SOME PRECISIONS ON THE FOURIER–BOREL TRANSFORM AND INFINITE ORDER DIFFERENTIAL EQUATIONS

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Let $f(z)$ be an entire function (of several variables). We define the function

$$M_f(r) = \max_{\|z\|=r} |f(z)|,$$

which is increasing. The *order* of $f(z)$ is the constant (perhaps infinite)

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log r}.$$

If $\rho < +\infty$, we define a proximate order as a function $\rho(r)$ such that

(1) $\quad \rho(r) \to \rho \quad \text{and} \quad \rho'(r) r \log r \to 0 \quad \text{as} \quad r \to \infty.$

We can also assume the additional condition

(2) $\quad \rho''(r) r^2 \log r \to 0 \quad \text{as} \quad r \to \infty.$

If $L(r) = r^{\rho(r)-\rho}$, then we have

(3) $\quad \lim_{r\to\infty} \frac{L(kr)}{L(r)} = 1 \quad \text{uniformly for} \quad 0 < a \leqq k \leqq b < +\infty.$

We define the *type* of $f(z)$ with respect to $\rho(r)$ by

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log M_f(r)}{r^{\rho(r)}}$$

and $f(z)$ is said to be of (a) *minimal*, (b) *normal*, or (c) *maximal* type if (a) $\sigma = 0$, (b) $0 < \sigma < +\infty$, or (c) $\sigma = +\infty$, respectively. For every function $f(z)$ of order ρ, there exists a proximate order $\rho(r)$ such that $f(z)$ has normal type with respect to $\rho(r)$ [4].

If $\rho > 1$, we assume that $\rho(r) > 1$ and $(d/dr)(r^{\rho(r)-1}) > 0$ for all r. Since this holds eventually, this assumption involves no loss of generality. Then the equation $t = r^{\rho(r)-1}$ has a unique solution r for all $t \geqq 0$. We define the *dual* proximate order $\rho^*(t)$ by $\rho^*(t) = \rho(r)/(\rho(r)-1)$, where r is this unique solution. It is an easy calculation to check that $\rho^*(t)$ satisfies (1) and (2) and that $\rho^{**}(r) = \rho(r)$.

For a real-valued continuous function $q(z)$, we define the Banach space B_q to be the set of entire functions such that

$$|f(z)\exp(-q(z))| \to 0 \quad \text{as} \quad \|z\| \to \infty,$$

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where the norm in B_q is the sup norm. If $q_n(z)$ is a decreasing (resp. increasing) sequence of functions, we define the set $F = \bigcap B_{q_n}$ (resp. $E = \bigcup B_{q_n}$), which we equip with the projective limit (resp. inductive limit) topology. We designate the dual space of continuous linear functionals by F' (resp. E'). F is a Fréchet space, and E' can be given the topology of a Fréchet space under the norm topologies

$$\|v\|_n = \sup_{\substack{f \in B_{q_n} \\ \|f\|_n = 1}} |v(f)|.$$

In particular, let $p(z)$ be a pseudonorm (i.e., $p(z_1+z_2) \leqq p(z_1)+p(z_2)$, $p(tz) = tp(z)$ for $t \geqq 0$) and let $p_n(z) = p(z)+\|z\|/n$ with $p_n'(u) = \sup_{p_n(z) \leqq 1} \operatorname{Re}\langle u, z\rangle$ be the dual norm. We designate by $F_p^{\rho(r)}$ (resp. $E_{p'}^{\rho(r)}$) the space we get by taking $q_n(z) = (p_n(z))^{\rho(\|z\|)}$ (resp. $q_n(z) = (p_n'(z))^{\rho(\|z\|)}$). It is clear from (3) that we can replace $q_n(z)$ by $q_n'(z) = (p_n(z))^{\rho(p_n(z))}$ (resp. $(p_n'(z))^{\rho(p_n'(z))}$) and still obtain the same topological vector spaces. For $v \in (F_p^{\rho(r)})'$ (resp. $(E_{p'}^{\rho(r)})'$), we define $f_v(u)$ by

(4) $\quad f_v(u) = v(\exp\langle u, z\rangle),$

which is an entire function of u, called the Fourier–Borel transform of v.

In [7], A. Martineau showed that, if $\rho > 1$ is a constant and $p(z)$ is a complex norm (i.e., $p(\lambda z) = |\lambda| p(z)$), then the Fourier–Borel transform establishes an isomorphism between the spaces $(F_p^\rho)'$ and $E_{\tau p'}^{\rho^*}$ (for a suitable constant τ). He introduced the notion of a constant coefficient differential operator of infinite order $\breve{\alpha}$, and, using the Fourier–Borel transform, showed that, for every $f \in F_p^\rho$, the equation $\breve{\alpha}(x) = f$ has a solution $g \in F_p^\rho$. In [**1**], the author extended this result to the case of complex pseudonorms and proximate orders (as well as to the case $\rho < 1$).

