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TRANSLATION GENERALIZED QUADRANGLES FOR WHICH THE TRANSLATION DUAL ARISES FROM A FLOCK

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Abstract. It is shown that each finite translation generalized quadrangle (TGQ) $\mathcal{S}$, which is the translation dual of the point-line dual of a flock generalized quadrangle, has a line $[\infty]$ each point of which is a translation point. This leads to the fact that the full group of automorphisms of $\mathcal{S}$ acts 2-transitively on the points of $[\infty]$, and the observation applies to the point-line duals of the Kantor flock generalized quadrangles, the Roman generalized quadrangles and the recently discovered Penttila-Williams generalized quadrangle. Moreover, by previous work of the author, the non-classical generalized quadrangles (GQ's) which have two distinct translation points, are *precisely* the TGQ's of which the translation dual is the point-line dual of a non-classical flock GQ.

We emphasize that, for a long time, it has been thought that every non-classical TGQ which is the translation dual of the point-line dual of a flock GQ has only one translation point. There are important consequences for the theory of *generalized ovoids* (or *eggs*) in $\mathbf{PG}(4n-1, q)$, the study of *span-symmetric generalized quadrangles*, *derivation* of flocks of the quadratic cone in $\mathbf{PG}(3, q)$, *subtended ovoids* in generalized quadrangles, and the understanding of automorphism groups of certain generalized quadrangles. Several problems on these topics will be solved completely.

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1. Introduction.

1.1. Standard preliminaries. A (finite) *generalized quadrangle* (GQ) *of order* (s, t) is an incidence structure $\mathcal{S} = (P, B, I)$ in which P and B are disjoint (non-empty) sets of objects called *points* and *lines* respectively, and for which I is a symmetric point-line incidence relation satisfying the following axioms.

1. Each point is incident with $t + 1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.
2. Each line is incident with $s + 1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.
3. If p is a point and L is a line not incident with p, then there is a unique point-line pair (q, M) such that $pIMIqIL$.

If $s = t$, then $\mathcal{S}$ is also said to be *of order* s.

Generalized quadrangles were introduced by Jaques Tits [**39**] in his celebrated work on triality. For notations and definitions not explicitly given here, see the monograph [**18**] of S. E. Payne and J. A. Thas, denoted FGQ throughout.

Let $\mathcal{S} = (P, B, I)$ be a (finite) thick generalized quadrangle of order (s, t). Then $|P| = (s+1)(st+1)$ and $|B| = (t+1)(st+1)$ [**18**]; also, $s \leq t^2$ and, dually, $t \leq s^2$.

There is a point-line duality for GQ's of order (s, t) for which in any definition or theorem the words "point" and "line", and the parameters s and t are interchanged. Normally, we assume without further notice that the dual of a given theorem or definition has also been given. If $\mathcal{S}$ is a GQ of order $(s, 1)$, then $\mathcal{S}$ is also called a *grid with parameters* $s+1, s+1$. Let p and q be (not necessarily distinct) points of the GQ $\mathcal{S}$; we write $p \sim q$ and say that p and q are *collinear*, provided that there is some line L so that $pILIq$; so $p \not\sim q$ means that p and q are not collinear. Dually, for $L, M \in B$, we write $L \sim M$ or $L \not\sim M$ according as L and M are *concurrent* or *non-concurrent*. If $p \neq q \sim p$, the line incident with both is denoted by pq, and if $L \sim M \neq L$, the point which is incident with both is sometimes denoted by $L \cap M$. For $p \in P$, we put $p^\perp = \{q \in P \parallel q \sim p\}$, and we note that $p \in p^\perp$. For a pair of distinct points $\{p, q\}$, the *trace* of $\{p, q\}$ is defined by $p^\perp \cap q^\perp$, and we denote this set by $\{p, q\}^\perp$. Then $|\{p, q\}^\perp| = s+1$ or $t+1$, according as $p \sim q$ or $p \not\sim q$. More generally, if $A \subset P$, the set $A^\perp$ is defined by $A^\perp = \bigcap\{p^\perp \parallel p \in A\}$. For $p \neq q$, the *span* of the pair $\{p, q\}$ is $\{p, q\}^{\perp\perp} = \{r \in P \parallel r \in s^\perp$ for all $s \in \{p, q\}^\perp\}$. Then $|\{p, q\}^{\perp\perp}| = s+1$ or $|\{p, q\}^{\perp\perp}| \leq t+1$ according as $p \sim q$ or $p \not\sim q$. If $p \sim q$, $p \neq q$, or if $p \not\sim q$ and $|\{p, q\}^{\perp\perp}| = t+1$, we say that the pair $\{p, q\}$ is *regular*. The point p is *regular* provided $\{p, q\}$ is regular for every $q \in P \setminus \{p\}$. One easily proves that either $s = 1$ or $t \leq s$ if $\mathcal{S}$ has a regular pair of non-collinear points, see FGQ. Regularity for lines is defined dually.

A *subquadrangle*, or also *subGQ*, $\mathcal{S}' = (P', B', I')$ of a GQ $\mathcal{S} = (P, B, I)$ of order (s, t), with $s, t > 1$, is a GQ for which $P' \subseteq P$, $B' \subseteq B$, and where I' is the restriction of I to $(P' \times B') \cup (B' \times P')$. An *ovoid* of $\mathcal{S}$ is a set of $st+1$ mutually non-collinear points. Dually, one defines *spreads*. Suppose $\mathcal{S}$ is a GQ of order (s, t), $s \neq 1 \neq t$, which contains a subGQ $\mathcal{S}'$ of order (s, t'), $t' > 1$, and let z be a point of $\mathcal{S} \setminus \mathcal{S}'$. Then we know by [**18**, 2.2.1] that z is collinear with the points of an ovoid $\mathcal{O}_z$ of $\mathcal{S}'$. We say that $\mathcal{O}_z$ is 'subtended by z', and that $\mathcal{O}_z$ is a *subtended ovoid*.

Suppose $\mathcal{S} = (P, B, I)$ is a finite generalized quadrangle of order (s, t), $s \neq 1 \neq t$, and consider a point x. Then $\mathcal{S} = \mathcal{S}^{(x)}$ is said to be a *translation generalized quadrangle* (TGQ) with *base point* x if there is an abelian group G of collineations of $\mathcal{S}$ fixing x linewise and which acts regularly on the set of points which are not collinear with x (denoted $P \setminus x^\perp$). The GQ $\mathcal{S}$ will also be denoted by $(\mathcal{S}^{(x)}, G)$. If the group G is not necessarily abelian, then $(\mathcal{S}^{(x)}, G)$ is in general called an *elation generalized quadrangle* (EGQ) with *base point* or *elation point* x. If $(\mathcal{S}^{(x)}, G)$ is a TGQ, then the group G is uniquely defined [**18**]; for EGQ's, this is not necessarily the case. Note that a collineation of a GQ which fixes a point linewise is often called a *whorl* about that point. Suppose L is a line of a GQ $\mathcal{S}$ of order (s, t), $s, t \neq 1$. A *symmetry about* L is an automorphism of the GQ which fixes every line of $\mathcal{S}$ which meets L; in the notation of FGQ, this set is written as $L^\perp$. The line L is called an *axis of symmetry* if there is a full group H of symmetries of size s about L; in such a case, if $M \in L^\perp \setminus \{L\}$, then H acts regularly on the points of M not incident with L. A point through which each line is an axis of symmetry is a *translation point*. Every line of the classical GQ's $\mathcal{Q}(4, s)$ and $\mathcal{Q}(5, s)$ arising from a nonsingular quadric of Witt index 2 in respectively $\mathbf{PG}(4, s)$ and $\mathbf{PG}(5, s)$, see FGQ, is an axis of symmetry. Finally, if x is a translation point of the GQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$, then the group G which is generated by the symmetries about the lines incident with x is abelian and acts regularly on the points of $\mathcal{S}$ which are not collinear with x. Hence $\mathcal{S}^{(x)}$ is a TGQ with base point x, see Chapter 8 of FGQ.

The converse is also true; each line through a translation point of a TGQ is an axis of symmetry.

1.2. Goal of the paper. Suppose $\mathcal{F}$ is a *flock* of the quadratic cone $\mathcal{K}$ in $\mathbf{PG}(3, q)$, that is, a partition of $\mathcal{K}$ minus its vertex in q nonsingular conics. Then there is a GQ $\mathcal{S}(\mathcal{F})^D$, with $\mathcal{S}(\mathcal{F})^D$ the point-line dual of $\mathcal{S}(\mathcal{F})$, of order (q, q^2) which is associated to $\mathcal{F}$, see Section 1.4. Suppose $\mathcal{S} = \mathcal{S}(\mathcal{F})^D$ is the GQ which arises from the flock $\mathcal{F}$, and suppose that $\mathcal{S}(\mathcal{F})^D$ is a TGQ (that is, suppose that $\mathcal{F}$ is *derived* from a *semifield flock*). Let $(\mathcal{S}(\mathcal{F})^D)^*$ be the TGQ which is the *translation dual* (for the definition, see the next section) of the TGQ $\mathcal{S}(\mathcal{F})^D$. Suppose that x is the *base point* (that is, the translation point) of $(\mathcal{S}(\mathcal{F})^D)^*$. Then there is some 'special' line $[\infty]Ix$ which is fixed by the full automorphism group of $(\mathcal{S}(\mathcal{F})^D)^*$ if the GQ is not classical, see [**19**, 3.3]. If q is even, then it was proved by Johnson [**6**] that the corresponding TGQ's $\mathcal{S}(\mathcal{F})^D$ and $(\mathcal{S}(\mathcal{F})^D)^*$ are classical.

