in X, that $\rho(B) \supset \mathbb{R}^+$, and that $\|(\gamma I - B)^{-1}\| \leq 1/\gamma$ for every $\gamma > 0$. It then follows from the Hille–Yosida theorem [9] that there is a C_0-semigroup of contractions $\hat{T}$ with infinitesimal generator $B = \Delta I - A$. From this, we get

$$\begin{aligned}
\lim_{t\to 0^+} \frac{e^{-\Delta t}\hat{T}(t)x - x}{t} &= \lim_{t\to 0^+} \frac{e^{-\Delta t}\hat{T}(t)x - e^{-\Delta t}x}{t} + \lim_{t\to 0^+} \frac{e^{-\Delta t}x - x}{t} \\
&= \lim_{t\to 0^+} e^{-\Delta t}\left(\frac{\hat{T}(t)x - x}{t}\right) + \lim_{t\to 0^+}\left(\frac{e^{-\Delta t} - 1}{t}\right)x \\
&= \Delta x - A(x) - \Delta x \\
&= -A(x) \quad \text{for all } x \in D(-A) = D(A).
\end{aligned}$$

Then, $-A$ is the infinitesimal generator of the C_0-semigroup $e^{-\Delta t}\hat{T}(t)$. From this, we get that $T(t)$ is invertible with inverse $(T(t))^{-1} = e^{-\Delta t}\hat{T}(t)$ for all $t \geq 0$ (see the proof of

Theorem 6.3 in [9, p. 23]). Since $\hat{T}$ is a semigroup of contractions,

$$\|(T(t))^{-1}\| = \|e^{-\Delta t}\hat{T}(t)\| = e^{-\Delta t}\|\hat{T}(t)\| \leq e^{-\Delta t} \quad \text{for all } t \geq 0,$$

proving the lemma. □

To deal with the example in the next section, we will use the following lemma.

LEMMA 2.6. *Let T be a C_0-semigroup of linear operators of a Banach space. If there is $\Delta > 0$ such that*

$$(T(t))^{-1} \textit{ exists and } \|(T(t))^{-1}\| \leq e^{-\Delta t} \quad \textit{for all } t \geq 0, \tag{2.1}$$

then T is positively expansive.

Proof. Given $\epsilon > 0$, choose $0 < \delta < 1$ satisfying

$$\min\left\{\frac{\ln(1+\delta)}{\Delta}, -\frac{\ln(1-\delta)}{\Delta}\right\} < \epsilon. \tag{2.2}$$

Choose $x, y \in X$ satisfying

$$\|T(t)x - T(s(t))y\| \leq \delta\|x\| \quad \text{for all } t \in \mathbb{R}_0^+, \tag{2.3}$$

for some continuous function $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $s(0) = 0$. By Lemma 2.4, one can assume $\|x\| = 1$. We have to prove

$$T(t_*)x = y \quad \text{or} \quad T(t_*)y = x \quad \text{for some } 0 \leq t_* \leq \epsilon.$$

By hypothesis, we have that $(T(t))^{-1}$ exists for every $t \geq 0$. Then, T can be embedded into a group of linear operators (cf. [9, Lemma 6.4]). For simplicity, we still denote this group by T.

Applying equations (2.3) and (2.1), we get

$$\|x - T(s(t) - t)y\| = \|(T(t))^{-1}(T(t)x - T(s(t))y)\| \leq e^{-\Delta t}\delta \to 0,$$

proving

$$T(s(t) - t))y \to x \quad \text{as } t \to \infty. \tag{2.4}$$

Since $x \neq 0$, this immediately implies $y \neq 0$.

Let us prove that the map $t \in \mathbb{R}^+ \mapsto s(t) - t$ is bounded.

Otherwise, $s(t_n) - t_n \to \pm\infty$ as $n \to \infty$ for some sequence $t_n \to \infty$. By equation (2.1), if $s(t_n) - t_n \to \infty$,

$$\|T(s(t_n) - t_n)y\| \geq e^{\Delta(s(t_n)-t_n)}\|y\| \to \infty,$$

whereas if $s(t_n) - t_n \to -\infty$,

$$\|T(s(t_n) - t_n)y\| \leq e^{\Delta(s(n)-t_n)}\|y\| \to 0.$$

In any case, we contradict equation (2.4), therefore, $t \mapsto s(t) - t$ is bounded.