We shall be primarily interested here in showing that the isomorphism proven by Martineau is still valid for arbitrary pseudonorms (not necessarily complex), for $\rho > 1$, and for all proximate orders. It is then a simple matter to apply his reasoning to the case of differential equations of infinite order in order to get a more precise estimate of the growth of solutions.

Before turning to the main theorems, we first collect some results which we shall need later.

PROPOSITION 1. *Let E, F be two Fréchet spaces and β a continuous linear map of E into F. The two following statements are equivalent*:

(i) *β is onto.*
(ii) *${}^t\beta : F' \to E'$ (the transpose map) is one-to-one and its image ${}^t\beta(F')$ is weakly closed in E'.*

Proof. See [**8**].

PROPOSITION 2. *Every element of the dual space of $F_p^{\rho(r)}$ can be represented by integration with respect to a measure μ such that*

(5) $\mu \cdot \exp(p_n(z))^{\rho(\|z\|)}$ *is a bounded measure for some n.*

Every element of the dual space of $E_{p'}^{\rho(r)}$ *can be represented by integration with respect to a measure* μ *such that*

(6) $\mu \cdot \exp(p'_n(z))^{\rho(\|z\|)}$ *is a bounded measure for all n.*

(The representations are not unique.)

Proof. The proof can be found in [7].

COROLLARY. *If we equip* $(E_{p'}^{\rho(r)})'$ *with its Fréchet space topology, then* $((E_{p'}^{\rho(r)})')' = E_{p'}^{\rho(r)}$.

Proof. The dual space is clearly a family of functions containing $E_{p'}^{\rho(r)}$, and, by considering the Dirac measures associated with every point, it is clear that every function $h(z)$ in the dual satisfies the condition

$$\sup_z |h(z)\exp(-q_n(z))| \leqq M < +\infty$$

for n sufficiently large. Thus it remains to show that $h(z)$ is holomorphic.

For a given complex line $u\lambda$ through the point z, we let γ be a rectifiable closed compact curve in $u\lambda$ and α represent integration around γ. Then $\alpha(f) = 0$ for every $f \in E_{p'}^{\rho(r)}$; so $\alpha(h) = 0$ for every $h \in ((E_{p'}^{\rho(r)})')'$. Thus h is holomorphic in every complex line through z and hence holomorphic in $\mathbb{C}^n$.

LEMMA. *Let* $\rho(r)$ *be a proximate order, with* $\rho > 1$. *If* $\eta(r)$ *is a nonnegative function such that* $\lim_{r\to\infty} \eta(r) r^{-\rho(r)} = 0$, *there exists a positive function* $\xi(r)$ *with nonnegative first and second derivatives such that* $\xi(r) \geqq \eta(r)$ *and* $\lim_{r\to\infty} \xi(r) r^{-\rho(r)} = 0$.

Proof. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers approaching zero and $\{r_n\}$ a sequence of numbers such that $\eta(r) \leqq \varepsilon_{n+1} r^{\rho(r)}$ for $r \geqq r_n$. We assume, without loss of generality, that both $dr^{\rho(r)}/dr$ and $d^2 r^{\rho(r)}/dr^2$ are everywhere positive (by (1) and (2), this holds eventually).

We construct a function $\xi_1(r)$ to be piecewise linear. The construction will be carried out by induction. For $n = 1$, we choose for $\xi_1(r)$ a constant such that $\xi_1(r) = \max_{r \leqq r_2} (\eta(r), \varepsilon_1 r^{\rho(r)})$. Having constructed $\xi_1(r)$ for $r \leqq r_n$ with the property that $\xi_1(r) \geqq \varepsilon_{n-1} r^{\rho(r)}$ for $r_{n-1} \leqq r \leqq r_n$, we construct $\xi_1(r)$ for $r_n \leqq r \leqq r_{n+1}$. We continue $\xi_1(r)$ linearly unless there exists an R_n, with $r_n \leqq R_n \leqq r_{n+1}$, such that $\xi_1(R_n) = \varepsilon_{n-1} R_n^{\rho(R_n)}$. If this occurs, we continue $\xi_1(r)$ past R_n by taking $\delta > 0$ and taking the tangent to the curve $\varepsilon_{n-1} r^{\rho(r)}$ at R_n; at $R_n + q\delta$, for q an integer, we extend this continuation as a continuous function by making a linear extension with slope $(d/dr)\{\varepsilon_{n-1} r^{\rho(r)}\}|_{R_n+q\delta}$. By choosing δ sufficiently small, we shall have $\xi_1(r) \geqq \varepsilon_n r^{\rho(r)}$ in the interval $r_n \leqq r \leqq r_{n+1}$. This establishes the induction. Furthermore, it is clear that $\xi_1(r) \geqq \eta(r)$, and that, given n, for r sufficiently large, $\xi_1(r) \leqq \varepsilon_n r^{\rho(r)}$.