It is the main objective in this paper to prove that, for q odd, the point x of the TGQ $(\mathcal{S}(\mathcal{F})^D)^*$ is never fixed by the automorphism group of the GQ; this observation contradicts a well-established conviction that the flag $(x, [\infty])$ is fixed by the full automorphism group of $(\mathcal{S}(\mathcal{F})^D)^*$. Our result implies that the line $[\infty]$ is a line of translation points, and hence that the automorphism group of $(\mathcal{S}(\mathcal{F})^D)^*$ acts 2-transitively on the points of $[\infty]$. So, if $yI[\infty]$ is arbitrary, then $\mathcal{S}^{(y)}$ is a TGQ with base point y.

Several important applications are deduced.

1.3. $T(n, m, q)$'s and translation duals of TGQ's. In this paragraph, we introduce the important notion of *translation dual* of a translation generalized quadrangle.

Suppose $H = \mathbf{PG}(2n + m - 1, q)$ is the finite projective $(2n + m - 1)$-space over $\mathbf{GF}(q)$, and let H be embedded in a $\mathbf{PG}(2n + m, q)$, say H'. Now define a set $\mathcal{O} = \mathcal{O}(n, m, q)$ of subspaces as follows: $\mathcal{O}$ is a set of $q^m + 1$ $(n - 1)$-dimensional subspaces of H, denoted $\mathbf{PG}(n - 1, q)^{(i)}$, every three of which generate a $\mathbf{PG}(3n - 1, q)$ and such that for every $i = 0, 1, \ldots, q^m$ there is a subspace $\mathbf{PG}(n + m - 1, q)^{(i)}$ of H of dimension $n + m - 1$, which contains $\mathbf{PG}(n - 1, q)^{(i)}$ and which is disjoint from any $\mathbf{PG}(n - 1, q)^{(j)}$ if $j \neq i$. If $\mathcal{O}$ satisfies all these conditions for $n = m$, then $\mathcal{O}$ is called a *pseudo-oval* or a *generalized oval* or an $[n - 1]$-*oval* of $\mathbf{PG}(3n - 1, q)$. A generalized oval of $\mathbf{PG}(2, q)$ is just an oval of $\mathbf{PG}(2, q)$. For $n \neq m$, $\mathcal{O}(n, m, q)$ is called a *pseudo-ovoid* or a *generalized ovoid* or an $[n - 1]$-*ovoid* or an *egg* of $\mathbf{PG}(2n + m - 1, q)$. A $[0]$-ovoid of $\mathbf{PG}(3, q)$ is just an ovoid of $\mathbf{PG}(3, q)$. The spaces $\mathbf{PG}(n + m - 1, q)^{(i)}$ are the *tangent spaces* of $\mathcal{O}(n, m, q)$, or just the *tangents*. Sometimes we will also use the terms "egg" or "generalized ovoid" for the case $n = m$. Generalized ovoids were introduced for the case $n = m$ by J. A. Thas in [**23**], and generalized by S. E. Payne and J. A. Thas in FGQ, Chapter 8. Then S. E. Payne and J. A. Thas prove in [**24**, **18**] that from any egg $\mathcal{O}(n, m, q)$ there arises a GQ $T(n, m, q) = T(\mathcal{O})$ which is a TGQ of order (q^n, q^m), for some special point (∞). This goes as follows.

The points are of three types.

1. A symbol (∞).
2. The subspaces $\mathbf{PG}(n + m, q)$ of H' which intersect H in a $\mathbf{PG}(n + m - 1, q)^{(i)}$.
3. The points of $H' \setminus H$.

The lines are of two types.

1. The elements of the egg $\mathcal{O}(n, m, q)$.

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2. The subspaces $\mathbf{PG}(n, q)$ of $\mathbf{PG}(2n+m, q)$ which intersect H in an element of the egg.

Incidence is defined as follows: the point (∞) is *incident* with all the lines of Type (1) and with no other lines; a point of Type (2) is *incident* with the unique line of Type (1) contained in it and with all the lines of Type (2) that it contains (as subspaces), and finally, a point of Type (3) is *incident* with the lines of Type (2) that contain it.

Conversely, any TGQ can be seen in this way, that is, as a $T(n, m, q)$ associated to an egg $\mathcal{O}(n, m, q)$ in $\mathbf{PG}(2n+m-1, q)$. Hence, *the study of translation generalized quadrangles is equivalent to the study of generalized ovoids*.

For a TGQ of order (s, t), there are natural q, k and n, where k is odd and q is a prime power, so that either $s = t = q^n$ or $s = q^{nk}$ and $t = q^{n(k+1)}$, and if q is even, then $k = 1$ [**18**]. Each TGQ $\mathcal{S}$ of order $(s, s^{\frac{k+1}{k}})$ (where $s = q^n$ for some prime power q), with translation point (∞), where k is odd and $s \neq 1$, has a *kernel* $\mathbb{K}$, which is a field with a multiplicative group isomorphic to the group of all collineations of $\mathcal{S}$ fixing the point (∞), and any given point not collinear with (∞), linewise. We have $|\mathbb{K}| \leq s$, see FGQ. The field $\mathbf{GF}(q)$ is a subfield of $\mathbb{K}$ if and only if $\mathcal{S}$ is of type $T(n, m, q)$ [**18**]. The TGQ $\mathcal{S}$ is isomorphic to a $T_3(\mathcal{O})$ of Tits with $\mathcal{O}$ an ovoid of $\mathbf{PG}(3, s)$ if and only if $|\mathbb{K}| = s$. Completely similar remarks can be made for the case $s = t$, and in that case, the TGQ is isomorphic to a $T_2(\mathcal{O})$ of Tits with $\mathcal{O}$ an oval of $\mathbf{PG}(2, s)$ if and only if $|\mathbb{K}| = s$.

If $n \neq m$, then by 8.7.2 of [**18**] the $q^m + 1$ tangent spaces of $\mathcal{O}(n, m, q)$ form an egg $\mathcal{O}^*(n, m, q)$ in the dual space of $\mathbf{PG}(2n+m-1, q)$. So in addition to $T(n, m, q)$ there arises a TGQ $T(\mathcal{O}^*)$, also denoted $T^*(n, m, q)$, or $T^*(\mathcal{O})$. The TGQ $T^*(\mathcal{O})$ is called the *translation dual* of the TGQ $T(\mathcal{O})$. The GQ's $T_3(\mathcal{O})$ and $\mathcal{S}(\mathcal{F})^D$, where $\mathcal{F}$ is a Kantor flock, see Section 4, are the only known TGQ's of order (q^n, q^m), $n \neq m$, which are isomorphic to their translation dual (and probably they are the only such ones). The TGQ $T(\mathcal{O})$ and its translation dual $T(\mathcal{O}^*)$ have isomorphic kernels, see e.g. [**35**] for a proof.

A TGQ $T(\mathcal{O})$ with $t = s^2$, $s = q^n$, is called *good* at an element $\pi \in \mathcal{O}$ (or is *good* at the corresponding line through (∞) of the TGQ) if for every two distinct elements π' and π'' of $\mathcal{O} \setminus \{\pi\}$ the $(3n-1)$-space $\pi\pi'\pi''$ contains exactly $q^n + 1$ elements of $\mathcal{O}$. In that case, it is easy to see that $\pi\pi'\pi''$ is skew to the other elements; use for instance [**18**, 8.7.2 (iii)]. If the egg $\mathcal{O}$ contains a good element, then the egg is called *good*.

NOTE. In what follows, if $\mathcal{S} = T(\mathcal{O})$ is a TGQ for some translation point x, then by $\mathcal{S}^*$ we will sometimes denote the translation dual $T(\mathcal{O}^*)$ of $T(\mathcal{O})$ (if it is defined).

We shall often use the notation of this section without further notice.

1.4. Flock generalized quadrangles. Suppose $(\mathcal{S}^{(p)}, G)$ is an EGQ of order (s, t), $s \neq 1 \neq t$, with elation point p and elation group G, and let q be a point of $P \setminus p^\perp$. Let $L_0, L_1, \ldots, L_t$ be the lines incident with p, and define r_i and M_i by $L_i I r_i I M_i I q$, $0 \leq i \leq t$. Put $H_i = \{\theta \in G \parallel M_i^\theta = M_i\}$, $H_i^* = \{\theta \in G \parallel r_i^\theta = r_i\}$, and $\mathcal{J} = \{H_i \parallel 0 \leq i \leq t\}$. Then $|G| = s^2 t$ and $\mathcal{J}$ is a set of $t+1$ subgroups of G, each of order s. Also, for each i, H_i^* is a subgroup of G of order st containing H_i as a subgroup. Moreover, the following two conditions are satisfied:

(**K1**) $H_i H_j \cap H_k = \mathbf{1}$ for distinct i, j and k;

(**K2**) $H_i^* \cap H_j = \mathbf{1}$ for distinct i and j.

Conversely, if G is a group of order $s^2 t$ and $\mathcal{J}$ (respectively $\mathcal{J}^*$) is a set of $t+1$ subgroups H_i (respectively H_i^*) of G of order s (respectively of order st), and if the

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conditions (K1) and (K2) are satisfied, then the H_i^* are uniquely defined by the H_i, and $(\mathcal{J}, \mathcal{J}^*)$ is said to be a *4-gonal family of type* (s, t) *in* G. Sometimes we will also say that $\mathcal{J}$ is a 4-*gonal family of type* (s, t) *in* G if this seems convenient.

Let $(\mathcal{J}, \mathcal{J}^*)$ be a 4-gonal family of type (s, t) in the group G of order s^2t, $s \neq 1 \neq t$. Define an incidence structure $\mathcal{S}(G, \mathcal{J})$ as follows.