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Next we prove that $s(t) - t$ converges as $t \to \infty$.

Otherwise, it would exist $t^1, t^2 \in \mathbb{R}$ with $t^1 > t^2$ (say) such that $s(t_k^i) - t_k^i \to t^i$ for some sequences $t_k^i \to \infty$ ($i = 1, 2$). It follows that $T(t^1)y = T(t^2)y$. However, then

$$\|y\| = \|T(n(t^2 - t^1))y\| \leq e^{n(t^2 - t^1)\Delta}\|y\|$$

for all $n \in \mathbb{N}$ by equation (2.5) and hence $y = 0$, again a contradiction. Then, $s(t) - t$ converges to $-t_0$ (say) as $t \to \infty$.

It follows that $y = T(t_0)x$ by equation (2.4).

To estimate t_0, we replace $t = 0$ in equation (2.3) to get

$$\|x - T(t_0)x\| \leq \delta.$$

Since $\|x\| = 1$, we get

$$1 - \delta \leq \|T(t_0)x\| \leq 1 + \delta.$$

If $t_0 \leq 0$, $\|T(t_0)x\| \leq e^{\Delta t_0}$ by equation (2.1). So, $1 - \delta \leq e^{\Delta t_0}$ and thus $\ln(1 - \delta)/\Delta \leq t_0 \leq 0$. Then, $t_* = -t_0$ satisfies $0 \leq t_* \leq -\ln(1 - \delta)/\Delta \leq \epsilon$ (by equation (2.2)) and

$$T(t_*)y = T(-t_0)y = x.$$

If $t_0 \geq 0$, $\|T(t_0)x\| \geq e^{\Delta t_0}$ by equation (2.1) once more. So, $e^{\Delta t_0} \leq 1 + \delta$ and thus $0 \leq t_0 \leq \ln(1 + \delta)/\Delta$. Then, $t_* = t_0$ satisfies $0 \leq t_* \leq \epsilon$ (by equation (2.2)) and

$$T(t_*)x = T(t_0)x = y.$$

This proves the result. □

3. *Proof of the theorems*

Proof of Theorem 1.1. Suppose that L is positively expansive and let δ be given by the corresponding definition. If $x, y \in X$, $\|x\| = 1$ and $\|L^n(x) - L^n(y)\| \leq \epsilon$ for all $n \geq 0$, then $x = y$. Therefore, L satisfies the conclusion of the theorem by Lemma 2.1.

Conversely, suppose that L satisfies the conclusion of the theorem. Then, it satisfies item (1) of Lemma 2.1 and let δ be given by this item. Take $x \in X \setminus \{0\}$ such that $\sup_{n \geq 0} \|L^n(x)\| < \infty$. Then, there is $0 < K < \infty$ such that $\|L^n(x)\| \leq K$ for every $n \geq 0$. Since

$$\left\| L^n\left(\frac{x}{\|x\|}\right) - L^n\left(\left(\frac{1}{\|x\|} - \frac{\delta}{2K}\right)x\right)\right\| = \frac{\delta}{2K}\|L^n(x)\| \leq \frac{\delta}{2} < \delta \quad \text{for all } n \geq 0,$$

one has $(1/\|x\| - \delta/2K)x = x/\|x\|$ by Lemma 2.1. Then, $(\delta/2K)x = 0$ so $x = 0$, a contradiction. We conclude that $\sup_{n \geq 0} \|L^n(x)\| = \infty$ for every $x \in X \setminus \{0\}$ and so L is positively expansive by [1, Proposition 19, p. 806]. □

Proof of item (1) of Theorem 1.3. Suppose that T is positively expansive. Assume by contradiction that there is $\lambda \in \sigma_p(A)$ such that $\mathrm{Re}(\lambda) \leq 0$. Since $\lambda \in \sigma_p(A)$, there is

$x \in D(A)$ with $\|x\| = 1$ such that $Ax = \lambda x$. By [9, Theorem 2.4], we have

$$\begin{aligned}\frac{d}{dt}(e^{-\lambda t}T(t)x) &= \frac{d}{dt}(e^{-\lambda t})T(t)x + e^{-\lambda t}\frac{d}{dt}(T(t)x)\\ &= -\lambda e^{-\lambda t}T(t)x + e^{-\lambda t}T(t)Ax\\ &= -\lambda e^{-\lambda t}T(t)x + \lambda e^{-\lambda t}T(t)x\\ &= 0,\end{aligned}$$

proving $T(t)x = e^{\lambda t}x$ for $t \geq 0$.