Let $\alpha(r)$ be a nonnegative C^∞ function with compact support depending only on $|r|$, such that $\int \alpha(r)\,dr = 1$. Then $\xi(r) = \int \xi_1(r')\alpha(r-r')\,dr'$ satisfies the requirements of the lemma.

THEOREM 1. *The Fourier–Borel transform given by* (4) *establishes an isomorphism between the spaces*

(i) $(F_p^{\rho(r)})'$ *and* $E_{\tau p'}^{\rho^*(r)}$,

and between the spaces

(ii) $(E_{p'}^{\rho(r)})'$ *and* $F_{\tau p}^{\rho^*(r)}$,

where

$$\tau = \frac{\rho}{(\rho-1)^{(\rho-1)/\rho}}.$$

Proof. Let $v \in (F_p^{\rho(r)})'$. Then, by Proposition 2, there exists an n such that

$$|v(f)| \leqq C_v \sup_z |f(z)\exp(-p_n(z))^{\rho(p_n(z))}|,$$

for some constant C_v. Thus

$$\begin{aligned} |f_v(u)| &\leqq C_v \sup_z |\exp\langle u, z\rangle - (p_n(z))^{\rho(p_n(z))}| \\ &\leqq C_v \exp(\sup_{t \geqq 0}(\sup_{p_n(z)=1}\{\mathrm{Re}\langle u, z\rangle\}t - t^{\rho(t)})) \\ &\leqq C_v \exp(\sup_{t\geqq 0} p_n'(u)t - t^{\rho(t)}). \end{aligned}$$

Now

$$\frac{d}{dt}(p_n'(u)t - t^{\rho(t)}) = p_n'(u) - \left(\rho'(t)\log t + \frac{\rho(t)}{t}\right)t^{\rho(t)}.$$

It follows from (1) that, for large values of $\|u\|$, this function takes on an absolute maximum.

For arbitrary $\delta > 0$, it follows from (1) that, for $\|u\|$ sufficiently large (depending on δ), the maximum occurs at $t_u^{\rho(t_u)-1} = p_n'(u)/(\rho+\xi(u))$ for $|\xi(u)| < \delta$ and equals

$$p_n'(u)^{\rho(t_u)/(\rho(t_u)-1)}\left\{\left(\frac{1}{\rho+\xi(u)}\right)^{1/(\rho(t_u)-1)} - \left(\frac{1}{\rho+\xi(u)}\right)^{\rho(t_u)/(\rho(t_u)-1)}\right\},$$

which is less than or equal to

$$p_n'(u)^{\rho^*(k(u)p_n'(u))}(\tau+\varepsilon),$$

where $\varepsilon \to 0$ as $\delta \to 0$ and $0 < a \leqq k(u) \leqq b < \infty$. So the maximum for large $\|u\|$ is less than $\tau p_{n+1}'(u)^{\rho^*(p_{n}'_{+1}(u))}$ by (3). Thus the mapping $v \to f_v$ of $(F_p^{\rho(r)})'$ is into $E_{\tau p'}^{\rho^*(r)}$. Similarly, one shows that the mapping $v \to f_v$ of $(E_{p'}^{\rho(r)})'$ is into $F_{\tau p}^{\rho^*(r)}$.

To show that the mappings are onto, we use an adaptation of an argument of Hörmander [2, p. 100]. Let $K = \{z : \mathrm{Re}\langle u, z\rangle \leqq p(u), u \in \mathbb{C}^N\}$. We let x, a $2N$-tuple, represent the real coordinates of z. We define

$$\phi(v) = \sup_{x\in K}(x_1 \,\mathrm{Im}\, v_1 + \ldots + x_{2N}\,\mathrm{Im}\, v_{2N}),$$