Points of $\mathcal{S}(G, \mathcal{J})$ are of three kinds:

(i) elements of G;
(ii) right cosets H_i^*g, $g \in G$, $i \in \{0, \dots, t\}$;
(iii) a symbol (∞).

Lines are of two kinds:

(a) right cosets H_ig, $g \in G$, $i \in \{0, \dots, t\}$;
(b) symbols $[H_i]$, $i \in \{0, \dots, t\}$.

Incidence. A point g of Type (i) is incident with each line H_ig, $0 \leq i \leq t$. A point H_i^*g of Type (ii) is incident with $[H_i]$ and with each line H_ih contained in H_i^*g. The point (∞) is incident with each line $[H_i]$ of Type (b). There are no further incidences.

It is straightforward to check that the incidence structure $\mathcal{S}(G, \mathcal{J})$ is a GQ of order (s, t). Moreover, if we start with an EGQ $(\mathcal{S}^{(p)}, G)$ to obtain the family $\mathcal{J}$ as above, then we have that $(\mathcal{S}^{(p)}, G) \cong \mathcal{S}(G, \mathcal{J})$; for any $h \in G$ let us define θ_h by $g^{\theta_h} = gh$, $(H_ig)^{\theta_h} = H_igh$, $(H_i^*g)^{\theta_h} = H_i^*gh$, $[H_i]^{\theta_h} = [H_i]$, $(\infty)^{\theta_h} = (\infty)$, with $g \in G$, $H_i \in \mathcal{J}$, $H_i^* \in \mathcal{J}^*$. Then θ_h is an automorphism of $\mathcal{S}(G, \mathcal{J})$ which fixes the point (∞) and all lines of Type (b). If $G' = \{\theta_h \parallel h \in G\}$, then clearly $G' \cong G$ and G' acts regularly on the points of Type (i). Hence, a group of order s^2t admitting a 4-gonal family is an elation group of a suitable elation generalized quadrangle. This was first noted by W. M. Kantor [**7**].

Let $\mathbb{F} = \mathbf{GF}(q)$, q any prime power, and put $G = \{(\alpha, c, \beta) \parallel \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$. Define a binary operation on G by

$$(\alpha, c, \beta)(\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta\alpha'^T, \beta + \beta').$$

This makes G into a group whose centre is $C = \{(0, c, 0) \in G \parallel c \in \mathbb{F}\}$.

Let $\mathcal{C} = \{A_u \parallel u \in \mathbb{F}\}$ be a set of q distinct upper triangular (2×2)-matrices over $\mathbb{F}$. Then $\mathcal{C}$ is called a *q-clan* provided $A_u - A_r$ is *anisotropic* whenever $u \neq r$, i.e. $\alpha(A_u - A_r)\alpha^T = 0$ has only the trivial solution $\alpha = (0, 0)$. For $A_u \in \mathcal{C}$, put $K_u = A_u + A_u^T$. Let

$$A_u = \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix}, \qquad x_u, y_u, z_u, u \in \mathbb{F}.$$

For q odd, $\mathcal{C}$ is a q-clan if and only if

$$-\det(K_u - K_r) = (y_u - y_r)^2 - 4(x_u - x_r)(z_u - z_r) \tag{1}$$

is a non-square of $\mathbb{F}$ whenever $r, u \in \mathbb{F}$, $r \neq u$. For q even, $\mathcal{C}$ is a q-clan if and only if

$$y_u \neq y_r \text{ and } tr((x_u + x_r)(z_u + z_r)(y_u + y_r)^{-2}) = 1 \tag{2}$$

whenever $r, u \in \mathbb{F}$, $r \neq u$.

Now we can define a family of subgroups of G by

$$A(u) = \{(\alpha, \alpha A_u \alpha^T, \alpha K_u) \in G \parallel \alpha \in \mathbb{F}^2\},\ u \in \mathbb{F},$$

and

$$A(\infty) = \{(0, 0, \beta) \in G \parallel \beta \in \mathbb{F}^2\}.$$

Then put $\mathcal{J} = \{A(u) \parallel u \in \mathbb{F} \cup \{\infty\}\}$ and $\mathcal{J}^* = \{A^*(u) \parallel u \in \mathbb{F} \cup \{\infty\}\}$, with $A^*(u) = A(u)C$. So

$$A^*(u) = \{(\alpha, c, \alpha K_u) \in G \parallel \alpha \in \mathbb{F}^2, c \in \mathbb{F}\}, \ u \in \mathbb{F},$$

and

$$A^*(\infty) = \{(0, c, \beta) \parallel \beta \in \mathbb{F}^2, c \in \mathbb{F}\}.$$

With G, $A(u)$, $A^*(u)$, $\mathcal{J}$ and $\mathcal{J}^*$ as above, the following important theorem is a combination of results of S. E. Payne and W. M. Kantor.

THEOREM 1. (S. E. Payne [**13**], W. M. Kantor [**7**]) *The pair* $(\mathcal{J}, \mathcal{J}^*)$ *is a* 4*-gonal family for G if and only if* $\mathcal{C}$ *is a q-clan.*

Let $\mathcal{F}$ be a *flock* of the quadratic cone $\mathcal{K}$ with vertex v of $\mathbf{PG}(3, q)$, that is, a partition of $\mathcal{K} \setminus \{v\}$ into q disjoint irreducible conics.

In his celebrated paper on flocks [**25**], J. A. Thas notes that (1) and (2) are exactly the conditions for the planes of $\mathbf{PG}(3, q)$ with equation

$$x_u X_0 + z_u X_1 + y_u X_2 + X_3 = 0$$

to define a flock of the quadratic cone $\mathcal{K}$ with equation $X_0 X_1 = X_2^2$.

THEOREM 2. (J. A. Thas [**25**]) *To any flock of the quadratic cone of* $\mathbf{PG}(3, q)$ *corresponds an EGQ of order* (q^2, q).

DEFINITION. We say that a TGQ *arises from a flock* if it is the point-line dual of a flock GQ.

In 1976 it was shown independently by J. A. Thas and M. Walker [**40**] that with any flock $\mathcal{F}$ of the quadratic cone $\mathcal{K}$ of $\mathbf{PG}(3, q)$ there corresponds an affine translation plane of order q^2. The flock is called a *semifield flock* if the corresponding plane is a semifield plane. In such a case the point-line dual of the corresponding GQ $\mathcal{S}(\mathcal{F})$ is a TGQ; if $\mathcal{S}(\mathcal{F})$ is not classical, then the point (∞) of the GQ $\mathcal{S}(\mathcal{F})$ is a line of Type (b) of the TGQ, that is, an element of the corresponding generalized ovoid. To each semifield flock of the quadratic cone $\mathcal{K}$ in $\mathbf{PG}(3, q)$, q odd, there corresponds a so-called *translation ovoid* of $\mathcal{Q}(4, q)$ [**3**], and conversely, see J. A. Thas [**27**] and G. Lunardon [**12**].

2. Proof of the main theorem. Suppose that $\hat{f}$ is a biadditive and symmetric map of $\mathbb{F}^2 \times \mathbb{F}^2$ onto $\mathbb{F}$, where $\mathbb{F} = \mathbf{GF}(q)$, q odd. Suppose $\hat{g}_t$ is a map of $\mathbb{F}^2$ onto $\mathbb{F}$ so that, for all $d, t, u \in \mathbb{F}$ and $\alpha, \gamma \in \mathbb{F}^2$, the following conditions are satisfied:

1. $\hat{g}_t(\alpha + \gamma) - \hat{g}_t(\alpha) - \hat{g}_t(\gamma) = \hat{f}(t\alpha, \gamma) = \hat{f}(t\gamma, \alpha)$;
2. $\hat{g}_{t+u}(\alpha) = \hat{g}_t(\alpha) + \hat{g}_u(\alpha)$;
3. $\hat{g}_t(\gamma) = 0$, where t is nontrivial, implies that $\gamma = (0, 0)$;
4. Condition (10) of [**18**, Section 10.4] holds.

Put $G = \mathbb{F}^4 = \{(r, c, b, d) \parallel r, c, b, d \in \mathbb{F}\}$ with coordinatewise addition. Define subgroups in the following way: $B(\infty) = \{(r, 0, 0, 0) \in G \parallel r \in \mathbb{F}\}$; $B^*(\infty) = \{(r, 0, \gamma) \in$

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$G \parallel r \in \mathbb{F}, \gamma \in \mathbb{F}^2\}$. For $\gamma \in \mathbb{F}^2$, write $B(\gamma) = \{(-\hat{g}_c(\gamma), c, -c\gamma) \in G \parallel c \in \mathbb{F}\}$. Put $\gamma = (g_1, g_2) \in \mathbb{F}^2$. Define $B^*(\gamma)$ in the usual way (see [**15**, Section VI]; this will be inessential for the sequel). Then $(\mathcal{J}, \mathcal{J}^*)$ is a 4-gonal family for G [**18**], with $\mathcal{J} = \{B(\gamma) \parallel \gamma \in \mathbb{F}^2 \cup \{\infty\}\}$ and $\mathcal{J}^* = \{B^*(\gamma) \parallel \gamma \in \mathbb{F}^2 \cup \{\infty\}\}$. In the usual way, we then have a TGQ $\mathcal{S} = (\mathcal{S}^{(\infty)}, G) = \mathcal{S}(G, \mathcal{J})$ (which satisfies some additional properties, see further) of order (q, q^2). Moreover, any TGQ which is the translation dual of the point-line dual of a flock GQ can be represented in this way, see S. E. Payne [**15**] for the case of the Roman GQ's (cf. Section 4), and M. Lavrauw and T. Penttila [**11**] for the general case. Actually, we will show that the TGQ's as defined above are *precisely* the TGQ's for which the translation dual is the point-line dual of a flock GQ.

