ON THE CONSTRUCTION OF THE KUROSH LOWER RADICAL CLASS FOR ASSOCIATIVE RINGS

DARYL KREILING and TERRY L. JENKINS

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All rings considered are to be associative. For definitions not included in this paper see [2].

Let R be a ring and S a subring of R. We say that S is accessible through zero extensions if there exists

$$S = A_0 \lhd A_1 \lhd \cdots \lhd A_n = R$$

such that $(A_i/A_{i-1})^2 = (0)$, $i = 1, 2, \cdots n$.

We require the following lemma due to S. E. Dickson.

LEMMA. [1, p. 447]. *Let R_1 be a homomorphically closed class of rings containing the zero rings and having the additional property that if $I \in R_1$ is an ideal of A with $(A/I)^2 = (0)$, it follows that $A \in R_1$ (i.e., R_1 is closed under extensions by zero rings), then $R_2 = \{A|$ each non-zero homomorphic image of A contains a nonzero idea from $R_1\}$ is the lower radical class containing R_1.*

Let P be a non-empty, homomorphically closed class of rings. Define the class P_1 by,

$P_1 = \{A|\ A$ has a subring S accessible through zero extensions, where $S \in P\}$.

THEOREM 1. *With P_1 as defined above P_1 is homomorphically closed, contains the class of all zero rings and is closed under extensions by zero rings.*

PROOF. Let $A \in P_1$ and let ϕ be any homomorphism of A. Then there exists

$$A_0 \lhd A_1 \lhd \cdots \lhd A_n = A$$

where $A_0 \in P$ and $(A_i/A_{i-1})^2 = (0)$. Hence,

$$A_0\phi \lhd A_1\phi \lhd \cdots \lhd A_n\phi = A\phi,$$

$A_0\phi \in P$ and $(A_i\phi/A_{i-1}\phi)^2 = (0)$ so $A\phi \in P_1$ and P_1 is homomorphically closed.

If A is a zero ring, $(0) \lhd A$, $(0) \in P$ and $(A/(0))^2 = (0)$ so $A \in P_1$.

Finally, let A be a ring such that there exists $I \lhd A$, $I \in P_1$ and $(A/I)^2 = (0)$. Since $I \in P_1$ there exists

$$I_0 \lhd I_1 \lhd \cdots \lhd I_n = I \lhd A,$$

$I_0 \in P$, $(I_i/I_{i-1})^2 = (0)$, $i = 1, 2, \cdots n$ and $(A/I)^2 = (0)$. Hence, $A \in P_1$.

Let $L(P)$ denote the Kurosh lower radical class determined by P. Define $P_2 = \{A|$ each nonzero homomorphic image of A contains a nonzero ideal from $P_1.\}$

COROLLARY 1. *P_2 is a radical class containing P and $L(P) \subseteq P_2$.*

PROOF. That P_2 is a radical class follows from the Lemma and Theorem 1. Since $L(P)$ is the smallest radical class containing P, $L(P) \subseteq P_2$.

COROLLARY 2. *If $L(P)$ (in particular if P) contains the class of zero rings, then $P_2 = L(P)$.*

PROOF. Since $L(P)$ contains all zero rings and is closed under extensions ($I \in L(P)$, $R/I \in L(P)$ implies $R \in L(P)$) [4, p. 13], we have that $P_1 \subseteq L(P)$. Since $L(P)$ is a radical class $P_2 \subseteq L(P)$. Hence, $P_2 = L(P)$.

REMARK. It is not true, in general, that $P_2 = L(P)$. For example, let P be the class of nontrivial simple rings plus the zero ring. Then $L(P)$ contains no simple zero rings but P_2 contains all zero rings. Hence in this case $L(P) \neq P_2$.

THEOREM 2. *If the class P is hereditary then P_2 is hereditary.*

PROOF. Let $A \in P_1$ and let I be an ideal of A. Since $A \in P_1$ there exists

$$A_0 \lhd A_1 \lhd \cdots \lhd A_n = A,$$

where $A_0 \in P$ and $(A_i/A_{i-1})^2 = (0)$. Now

$$(A_0 \cap I) \lhd (A_1 \cap I) \lhd \cdots \lhd (A_n \cap I) = I.$$

Since P is hereditary and since $(A_0 \cap I) \lhd A_0$, $A_0 \cap I \in P$.

Moreover,

$$((A_i \cap I)/(A_{i-1} \cap I))^2 = (0)$$

so $I \in P_1$. Thus P_1 is hereditary. Leavitt [3, p. 29] has shown that if P_1 is hereditary, then P_2 is hereditary.

REMARK. It is known that a radical class of associative rings contains all zero rings if and only if it contains all nilpotent rings [4, p. 18]. Since the lower Baer radical class, B, is the smallest radical class containing the nilpotent rings [2, p. 59], we have that $B \subseteq P_2$.

THEOREM 3. *If P contains no complete matrix rings over division rings then P_2 contains none. Hence, if A is a ring with descending chain condition (d.c.c.) on left ideals and $W(A)$ denotes the classical Wedderburn radical of A, then $W(A) = P_2(A)$.*

PROOF. If P contains no complete matrix ring over a division ring, then from the definition of P_1 it is clear that P_1 contains none. From the definition of P_2 it is then clear that P_2 contains no complete matrix rings over division rings. It then follows from [4, p. 19] that if A is a ring with d.c.c. on left ideals, then $W(A) = P_2(A)$.

References

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[2] N. J. Divinsky, *Rings and Radicals* (University of Toronto Press, Toronto, 1965).
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[4] Terry L. Jenkins, *The Theory of Radicals and Radical Rings* (Doctoral Dissertation, University of Nebraska, 1965).

Western Illinois University
University of Wyoming