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which is plurisubharmonic in the variable v. Then $\theta(v) = (\phi(v))^{\rho(\phi(v))}$ is also plurisubharmonic for $\phi(v)$ sufficiently large, for, given any complex line (which we assume, without loss of generality, to be the line $\lambda(v_1, 0, \ldots, 0)$), we have

$$\frac{d^2\theta(v)}{dv_1\, d\bar{v}_1} = \left(\rho''(\phi)\log\phi + 2\frac{\rho'(\phi)}{\phi} - \frac{\rho(\phi)}{\phi^2} + \left[\rho'(\phi)\log\phi + \frac{\rho(\phi)}{\phi}\right]^2\right)\phi^{\rho(\phi)}\frac{d\phi}{dv_1}\frac{d\phi}{d\bar{v}_1} + \left(\rho'(\phi)\log\phi + \frac{\rho(\phi)}{\phi}\right)\phi^{\rho(\phi)}\frac{d^2\phi}{dv_1 d\bar{v}_1}.$$

By adjusting $\rho(r)$ on a bounded set of r, if necessary, we may assume that $\theta(v)$ is everywhere plurisubharmonic.

Let $F(u) \in F_p^{\rho(r)}$, and let $\eta(r) = \sup_{\|u\|=r} (\sup(\log|F(u)| - p(u)^{\rho(p(u))}, 0))$. For $\varepsilon > 0$, we let $p_\varepsilon(u) = \sup(p(u), \varepsilon\|u\|)$. Then, since $p_\varepsilon(u)$ is continuous and

$$\overline{\lim_{r\to\infty}} \frac{\log|F(ru)|}{r^{\rho(r)}} \leqq p_\varepsilon(u)^\rho,$$

it follows from Hartog's Theorem applied to plurisubharmonic functions (cf. [**5**, **6**, **3**, Corollary to Theorem 5.4.1]) that, for the compact set $\|u\| = 1$,

$$\frac{\log|F(ru)|}{r^{\rho(r)}} \leqq p_\varepsilon(u)^\rho + \varepsilon$$

for r sufficiently large. This implies, by (3), that $\log|F(z)| \leqq p_\varepsilon(z)^{\rho(p_\varepsilon(z))} + \varepsilon\|z\|^{\rho(\|z\|)}$ for $\|z\|$ sufficiently large, which in turn implies that $\lim_{r\to\infty} \eta(r)r^{-\rho(r)} = 0$. Thus, by the lemma, there exists a positive function $\xi(r)$ with nonnegative first and second derivatives such that $\lim_{r\to\infty} \xi(r)r^{-\rho(r)} = 0$ and $\xi(r) \geqq \eta(r)$. Let $\phi^*(v) = \sup_{\|x\|\leqq 1} (x_1 \operatorname{Im} v_1 + \ldots + x_{2N} \operatorname{Im} v_{2N})$ and $\xi(\phi^*(v)) = \xi^*(v)$, which is plurisubharmonic.

Let Σ be the N-dimensional subspace, $v = (iu_1, -u_1, \ldots, iu_N, -u_N)$ and w be the function $(iu_1, -u_1, \ldots, iu_N, -u_N) \to F(u_1, \ldots, u_N)$. Then $|w(v)| \leqq C_0 \exp(\theta(v) + \xi^*(v))$ on Σ. Thus, if $\theta'(v) = \theta(v) + \xi^*(v) + \log(1 + \|v\|^2)^N$, then $\int_\Sigma |w(v)|^2 \exp(-2\theta'(v))\, d\sigma(v) < \infty$, where $d\sigma$ indicates the Lebesgue measure.

By a modification of the proof of Theorem 4.4.3 of [**2**] (due to A. Martineau; cf. [**5**] or [**3**, Theorem 5.3.3]), we have the following result: If ψ is a plurisubharmonic function in $\mathbb{C}^m$ and f is holomorphic in $\mathbb{C}^k$ $(k < m)$ such that $\int_{\mathbb{C}^k} |f|^2 \exp(-\psi)\, d\sigma < \infty$, then there exists g, holomorphic in $\mathbb{C}^m$, such that $g = f$ on $\mathbb{C}^k$ and $\int_{\mathbb{C}^m} |g|^2 \exp(-\psi')(1 + \|z\|^2)^{-3(m-k)}\, d\sigma < \infty$, where $\psi'(z) = \sup_{\|z'-z\|\leqq 2(m-k)} \psi(z')$.