For convenience, we will work with the point-line dual $\mathcal{S}^D$ of $\mathcal{S}$. This GQ can be represented [**15**] by the 4-gonal family $\mathcal{J} = \{A(t) \parallel t \in \mathbb{F} \cup \{\infty\}\}$ in the group $H = \{(\alpha, c, \beta) \parallel \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$, where $A(t) = \{(\alpha, \hat{g}_t(\alpha), t\alpha) \parallel \alpha \in \mathbb{F}^2\}$, $A(\infty) = \{(0, 0, \beta) \parallel \beta \in \mathbb{F}^2\}$, and where the group operation of H is defined by

$$(\alpha, c, \beta)(\alpha', c', \beta') = (\alpha + \alpha', c + c' + \hat{f}(\beta, \alpha'), \beta + \beta').$$

The corresponding groups $A^*(t)$, with $t \in \mathbb{F} \cup \{\infty\}$, are defined by $A^*(t) = \{(\alpha, c, \alpha t) \parallel \alpha \in \mathbb{F}^2, c \in \mathbb{F}\}$ and $A^*(\infty) = \{(0, c, \beta) \parallel \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$. With this representation, $\mathcal{S}^D$ is a TGQ with base line $[A(\infty)]$ [**15**].

THEOREM 3. *Suppose that $\mathcal{S} = \mathcal{S}^{(\infty)}$ is a TGQ with base point (∞), which is the translation dual of the point-line dual of a flock GQ of order (q^2, q), q odd. Then there is a line $L I (\infty)$ so that every point on L is a translation point. In particular, the group of automorphisms of $\mathcal{S}$ acts 2-transitively on the points of L.*

Proof. As noted before, any TGQ $\mathcal{S}$ of order (q, q^2), q odd, which is the translation dual of the point-line dual of a flock GQ, can be represented in the aforementioned way. Dualize to obtain $\mathcal{S}^D$, and use the 4-gonal family $\mathcal{J} = \{A(t) \parallel t \in \mathbb{F} \cup \{\infty\}\}$ in the group $H = \{(\alpha, c, \beta) \parallel \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$, as above. For arbitrary $v \in \mathbb{F}$, $v \neq 0$, define a collineation θ_v of $\mathcal{S}^D$ as follows:

$$(\alpha, c, \beta) \longrightarrow (\alpha + v^{-1}\beta, c + \hat{g}_{v^{-1}}(\beta), \beta).$$

It is easily checked that θ_v is indeed a nontrivial collineation of $\mathcal{S}^D$ (first note that θ_v induces a nontrivial automorphism of H, and observe that $A(t)$ is mapped onto $A(\frac{t}{1+t/v})$ if $t \neq -v$, that $A(-v)$ is mapped onto $A(\infty)$, and that $A(\infty)$ is mapped onto $A(v)$), and θ_v fixes any point of $(A^*(0))^\perp$. An easy way to see this is the following: each point on $[A(0)]$ is fixed, the point $(\bar{0}, 0, \bar{0})$ is fixed, and since clearly the order of θ_v is p where $q = p^h$ for the prime p, at least one line through $A^*(0)$ different from $[A(0)]$ and $(\bar{0}, 0, \bar{0})A^*(0)$ is fixed. Also, any point of $(\infty)^\perp$ is well-known to be regular; see Remark 5 (iii) for an easy proof. Then apply the result of K. Thas [**34**] to obtain that θ_v indeed is a symmetry about $A^*(0)$. Hence θ_v is a nontrivial symmetry about $A^*(0)$, and $A^*(0)$ is a centre of symmetry. Since $\mathcal{S}^D$ is a TGQ with base line $[A(\infty)]$ (so every point on $[A(\infty)]$ is a centre of symmetry), there easily follows that any point of $(\infty)^\perp$ is a centre of symmetry, and hence $\mathcal{S}^D$ is a TGQ for every base line through (∞). There

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follows that there is some line L through (∞) in $\mathcal{S}^{(\infty)}$ so that each point on L is a translation point, and the theorem easily follows.

NOTE. The fact that each point of $(\infty)^{\perp}$ is regular in the proof of Theorem 3 is inessential for that proof. Without this information, it still follows that θ_v, $v \neq 0$, is a collineation of $\mathcal{S}^D$ which does not fix $[A(\infty)]$.

As a direct corollary, we obtain the following representation method for TGQ's of which the translation dual arises from a flock.

THEOREM 4. *Suppose that $\hat{f}$ is a biadditive and symmetric map of $\mathbb{F}^2 \times \mathbb{F}^2$ onto $\mathbb{F}$, where $\mathbb{F} = \mathbf{GF}(q)$, q odd. Suppose $\hat{g}_t$ is a map of $\mathbb{F}^2$ onto $\mathbb{F}$ so that, for all $d, t, u \in \mathbb{F}$ and $\alpha, \gamma \in \mathbb{F}^2$, the Conditions* (1)–(4) *are satisfied. If the GQ $\mathcal{S}$ arises from the* 4*-gonal family $(\mathcal{J}, \mathcal{J}^*)$ as above, then the TGQ $\mathcal{S}$ of order (q, q^2) is the translation dual of the point-line dual of a flock GQ of order (q^2, q), and conversely, any* (*non-classical*) *TGQ of which the translation dual arises from a flock arises in this way.*

Proof. By Theorem 3, we know that $\mathcal{S}$ has a line of translation points. The theorem now follows from the main theorem of [**37**].

REMARK 5. (i) If a GQ has non-collinear translation points, then it is well-known that it is classical, see e.g. [**38**].

(ii) In Theorem 3, we could also have stated that $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ is a TGQ which is good at some element $\pi \in \mathcal{O}$ (which corresponds to the line $LI(\infty)$). Since q is odd, by the main theorem of J. A. Thas [**28**] (which is stated there in a more general way), $\mathcal{S}^{(\infty)}$ is then the translation dual of the point-line dual of a semifield flock GQ of order (q^2, q). The converse is 'trivially' true. Below, we will therefore make no distinction between TGQ's in odd characteristic which are good at some line containing the translation point (∞), and TGQ's which are the translation dual of a TGQ of order (q, q^2) arising from a flock.

(iii) There is also a purely geometrical proof without the use of coordinates, as was noted to us by J. A. Thas, which uses recent developments in the study of nets and GQ's. We give a sketch of that proof. Suppose $\mathcal{S}^{(\infty)}$ is a TGQ of order (q, q^2) which is good at its line $LI(\infty)$. Then by J. A. Thas [**26**], there are $q^3 + q^2$ subGQ's of order q, all isomorphic to $\mathcal{Q}(4, q)$, which contain the flag $((\infty), L)$. It follows immediately that L is regular, as $\mathcal{S}^{(\infty)}$ is a TGQ. Now suppose $M \sim L \neq MImIL$. It is clear that if $N \sim L \neq N$ and $N \not\sim M$, then $\{M, N\}$ is a regular pair of lines (M and N are contained in one of the classical subGQ's). Now suppose $U \not\sim M$ is not a line of $L^{\perp}$. Consider an arbitrary point u of L different from m, and let V be the unique line of $\mathcal{S}$ for which $uIV \sim U$. Since there are $q^3 + q^2$ classical subGQ's of $\mathcal{S}$ which contain L, it follows that there is a (necessarily) unique such classical subGQ of $\mathcal{S}$ of order q which contains L, M, V and U (this is also immediate by representing $\mathcal{S}^{(\infty)}$ as $T(\mathcal{O})$, with $T(\mathcal{O})$ good at L). Hence the pair $\{M, V\}$ is regular, and M is a regular line. It follows that every point on L is coregular. Consider any such coregular point pIL. From the regular line L there arises a net $\mathcal{N}_L$, see [**18**, Chapter 1], and $\mathcal{N}_L$ is a $\mathcal{P}$-net as in [**29**] with $\mathcal{P}$ the parallel class of $\mathcal{N}_L$ defined by p, since the dual of $\mathcal{N}_L$ satisfies the Axiom of Veblen [**32**] (and so $\mathcal{N}_L$ is the dual of a H_s^3 by [**30**]). Hence by [**29**], every point on L is a translation point, and so every line of $L^{\perp}$ is an axis of symmetry.

(iv) In some sense, Theorem 3 explains the intrinsic difference between a TGQ which arises from a flock and its translation dual, if the flock is not a Kantor flock (see below).

(v) A proof of Theorem 3 is also implicitly contained in K. Thas [**38**].

3. TGQ's for which the translation dual arises from a flock, and span-symmetric generalized quadrangles. Theorem 3 implies that if $\mathcal{S}$ and L are as in Theorem 3, then every line of $L^\perp$ is an axis of symmetry, and then for every two non-concurrent lines U and V in $L^\perp$, the GQ $\mathcal{S}$ is *span-symmetric* (see below) with *base-span* $\{U, V\}^{\perp\perp}$. We start with an introduction to span-symmetric generalized quadrangles.

3.1. Span-symmetric generalized quadrangles. Suppose $\mathcal{S}$ is a GQ of order (s, t), $s \neq 1 \neq t$, and suppose L and M are distinct non-concurrent axes of symmetry. Then it is easy to see (by transitivity) that every line of $\{L, M\}^{\perp\perp}$ is an axis of symmetry, and $\mathcal{S}$ is called a *span-symmetric generalized quadrangle* (SPGQ) with *base-span* $\{L, M\}^{\perp\perp}$.

Let $\mathcal{S}$ be a span-symmetric GQ of order (s, t), $s, t \neq 1$, with base-span $\{L, M\}^{\perp\perp}$.

In this paper, we will use the following *notation*.