Now let $\delta > 0$ be given by the definition of positive expansiveness for $\epsilon = 1$. We divide the proof in two parts depending on whether $\mathrm{Re}(\lambda) < 0$ or $\mathrm{Re}(\lambda) = 0$.

If $\mathrm{Re}(\lambda) < 0$, take $\alpha \in \mathbb{C} \setminus \{1\}$ with $|\alpha| = 1$ such that $|1 - \alpha| < \delta$. Taking $s(t) = t$, we get $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ continuous with $s(0) = 0$ such that

$$\|T(t)x - T(s(t))\alpha x\| = \|e^{\lambda t}x - e^{\lambda t}\alpha x\| = e^{\mathrm{Re}(\lambda)t} \cdot |1 - \alpha| \cdot \|x\| < \delta\|x\| \quad \text{for all } t \geq 0.$$

It would follow that $T(t_*)x = \alpha x$ or $T(t_*)\alpha x = x$ and so $e^{\lambda t_*}x = \alpha x$ or $e^{\lambda t_*}\alpha x = x$ for some $0 \leq t_* \leq 1$. If $t_* = 0$, then $\alpha = 1$ contradicting $\alpha \neq 1$ and if $t_* > 0$, we would have $1 = |\alpha| = |e^{\lambda t_*}| = e^{\mathrm{Re}(\lambda)t_*} < 1$ or $1 = |\alpha e^{\lambda t_*}| = e^{\mathrm{Re}(\lambda)t_*} < 1$, again a contradiction.

If $\mathrm{Re}(\lambda) = 0$, we take $\alpha \in \mathbb{C}$ with $|\alpha| > 1$ such that $|1 - \alpha| < \delta$. As before, we take $s(t) = t$ to get

$$\|T(t)x - T(s(t))\alpha x\| = \|e^{\lambda t}x - e^{\lambda t}\alpha x\| = |1 - \alpha|\|x\| < \delta\|x\| \quad \text{for all } t \geq 0.$$

Hence, $e^{\lambda t_*}x = \alpha x$ or $e^{\lambda t_*}\alpha x = x$ for some $0 \leq t_* \leq 1$. In any case, we would have $|\alpha| = 1$, again a contradiction. Therefore, $\mathrm{Re}(\lambda) > 0$ for every $\lambda \in \sigma_p(A)$.

Now we prove that T cannot be uniformly bounded. Suppose by contradiction that it is.

We claim that $T(t)x = x$ for every $x \in X$ and $t \geq 0$. Otherwise, apply Lemma 2.2 to get $x \in X \setminus \{0\}$ and $\Delta > 0$ such that $t \mapsto T(t)$ is injective in $[0, \Delta]$. Since X has dimension ≥ 2, the curve $T([0, \Delta])x = \{T(t)x : 0 \leq t \leq \Delta\}$ is nowhere dense by invariance of domain.

Now take $\delta > 0$ from the positive expansiveness of T for $\epsilon = \Delta/2$. Let M from the uniform boundedness of T, $y \in X$ with $\|x - y\| \leq (\delta/M)\|x\|$, and $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be defined by $s(t) = t$ for $t \geq 0$. It follows that S is continuous, $S(0) = 0$, and

$$\|T(t)x - T(s(t)y\| = \|T(t)x - T(t)y\| \leq \|T(t)\|\,\|x - y\| \leq M\,\frac{\delta}{M}\|x\| = \delta\|x\|$$

for every $t \geq 0$. Then, $T(t_*)x = y$ or $T(t_*)y = x$ for some $0 \leq t_* \leq \epsilon$. However, $B(x(\delta/M)\|x\|) \cap T([0, \Delta])x$ is nowhere dense so there are $0 < t_a \leq \epsilon$, sequence $y_k \in B(x, (\delta/M)\|x\|)$, and a sequence $0 \leq t_*^k \leq \epsilon$ satisfying $T(t_k^*)y_k = x$ for all $k \in \mathbb{N}$ and $y_k \to T(t_a)x$ as $k \to \infty$. By compactness, we can assume that $t_*^k \to \hat{t}$ for some