Applying this to the present case, we can find an entire function W in $\mathbb{C}^{2N}$ such that $W = w$ on Σ and

$$\int |W(v)|^2 \exp(-2\theta''(v))(1+\|v\|)^{-6N} d\sigma(v) < \infty,$$

where $\theta''(v) = \sup_{\|v-v'\| \leqq 2N} \theta'(v)$. From this we conclude, by Schwarz's Lemma (cf. [5]), that there exists a constant C_0' such that

$$|W(v)| \leqq C_0'(1+\|v\|)^{3N} \exp \theta'''(v),$$

where $\theta'''(v) = \sup_{\|v-v'\|<2N+1} \theta'(v')$, and hence, following the same reasoning as in [2, Theorem 4.5.3], there exists a function $W'(v)$ such that $W'(v) = w(v)$ on Σ and

$$(7) \qquad |W'(v)| \leqq C_0''(1+\|v\|)^{-2N-1} \exp\left(\theta'''(v) + \varepsilon \sum_{i=1}^{2N} |\operatorname{Im} v_i|\right).$$

By the Paley–Wiener Theorem, if

$$\mu(x) = \frac{1}{(2\pi)^{2N}} \int_{R^{2N}} \exp i\langle x, v+iv'\rangle W'(v+iv')\, dv,$$

then $\mu(x)$ is continuous and independent of v', and the Fourier–Laplace transform of $\mu(x)$, $\int \exp\{-i(x_1 v_1 + \ldots + x_{2N} v_{2N})\}\mu(x)\, dx = W'(v)$; hence the Fourier–Borel transform of $\mu(x)$ is $F(u)$. In this case, it follows from (7) that

$$\mu(x) \leqq K_n \exp(\inf_u p_n(u)^{\rho(p_n(u))} - \operatorname{Re}\langle u, z\rangle)$$

and, by applying the same reasoning as above, we conclude that

$$\mu(x) \leqq K_n' \exp\{-\tau p_{n-1}'(u)^{\rho^*(p_{n-1}(u))}\}$$

for all n, which implies that $\mu(x)$ satisfies (6). Thus the map $(E_{\tau p'}^{\rho^*(r)})' \to F_p^{\rho(r)}$ is onto and, since

$$\tau^* = \frac{\rho^*}{(\rho^*-1)^{(\rho^*-1)/\rho^*}} = \frac{1}{\tau}$$

(where $1/\rho^* + 1/\rho = 1$) and $\rho^{**}(r) = \rho(r)$, $(E_{p'}^{\rho(r)})' \to F_{\tau p}^{\rho^*(r)}$ is onto. Similarly, one shows that the mapping of $(F_p^{\rho(r)})'$ into $E_{\tau p'}^{\rho^*(r)}$ given by (4) is onto.

The map $v \to f_v$ is thus a continuous mapping of the Fréchet space $(E_{p'}^{\rho(r)})'$ onto $F_{\tau p}^{\rho^*(r)}$, which implies, by Proposition 1, that the transpose map of $(F_{\tau p}^{\rho^*(r)})'$ into $E_{p'}^{\rho(r)}$ is one-to-one with closed image. In fact, we know that the map of $(F_{\tau p}^{\rho^*(r)})'$ is onto $E_{p'}^{\rho(r)}$, which implies in turn that the map of $(E_{p'}^{\rho(r)})'$ is one-to-one onto $F_{\tau p}^{\rho^*(r)}$, which establishes the desired isomorphisms.

COROLLARY. *In the space $F_p^{\rho(r)}$, the subspaces spanned by*

(i) $\exp\langle u, z\rangle$ *for* $u \in K$ *with* $\mathring{K} \neq \emptyset$,
(ii) $z^\alpha \exp\langle u_0, z\rangle$ *for all multi-indices of nonnegative integers* α,
are dense (in particular, the exponentials and the polynomials are dense).

Proof. For $v \in (F_p^{\rho(r)})'$, if $v(\exp\langle u, z\rangle) = 0$ for $u \in K$, then $f_v \equiv 0$, from which (i) follows. The function $v(z^\alpha \exp\langle u_0, z\rangle) = c_\alpha$, where c_α is the coefficient of $(u-u_0)^\alpha$ in the Taylor's series expansion of f_v at u_0. Thus, if $c_\alpha = 0$ for all α, then $f_v \equiv 0$, from which (ii) follows.

For $v \in F_p^{\rho(r)}$ such that f_v has type zero (with respect to $\rho^*(r)$), we define the convolution

$$v * \mu(f) = v_w[\mu_v(f(w+v))].$$

Then, by Theorem 1, the map $\mu \to v * \mu$ is a map of $(F_p^{\rho(r)})'$ into itself. We define a differential equation of infinite order to be

$$(\check{v}(f), \mu) = (f, v * \mu).$$

THEOREM 2. *If $v \in (F_p^{\rho(r)})'$ is such that f_v has minimal type with respect to $\rho^*(r)$, then the equation $\check{v}(x) = f$, for $f \in F_p^{\rho(r)}$, always has a solution in $F_p^{\rho(r)}$.*

Proof. The reader is referred to [7]; the proof that Martineau gives there for complex norms easily carries over to the present case.

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