First of all, the base-span is denoted by $\mathcal{L}$. The group which is generated by the symmetries about the lines of $\mathcal{L}$ is G, and sometimes G is called the *base-group*. This group clearly acts 2-transitively on the lines of $\mathcal{L}$ (recall that $s > 1$), and fixes every line of $\mathcal{L}^\perp$. The set of all the points which are on lines of $\mathcal{L}$ (or $\mathcal{L}^\perp$) is denoted by Ω, and we will refer to $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$, with I' being the restriction of I to $[\Omega \times (\mathcal{L} \cup \mathcal{L}^\perp)] \cup [(\mathcal{L} \cup \mathcal{L}^\perp) \times \Omega]$, as being *the base-grid*. In W. M. Kantor [**8**]; see also K. Thas [**37**], it is shown that if $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s, t), $s \neq 1 \neq t$, then $t \in \{s, s^2\}$ and s is a power of a prime. The following result is the solution of a longstanding conjecture, and completely determines the SPGQ's of order s. It was independently obtained by W. M. Kantor [**8**] and K. Thas [**36**].

THEOREM 6. (W. M. Kantor [**8**], K. Thas [**36**]) *Let $\mathcal{S}$ be a span-symmetric generalized quadrangle of order s, where $s \neq 1$. Then $\mathcal{S}$ is classical, that is, isomorphic to $\mathcal{Q}(4, s)$.*

As a nice group-theoretical analogue, we have that a group of order $s^3 - s$ has a 4-gonal basis (see FGQ, or [**36**]) if and only if it is isomorphic to $\mathbf{SL}(2, s)$.

Finally, the following strong theorem shows that SPGQ's of order (s, s^2) always have classical subGQ's of order s. It is an important tool in the classification of GQ's which have axes of symmetry, see [**37**, **38**].

THEOREM 7. (K. Thas [**37**, **38**]) *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s, s^2), $s \neq 1$, with base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$ and base-group G. Then $\mathcal{S}$ contains $s + 1$ subquadrangles isomorphic to the classical GQ $\mathcal{Q}(4, s)$ which mutually intersect in Γ. Also, G acts semi-regularly on $\mathcal{S} \setminus \Omega$, $|G| = s^3 - s$ and $G \cong \mathbf{SL}(2, s)$.*

REMARK 8. Note that if N is the kernel of the action of G on $\mathcal{L}$, then $G/N \cong \mathbf{PSL}(2, s)$.

3.2. Generalized quadrangles with two translation points. In [**37**], we focussed on a particular class of SPGQ's of order (s, s^2), namely those having a line M for which each line of $M^\perp$ is an axis of symmetry. The following result was obtained (using Theorem 7, other work of the author, results of Blokhuis et al. [**4**], and of J. A. Thas [**28**]).

THEOREM 9. (K. Thas [**37**]) *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s, t), $s \neq 1 \neq t$, with two distinct collinear translation points. Then we have one of the following:*

(i) *$s = t$, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(4, s)$;*

(ii) *$t = s^2$, s is even, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(5, s)$;*

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(iii) *$t = s^2$, $s = q^n$ with q odd, where* $\mathbf{GF}(q)$ *is the kernel of the TGQ* $\mathcal{S} = \mathcal{S}^{(\infty)}$ *with* (∞) *an arbitrary translation point of* $\mathcal{S}$*,* $q \geq 4n^2 - 8n + 2$ *and* $\mathcal{S}$ *is the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$ *where* $\mathcal{F}$ *is a Kantor flock;*

(iv) *$t = s^2$, $s = q^n$ with q odd, where* $\mathbf{GF}(q)$ *is the kernel of the TGQ* $\mathcal{S} = \mathcal{S}^{(\infty)}$ *with* (∞) *an arbitrary translation point of* $\mathcal{S}$*,* $q < 4n^2 - 8n + 2$ *and* $\mathcal{S}$ *is the translation dual of the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$ *for some flock* $\mathcal{F}$*.*

If a thick GQ $\mathcal{S}$ *has two non-collinear translation points, then* $\mathcal{S}$ *is always of classical type, i.e. isomorphic to one of* $\mathcal{Q}(4, s)$*,* $\mathcal{Q}(5, s)$*.*

As a direct corollary of Theorem 3 and Theorem 9, we obtain the following important result, which states that the generalized quadrangles of order (s, t), $s \neq t$, with two distinct translation points are *exactly* the TGQ's of which the translation dual is the point-line dual of a flock GQ.

THEOREM 10. *A generalized quadrangle* $\mathcal{S}$ *of order* (s, t)*,* $s \neq 1 \neq t \neq s$*, has two distinct collinear translation points if and only if* $\mathcal{S}$ *is a TGQ which is the translation dual of the point-line dual of a flock GQ. In particular, if s is even, then* $\mathcal{S} \cong \mathcal{Q}(5, s)$*. If* $\mathcal{S}$ *has non-collinear translation points, then* $\mathcal{S}$ *is always of classical type.*

4. The known examples. In this section, we review the known examples of (nonclassical) TGQ's of order (s, s^2), $s > 1$, which are the translation dual of the point-line dual of a flock GQ. There are two infinite classes and there is one sporadic example. Every observation made in this paper is valid for these examples.

KANTOR GENERALIZED QUADRANGLES. Let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathbf{PG}(3, q)$, q odd. Then the q planes π_t with equation $tX_0 - mt^\sigma X_1 + X_3 = 0$, $t \in \mathbf{GF}(q)$, m a given non-square in $\mathbf{GF}(q)$ and σ a given automorphism of $\mathbf{GF}(q)$, define a flock $\mathcal{F}$ of $\mathcal{K}$; see [**25**]. All the planes π_t contain the exterior point $(0, 0, 1, 0)$ of $\mathcal{K}$. The flock is *linear*, that is, all the planes π_t contain a common line, if and only if $\sigma = \mathbf{1}$. Conversely, every nonlinear flock $\mathcal{F}$ of $\mathcal{K}$ for which the planes of the q conics share a common point, is of the type just described, see [**25**]. The corresponding GQ $\mathcal{S}(\mathcal{F})$ was first discovered by W. M. Kantor, and is called a *Kantor (flock) generalized quadrangle*. The kernel $\mathbb{K}$ is the fixed field of σ, see [**22**]. The described quadrangle is a TGQ for some base line, and the following was shown by Payne in [**15**].

THEOREM 11. (S. E. Payne [**15**]) *Suppose a TGQ* $\mathcal{S} = T(\mathcal{O})$ *is the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$*,* $\mathcal{F}$ *a Kantor flock. Then* $T(\mathcal{O})$ *is isomorphic to its translation dual* $T^*(\mathcal{O})$*.*

In the sequel, Kantor quadrangles will be assumed to be nonclassical, and hence $\sigma \neq \mathbf{1}$.

ROMAN GENERALIZED QUADRANGLES. Let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathbf{PG}(3, q)$, with $q = 3^r$ and $r > 2$. Then the q planes π_t with equation $tX_0 - (mt + m^{-t}t^9)X_1 + t^3X_2 + X_3 = 0$, $t \in \mathbf{GF}(q)$, m a given non-square in $\mathbf{GF}(q)$, define a flock $\mathcal{F}$ of $\mathcal{K}$ which is called the *Ganley flock*; see [**15**]. The corresponding GQ $\mathcal{S}(\mathcal{F})$ is a TGQ for some base line, and so the dual $\mathcal{S}(\mathcal{F})^D$ of $\mathcal{S}(\mathcal{F})$ is isomorphic to some $T(\mathcal{O})$. By [**22**], the kernel $\mathbb{K}$ is isomorphic to $\mathbf{GF}(3)$. Payne [**15**] shows that $T(\mathcal{O})$ is not isomorphic to its translation dual $T(\mathcal{O}^*)$. Also, he proves that $T(\mathcal{O}^*)$ is a TGQ which does not arise from a flock. The GQ's $T(\mathcal{O}^*)$ were called by Payne the *Roman generalized quadrangles*.

THE PENTTILA-WILLIAMS GENERALIZED QUADRANGLE. Let $q = 3^5$. The q planes π_t with equation $tX_0 + 2t^9X_1 + t^{27}X_2 + X_3 = 0$, $t \in \mathbf{GF}(q)$, define a flock of the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathbf{PG}(3, q)$. This flock was constructed by L. Bader, G. Lunardon and I. Pinneri in [**1**], starting from the Penttila-Williams ovoid of $\mathcal{Q}(4, 3^5)$ [**21**], and relying on the construction of semifield flocks from translation ovoids of $\mathcal{Q}(4, q)$ (and conversely) as described in J. A. Thas [**27**], and G. Lunardon in [**12**]. Hence the 'corresponding' GQ (that is, the translation dual of $\mathcal{S}(\mathcal{F})^D$) is therefore referred to as the *Penttila-Williams generalized quadrangle*. The kernel of the Penttila-Williams GQ is isomorphic to $\mathbf{GF}(3)$ and $\mathcal{S}(\mathcal{F})^D$ is not isomorphic to $(\mathcal{S}(\mathcal{F})^D)^*$, see e.g. [**38**].

THEOREM 12. *The automorphism groups of the dual Kantor GQ's, the Roman GQ's and the Penttila-Williams GQ act* 2*-transitively on the points incident with the line of translation points.*

Proof. This follows immediately by Theorem 3.

NOTE. For the dual Kantor GQ's, this observation was already made by S. E. Payne in [**16**].

5. Consequences of the main result.

5.1. Isomorphisms of subtended ovoids in the TGQ's $(\mathcal{S}(\mathcal{F})^D)^*$, $\mathcal{F}$ a semifield flock. Suppose that $\mathcal{S}$ is a GQ of order (s, t), $s \neq 1 \neq t$, which contains a subGQ $\mathcal{S}'$ of order (s, t'), $t' > 1$, and let z be a point of $\mathcal{S} \setminus \mathcal{S}'$. Then it is well known that z is collinear with the points of an ovoid $\mathcal{O}_z$ of $\mathcal{S}'$. See § 1.1. We say that $\mathcal{O}_z$ is *subtended by* z, and $\mathcal{O}_z$ is a *subtended ovoid*.