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$0 \leq \hat{t} \leq \epsilon$. Then,

$$\begin{aligned}\|T(\hat{t}+t_a)x - x\| &\leq \|T(\hat{t})T(t_a)x - T(t_k^*)T(t_a)x\| \\ &\quad + \|T(t_k^*)T(t_a)x - T(t_k^*)y_k\| + \|T(t_k^*)y_k - x\| \\ &\leq \|T(\hat{t})T(t_a)x - T(t_k^*)T(t_a)x\| + M\|T(t_a)x - y_k\| \\ &\quad + \|T(t_k^*)y_k - x\| \to 0 \quad \text{as } k \to \infty,\end{aligned}$$

concluding that $T(\hat{t}+t_a)x = x$. Since $0 < \hat{t}+t_a \leq 2\epsilon = \Delta$ and $t \mapsto T(t)x$ is injective in $[0, \Delta]$, we get a contradiction. It follows that $T(t)x = x$ for every $t \geq 0$ and $x \in X$, and so T cannot be expansive. This contradiction proves that T cannot be uniformly bounded and so it cannot be equicontinuous too by Lemma 2.3. Proposition 7.18 in [**6**, p. 191] implies that T cannot be chaotic. This completes the proof of item (1). □

Proof of item (2) of Theorem 1.3. This item is a direct consequence of Lemmas 2.5 and 2.6. □

4. *An application*

In this section, we give a short application of our main result. The *Lasota* (or van Foerster–Lasota) *equation* is the linear partial differential equation of the first order,

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \lambda u, \quad t \geq 0,\ x \geq 0, \tag{4.1}$$

with initial condition

$$u(0, x) = v(x) \quad \text{for all } x \geq 0.$$

Here, $\lambda \in \mathbb{R}$ is a real parameter and v belongs to a certain space of functions defined on $\{x \geq 0\}$. It has been used as a mathematical description of the red blood cells population. Its multidimensional version has been used to model the structured, segregated model of microbial growth [**3**, **7**].

The solution of equation (4.1) generates the semigroup

$$T(t)v(x) = e^{\lambda t}v(xe^{-t}) \quad \text{for all } t \geq 0.$$

Here we will consider this semigroup in $C_b[0, \infty[$, the space of continuous bounded functions $f : [0, \infty[\to \mathbb{R}$ endowed with the supremum norm. It is clear that T is strongly continuous in the space. We get the following facts.

- If $\lambda < 0$, then 0 is a global attractor of T.
- If $\lambda = 0$, the semigroup T is uniformly bounded and so it cannot be expansive (by item (1) of Theorem 1.3).

Now we deal with the case $\lambda > 0$.

THEOREM 4.1. *The semigroup T of $C_b[0, \infty[$ derived from the Lasota equation with parameter $\lambda > 0$ is positively expansive.*

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Proof. Given $v \in C_b[0, \infty[$, define

$$S(t)v(x) = e^{-\lambda t}v(xe^t) \quad \text{for all } x, t \geq 0.$$

Since

$$(S(t) \circ T(t))v(x) = e^{-\lambda t}e^{\lambda t}v(xe^{-t}e^t) = v(x)$$

and likewise $(T(t) \circ S(t))v(x) = v(x)$ for every $v \in C_b[0, \infty[$ and $x, t \geq 0$, we have that $(T(t))^{-1} = S(t)$.

Moreover,

$$|(T(t))^{-1}v(x)| = e^{-\lambda t}|v(xe^t)| \leq e^{-\lambda t}\|v\|,$$

so $\|(T(t))^{-1}\| \leq e^{-\lambda t}$ for every $t \geq 0$. Then, T is positively expansive by Lemma 2.6. □

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