THEOREM 13. (M. Lavrauw [**9**, **10**]) *Let $\mathcal{O}$ be an egg of* $\mathbf{PG}(4n - 1, q)$ *which is good at the element π, q odd, and consider the TGQ $\mathcal{S}^{(\infty)} = T(\mathcal{O})$. Then all the ovoids of a fixed (arbitrary) subGQ* $\mathcal{Q}(4, q^n)$ *through the flag* $((\infty), [\infty])$ *which are subtended by a point of Type* (2) *are isomorphic translation ovoids. Moreover, these ovoids are isomorphic to the ovoid of* $\mathcal{Q}(4, q^n)$ *arising from the semifield flock which corresponds to the egg $\mathcal{O}$.*

Consider the TGQ $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ as in Theorem 13. Then there is a line $[\infty]I(\infty)$ which is a line of translation points by Theorem 3. Fix the subGQ $\mathcal{S}' = \mathcal{Q}(4, q^n)$ as before. Suppose that x is a point of Type (3) not in $\mathcal{S}'$ in the TGQ $T(\mathcal{O})$ (that is, $x \not\sim (\infty)$). Consider two non-concurrent lines U, V in $[\infty]^\perp \cap \mathcal{S}'$. Then U and V are axes of symmetry of $\mathcal{S}$, and the group G generated by all the symmetries about U and V fixes $\mathcal{S}'$ and $[\infty]$, and acts transitively on the points of $[\infty]$. Since there is a collineation in G which maps (∞) onto the unique point on $[\infty]$ which is collinear with x, there readily follows that x subtends an ovoid which is isomorphic to the translation ovoid subtended by the points of Type (2). We obtain the following result, which completely solves the isomorphism problem for subtended ovoids in classical subGQ's of order q^n which contain the flag $((\infty), [\infty])$ (that is, $((\infty), \pi)$), in TGQ's of order (q^n, q^{2n}) with a good element, q odd.

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THEOREM 14. *Let $\mathcal{O}$ be an egg of* $\mathbf{PG}(4n-1,q)$ *which is good at the element π, q odd, and consider the TGQ $\mathcal{S}^{(\infty)} = T(\mathcal{O})$. Then all subtended ovoids of a fixed (arbitrary) subGQ $\mathcal{Q}(4,q^n)$ through the flag $((\infty),[\infty])$ are isomorphic translation ovoids. Moreover, these ovoids are isomorphic to the ovoid of $\mathcal{Q}(4,q^n)$ arising from the semifield flock which corresponds to the egg $\mathcal{O}$.*

As an easy corollary, Theorem 5.4 of M. R. Brown [**5**] follows (this is the 'isomorphism part' of Theorem 14 for the Kantor GQ's).

NOTE. Using Theorem 3, it is also easy to prove the 'isomorphism part' of Theorem 13.

As the subGQ $\mathcal{Q}(4,q^n)$ through $[\infty]$ in Theorem 14 is arbitrary, we have the following result.

THEOREM 15. *Two TGQ's $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ and $(\mathcal{S}')^{(\infty)'} = T(\mathcal{O}')$ of order (q^n,q^{2n}), q odd, with $\mathcal{O}$ and $\mathcal{O}'$ good at π and π', are isomorphic if and only if the subtended ovoids of a fixed subGQ $\mathcal{Q}(4,q^n)$ of $\mathcal{S}$ through $[\infty]$, where $[\infty]$ corresponds to π, and the subtended ovoids of a fixed subGQ $\mathcal{Q}(4,q^n)$ of $\mathcal{S}'$ through $[\infty]'$, where $[\infty]'$ corresponds to π', are isomorphic translation ovoids of $\mathcal{Q}(4,q^n)$.*

COROLLARY 16. *The Penttila-Williams TGQ $\mathcal{S} = T(\mathcal{O}_{PW})$ is new.*

Proof. For the Penttila-Williams TGQ $\mathcal{S} = T(\mathcal{O}_{PW})$, we have that $\mathcal{O}_{PW}$ is good at some element. But the Penttila-Williams ovoid, which is isomorphic to the ovoid of $\mathcal{Q}(4,3^5)$ arising from the Penttila-Williams flock which corresponds to the egg $\mathcal{O}_{PW}$, is not isomorphic to any other known ovoid of $\mathcal{Q}(4,3^5)$ [**21**]; so by Theorem 15 the result follows.

COROLLARY 17. *The Penttila-Williams flock $\mathcal{F}_{PW}$ is new.*

6. Translation generalized quadrangles with isomorphic translation duals.

THEOREM 18. *Suppose that $\mathcal{S}^{(x)}$ is a non-classical TGQ which is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2,q). So q is odd. Then the full automorphism group of $\mathcal{S}^{(x)}$ does not fix x if and only if $\mathcal{S}^{(x)}$ is the point-line dual of a non-classical Kantor flock GQ.*

Proof. Suppose that the translation point x of $\mathcal{S}$ is not fixed by $Aut(\mathcal{S})$. Then as $\mathcal{S}$ is non-classical, all the translation points of $\mathcal{S}$ are incident with the same line $[\infty]Ix$. By [**37**], there are q^3+q^2 classical subGQ's of $\mathcal{S}$ of order q which contain $[\infty]$. That line $[\infty]$ is thus fixed by the full automorphism group of $\mathcal{S}$ (and it is the only line with that property), and hence it follows that $[\infty]$ corresponds to the special point (∞) of $\mathcal{S}(\mathcal{F})$, as (∞) is fixed by each automorphism of $\mathcal{S}(\mathcal{F})$, see [**19**]. But by [**32**], it then follows that $\mathcal{F}$ is a Kantor flock, as the dual net $\mathcal{N}^*_{(\infty)}$ satisfies the Axiom of Veblen, contradiction.

The following corollary characterizes the Kantor flock GQ's.

THEOREM 19. *Suppose that $\hat{f}$ is a biadditive and symmetric map of $\mathbb{F}^2 \times \mathbb{F}^2$ to $\mathbb{F}$, where $\mathbb{F} = \mathbf{GF}(q)$, q odd. Suppose $\hat{g}_t$ is a map of $\mathbb{F}^2$ to $\mathbb{F}$ so that, for all $d,t,u \in \mathbb{F}$ and $\alpha,\gamma \in \mathbb{F}^2$, the following conditions are satisfied:*

1. $\hat{g}_t(\alpha+\gamma) - \hat{g}_t(\alpha) - \hat{g}_t(\gamma) = \hat{f}(t\alpha,\gamma) = \hat{f}(t\gamma,\alpha)$;
2. $\hat{g}_{t+u}(\alpha) = \hat{g}_t(\alpha) + \hat{g}_u(\alpha)$;

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3. $\hat{g}_t(\gamma) = 0$, *where* t *is nontrivial, implies that* $\gamma = (0, 0)$*;*

4. $\hat{g}_t(\gamma(t-d)^{-1}) - \hat{g}_d(\gamma(t-d)^{-1}) + \hat{g}_d(-\gamma(d-u)^{-1}) - \hat{g}_u(-\gamma(d-u)^{-1}) = 0$ *implies* $\gamma = 0$ *if* t, d, u *are distinct* (*this is Condition* (10) *of* [**18**, Section 10.4]).

Assume also the following additional condition:

(5) *If for* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}^2$ *we have that*

$$0 = \hat{f}(\alpha_1(t-u), \beta_1) = \hat{f}(\alpha_1(t-u), \beta_2) = \hat{f}(\alpha_2(t-u), \beta_1),$$

then $\hat{f}(\alpha_2(t-u), \beta_2) = 0$ (*this is Condition* $V.2$ *of* [**15**]).

Let $\mathcal{S}$ *be the GQ which arises from the* 4*-gonal family* $(\mathcal{J}, \mathcal{J}^*)$ *as before. Then* $\mathcal{S}^D \cong \mathcal{S}(\mathcal{F})$ *with* $\mathcal{F}$ *a Kantor flock, and conversely.*

Proof. By Section 2 and Theorem 3, $\mathcal{S}$ is a TGQ for which $(\mathcal{S}^*)^D$ is a flock GQ, say $\mathcal{S}(\mathcal{F})$. Condition (5) is exactly the condition for $\mathcal{S}^D$ to be a flock GQ, see [**15**] (in fact, Condition (5) infers that $\mathcal{S}$ satisfies Property (G) at its point (∞), and then the main theorem of [**28**] implies that $\mathcal{S}^D$ is a flock GQ). The theorem now follows from Theorems 3 and 18.

Another corollary of Theorem 18 is the following result.

THEOREM 20. *Suppose* $T(\mathcal{O})$ *is a TGQ of order* (q, q^2)*,* q *odd, where* $\mathcal{O}$ *is good at some element. Then* $T(\mathcal{O})$ *is the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$ *if and only if* $\mathcal{F}$ *is a Kantor flock.*

Proof. This follows immediately by Theorem 3, Theorem 18 and the fact that as $\mathcal{O}$ is good at some element, $T(\mathcal{O})^*$ is the point-line dual of a flock GQ.

There are some immediate corollaries, which we state as theorems.

THEOREM 21. (i) *Suppose* $\mathcal{S} = T(\mathcal{O})$ *is a TGQ of order* (q, q^2)*,* q *odd, where* $\mathcal{O}$ *is good at some element. Then* $T(\mathcal{O}) \cong T(\mathcal{O}^*)$ *if and only if* $\mathcal{S}$ *is the point-line dual of a Kantor flock GQ.*

(ii) *Suppose* $\mathcal{S} = T(\mathcal{O})$ *is a TGQ of order* (q, q^2)*,* q *odd, which is the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$*. Then* $\mathcal{S}$ *is isomorphic to its translation dual if and only if* $\mathcal{F}$ *is a Kantor flock.*

Proof. Immediate.

THEOREM 22. *The point-line dual of* $\mathcal{S}(\mathcal{F}_{PW})$*, which is the translation dual of* $T(\mathcal{O}_{PW})$*, is new.*

Proof. This follows from the fact that $T(\mathcal{O}_{PW})$ is new, that for the point-line dual $T(\mathcal{O})$ of $\mathcal{S}(\mathcal{F}_{PW})$, $\mathcal{O}$ is good at no element (by Theorem 20), and that for two TGQ's $T(\mathcal{O}_1)$ and $T(\mathcal{O}_2)$, we have that $T(\mathcal{O}_1) \cong T(\mathcal{O}_2)$ if and only if $T(\mathcal{O}_1^*) \cong T(\mathcal{O}_2^*)$ by [**31**].

6.1. First notes on automorphism groups. The following theorem is taken from [**31**].

THEOREM 23. (J. A. Thas and K. Thas [**31**]) *Suppose* $\mathcal{S} = T(\mathcal{O})$ *is a TGQ of order* (q^n, q^m) *with base point* (∞)*, where* q *is odd if* $n = m$*, and suppose that* $G_{(\infty)}$ *is the stabilizer of* (∞) *in the automorphism group* G *of* $\mathcal{S}$*. Furthermore, suppose* $(\infty)'$ *is the base point of* $T(\mathcal{O}^*) = \mathcal{S}^*$*, and let* $G'_{(\infty)'}$ *be the stabilizer of* $(\infty)'$ *in the automorphism group* G' *of* $\mathcal{S}^*$*. Then* $|G_{(\infty)}| = |G'_{(\infty)'}|$.

The following theorem is a nice corollary.

COROLLARY 24. (1) *Suppose* $\mathcal{S} = T(\mathcal{O})$ *is a TGQ of order* (q^n, q^{2n})*, where* q *is even and where* $\mathcal{O}$ *is good at some element, and suppose that* $Aut(\mathcal{S})$ *is the automorphism group of* $\mathcal{S}$*. Suppose* (∞) *is the base point. Let* $\mathcal{S}^* = T(\mathcal{O}^*)$ *be the translation dual of* $\mathcal{S} = T(\mathcal{O})$ *with base point* $(\infty)'$*, and let* $Aut(\mathcal{S}^*)$ *be the automorphism group of* $\mathcal{S}^*$*. Then* $|Aut(\mathcal{S})| = |Aut(\mathcal{S}^*)|$.

(2) *Suppose* $\mathcal{S} = T(\mathcal{O})$ *is a TGQ of order* (q^n, q^{2n})*,* q *odd and* $\mathcal{O}$ *good at some element, which is not the point-line dual of a Kantor flock GQ, and suppose that* $Aut(\mathcal{S})$ *is the automorphism group of* $\mathcal{S}$*. Suppose* (∞) *is the base point. Let* $\mathcal{S}^* = T(\mathcal{O}^*)$ *be the translation dual of* $\mathcal{S} = T(\mathcal{O})$ *with base point* $(\infty)'$*, and let* $Aut(\mathcal{S}^*)$ *be the automorphism group of* $\mathcal{S}^*$*. Then* $|Aut(\mathcal{S})| = (q^n + 1)|Aut(\mathcal{S}^*)|$.

Proof. Suppose we are in Case (1). If $\mathcal{S}$ is classical, then the result is obvious, hence suppose this is not the case. As q is even, Theorem 9 asserts that the translation points of both $\mathcal{S}$ and $\mathcal{S}^*$ are fixed by their respective automorphism group. The result then follows from Theorem 23.

Suppose we are in Case (2). As $\mathcal{S} = T(\mathcal{O})$ is not a dual Kantor flock GQ, there follows that the translation point of $\mathcal{S}^*$ is fixed by $Aut(\mathcal{S}^*)$ (see Theorem 18). The assertion follows from Theorem 3 and Theorem 23.

We emphasize at this point that Corollary 24 implies that the size of the automorphism group of a TGQ in the even characteristic case always equals that of the automorphism group of its translation dual. There is another interesting corollary.

COROLLARY 25. *Suppose that* $\mathcal{S}$ *is the Roman GQ of order* (q, q^2)*,* $q = 3^h$*,* $h > 3$ *(so that* $\mathcal{S}$ *is the translation dual of the point-line dual of the flock GQ* $\mathcal{S}(\mathcal{F})$ *with* $\mathcal{F}$ *a Ganley flock). If* $Aut(\mathcal{S})$ *is the full automorphism group of* $\mathcal{S}$*, then*

$$|Aut(\mathcal{S})| = q^6(q + 1)(q - 1)2h.$$

Proof. This follows immediately by Corollary 24 and the fact that the full automorphism group of $\mathcal{S}(\mathcal{F})$ with $\mathcal{F}$ a Ganley flock has size $q^6(q - 1)2h$ when $q > 27$ [**16**].

For $q = 27$, $|\mathrm{Aut}(\mathcal{S}(\mathcal{F}))| = 4(q - 1)q^6 2h$ (Maska Law, private communication) so that $|\mathrm{Aut}(\mathcal{S})| = 4(q + 1)(q - 1)q^6 2h$.

6.2. Automorphism groups revisited. In this section, it is our objective to obtain a lower bound for the size of the full automorphism group of a nonclassical TGQ which is the translation dual of a TGQ that arises from a flock. The bound will be sharp, since equality will hold for the Roman GQ's.

Suppose that $\mathcal{S}$ is a nonclassical TGQ of order (q, q^2), q odd, which is the translation dual of a TGQ of order (q, q^2) arising from a flock. By Theorem 3, $\mathcal{S}$ has a line $[\infty]$ of translation points, and there are $q^3 + q^2$ classical subGQ's of order q all containing $[\infty]$ [**26**]. Consider a fixed subGQ $\mathcal{S}' \cong \mathcal{Q}(4, q)$ of order q through the line $[\infty]$, and note that each symmetry of $\mathcal{S}$ about a line of $\mathcal{S}' \cap [\infty]^{\perp}$ fixes $\mathcal{S}'$. It is then clear that $Aut(\mathcal{S})_{\mathcal{S}'} =: H$ (that is, the stabilizer of $\mathcal{S}'$ in the automorphism group of $\mathcal{S}$) acts transitively on the ordered pairs (x, y) of points in $\mathcal{S}' \setminus [\infty]$ for which $x \sim y$, xy not intersecting $[\infty]$. Hence we obtain that

$$q^4(q + 1) \quad \text{divides} \quad |H|.$$

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Note that H fixes $[\infty]$. Fix an ordinary quadrangle $\mathcal{A}$ in $\mathcal{S}'$ which contains $[\infty]$ as a side, and suppose $[\infty]$, U, V, W are the lines of $\mathcal{A}$, so that $U \not\sim [\infty]$. Consider the action of the elementwise stabilizer $H(\mathcal{A})$ of $\mathcal{A}$ in H on the lines of $X := \{U, [\infty]\}^{\perp\perp} \setminus \{U, [\infty]\}$. By Theorem 3, V and W are axes of symmetry of $\mathcal{S}$ (and $\mathcal{S}'$), and the group G generated by the symmetries about V and W fixes $\mathcal{S}'$ and every line of $\{V, W\}^{\perp}$, and is isomorphic to $\mathbf{SL}(2, q)$ by [**37**]. Hence the kernel of the action of $H(\mathcal{A})$ on the lines of X has a subgroup of order $q - 1$ (recall that the action of G on $\mathcal{S}' \setminus \{V, W\}^{\perp\perp}$ is faithful). Hence

$$q^4(q + 1)(q - 1) \quad \text{divides} \quad |H|.$$

Let $v = V \cap [\infty]$ and $w = W \cap U$. Then, as noted before, the group $\mathbf{W}(v, w)$ of whorls about v and w has size $|\mathbb{K}| - 1$, where $\mathbb{K}$ is the kernel of the TGQ, see [**18**, 8.6.5]. This group clearly fixes $\mathcal{S}'$, and acts semi-regularly on X, see [**18**, 8.1.1]. Thus $(|\mathbb{K}| - 1)q^4(q + 1)(q - 1)$ divides $|H|$.

It is also clear that each subGQ of order q of $\mathcal{S}$ which contains $[\infty]$ has an $Aut(\mathcal{S})$-orbit of size at least q^2, since $Aut(\mathcal{S})$ acts transitively on the pairs of non-concurrent lines in $[\infty]^{\perp}$, and hence

$$|Aut(\mathcal{S})| \geq q^6(q + 1)(q - 1)(|\mathbb{K}| - 1).$$

Recall at this point that, if x_1, x_2, x_3, x_4 are four collinear points of $\mathbf{PG}(n, q)$, with $|\{x_1, x_2, x_3, x_4\}| \geq 3$, then by $(x_1, x_2; x_3, x_4)$ we denote the usual *cross-ratio* given by

$$\frac{r_3 - r_1}{r_3 - r_2} : \frac{r_4 - r_1}{r_4 - r_2},$$

where the r_i, $i = 1, 2, 3, 4$, are non-homogeneous coordinates of the x_i on the line through x_1, x_2, x_3, x_4.

We also recall that, if a semilinear automorphism θ of $\mathbf{PG}(n, q)$ (i.e. $\theta \in \mathbf{P\Gamma L}(n + 1, q)$) preserves the cross-ratio of all 4-tuples of points on at least one line, then θ is a linear automorphism of $\mathbf{PG}(n, q)$ (i.e. $\theta \in \mathbf{PGL}(n + 1, q)$). Note that if a semilinear automorphism θ of $\mathbf{PG}(n, q)$ fixes some $\mathbf{PG}(k, q)$ in $\mathbf{PG}(n, q)$, $k > 0$, elementwise, then θ preserves the cross-ratio, and hence $\theta \in \mathbf{PGL}(n + 1, q)$.

By [**31**], we can consider $Aut(\mathcal{S})_{(\infty)}$, for any fixed point $(\infty)I[\infty]$, as a group of automorphisms of $\mathbf{PG}(4n, q)$ which fixes the corresponding egg $\mathcal{O} \subset \mathbf{PG}(4n - 1, q)$, and moreover, as the groups from the previous arguments, if restricted to $\mathbf{P\Gamma L}(4n + 1, q)_{\mathcal{O}}$, all fix at least one line of $\mathbf{PG}(4n - 1, q)$ pointwise, it is clear that

$$\left|Aut(\mathcal{S})_{(\infty)} \cap \mathbf{PGL}(4n + 1, q)\right| \geq q^6(q - 1)(|\mathbb{K}| - 1).$$

We now show that the latter bound is sharp. Suppose that $(\mathcal{S}(\mathcal{F})^D)^*$, with $\mathcal{F}$ a Ganley flock, is a Roman GQ of order (q, q^2), $q > 27$. Then

$$\left|Aut((\mathcal{S}(\mathcal{F})^D)^*)_{(\infty)} \cap \mathbf{PGL}(4n + 1, q)\right| = q^6(q - 1)2,$$

by the previous section and the preceding arguments, [**16**] and [**31**].

REMARK 26. (*On a special involution.*) Let $\mathcal{S}$ be a span-symmetric generalized quadrangle of order (s, s^2), $s > 1$ and s odd, with base-span $\mathcal{L}$ and base-group G. Then there is an involution θ in G which acts trivially on the points of the lines of $\mathcal{L}$ and which acts semiregularly on the other points (as the kernel of $G \cong \mathbf{SL}(2, s)$ on $\mathcal{L}$).

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Now suppose $\mathcal{O}$ is an egg of $\mathbf{PG}(4n-1, q) \subseteq \mathbf{PG}(4n, q)$, q odd, which is good at its element π. Then by Theorem 3, $T(\mathcal{O})$ is an SPGQ for every span $\{L, M\}^{\perp\perp}$ in $\pi^{\perp}$ (with the obvious notation), $L \not\sim M$. Hence [31] implies that for arbitrary $\pi' \in \mathcal{O} \setminus \{\pi\}$ and $p \in \mathbf{PG}(4n, q) \setminus \mathbf{PG}(4n-1, q)$, there is an involution of $\mathbf{PG}(4n, q)$ (which is an element of $\mathbf{PGL}(4n+1, q)$) which fixes $\pi\pi' p$ pointwise and $\mathcal{O}$ as a set, and which acts faithfully as an involution on the elements of $\mathcal{O} \setminus \{\pi, \pi'\}$.

6.3. Derivation of semifield flocks, BLT-sets and automorphisms. Let q be an odd prime power, and let $\mathcal{F} = \{\mathcal{C}_1, \ldots, \mathcal{C}_q\}$ be a flock of the quadratic cone $\mathcal{K}$ in $\mathbf{PG}(3, q)$ with vertex v. Let $\mathcal{K}$ be embedded in a nonsingular quadric $\mathcal{Q}$ of a $\mathbf{PG}(4, q)$ containing $\mathbf{PG}(3, q)$ as a hyperplane so that $\mathcal{K} = \mathbf{PG}(3, q) \cap \mathcal{Q}$. There are unique points $p_1, \ldots, p_q$ of $\mathcal{Q}$ for which $\mathcal{C}_i = v^{\perp} \cap p_i^{\perp}$, $1 \leq i \leq q$, where '$\perp$' is relative to $\mathcal{Q}$. Then the condition that $\mathcal{C}_1, \ldots, \mathcal{C}_q$ are disjoint is precisely the condition that $V = \{v, p_1, \ldots, p_q\}$ is a set of $q+1$ points of $\mathcal{Q}$ such that for $1 \leq i < j \leq q$, (v, p_i, p_j) is a triad (that is, a set of three mutually non-collinear points) on the GQ $\mathcal{Q}$, and for which $\{v, p_i, p_j\}^{\perp} = \emptyset$. The main theorem of [**2**] is that given such a set V, it is also true that for each triple (p_i, p_j, p_k), $0 \leq i < j < k \leq q$ (where $p_0 = v$), no point of $\mathcal{Q}$ is collinear (in $\mathcal{Q}$) with all three of the points. It follows that each point of $\mathcal{Q} \setminus V$ is collinear with 0 or 2 points of V. Such a set V of $q+1$ points of $\mathcal{Q}$ is called a *BLT-set of* $\mathcal{Q}$. L. Bader, G. Lunardon and J. A. Thas have showed in [**2**] that by using the BLT-set V, the flock $\mathcal{F}$ of the quadratic cone $\mathcal{K}$ may be interpreted as one of a set of $q+1$ flocks (also called a *BLT-set*) — recall that q is odd [**2**]. Each of these flocks corresponds to a line of the GQ $\mathcal{S}(\mathcal{F})$ through (∞); each of the q 'new' flocks is obtained by recoordinatizing the GQ $\mathcal{S}(\mathcal{F})$ so as to interchange the line $[A(\infty)]$ and some other line through (∞). It follows that two flocks of a BLT-set are projectively equivalent if and only if the corresponding pair of lines of $\mathcal{S}(\mathcal{F})$ is in the same orbit of the automorphism group of $\mathcal{S}(\mathcal{F})$ [**20**].

We directly obtain the following theorem, which solves the isomorphism problem of derivation for the flocks of a BLT-set in the semifield case.

THEOREM 27. *Suppose that $\mathcal{S}^{(\infty)}$ is a TGQ which is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q). Suppose $\{\mathcal{F}_0 = \mathcal{F}, \mathcal{F}_1, \ldots, \mathcal{F}_q\}$ is the BLT-set of $q+1$ flocks which is derived from $\mathcal{F}$. Then all these flocks are isomorphic if and only if $\mathcal{F}$ is a Kantor flock. If $\mathcal{F}$ is not a Kantor flock, then $\mathcal{F}_1, \ldots, \mathcal{F}_q$ are all isomorphic, but non-isomorphic to $\mathcal{F}$.*

Proof. If all these flocks are isomorphic, then this implies that $Aut(\mathcal{S}(\mathcal{F})^D)$ acts transitively on the points of the line $[\infty]$ which corresponds to the point (∞) of $\mathcal{S}(\mathcal{F})$. Hence $[\infty]$ is a line of translation points, and so $\mathcal{F}$ is a Kantor flock by Theorem 18. The theorem easily follows.

COROLLARY 28. *Suppose that $\mathcal{S}^{(\infty)}$ is the point-line dual of the flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q), $\mathcal{F}$ the Penttila-Williams flock. Suppose $\{\mathcal{F}_0 = \mathcal{F}, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_q\}$ is the BLT-set of $q+1$ flocks which is derived from $\mathcal{F}$. Then $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_q$ are all isomorphic, but non-isomorphic to $\mathcal{F}$. Hence $\mathcal{F}_1$ is a new flock.*

Proof. The fact that $\mathcal{F}_1$ is new follows from the fact that $\mathcal{F}_1$ is non-isomorphic to $\mathcal{F}$, that both flocks give rise to the same generalized quadrangle, and that the latter is new by Theorem 22.

REMARK 29. Some of the results just obtained are also contained in L. Bader, G. Lunardon and I. Pinneri [**1**], although they continuously use the Lemma 1 of that paper, for which the proof seems to be rather terse; see J. A. Thas and K. Thas [**31**] for an alternative approach. It was preferred to give new proofs of these results.

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REFERENCES

1. L. Bader, G. Lunardon and I. Pinneri, A new semifield flock, *J. Combin. Theory Ser. A* **86** (1999), 49–62.

2. L. Bader, G. Lunardon and J. A. Thas, Derivation of flocks of quadratic cones, *Forum Math.* **2** (1990), 163–174.

3. I. Bloemen, J. A. Thas and H. Van Maldeghem, Translation ovoids of generalized quadrangles and hexagons, *Geom. Dedicata* **72** (1998), 19–62.

4. A. Blokhuis, M. Lavrauw and S. Ball, On the classification of semifield flocks, *Adv. Math.*, to appear.

5. M. R. Brown, Semipartial geometries and generalized quadrangles of order (r, r^2), in *Finite geometry and combinatorics (Deinze, 1997), Bull. Belg. Math. Soc. Simon Stevin* **5** (1998), 187–205.

6. N. L. Johnson, Semifield flocks of quadratic cones, *Simon Stevin* **61** (1987), 313–326.

7. W. M. Kantor, Generalized quadrangles associated with $\mathbf{G}_2(q)$, *J. Combin. Theory Ser. A* **29** (1980), 212–219.

8. W. M. Kantor, Note on span-symmetric generalized quadrangles, *Adv. Geom.* **2** (2002), 197–200.

9. M. Lavrauw, *Scattered subspaces with respect to spreads, and eggs in finite projective spaces*, PhD Thesis, Technische Universiteit Eindhoven (Eindhoven, 2001).

10. M. Lavrauw, On semifield flocks, eggs and ovoids of $Q(4, q)$, submitted.

11. M. Lavrauw and T. Penttila, On eggs and translation generalised quadrangles, *J. Combin. Theory Ser. A* **96** (2001), 303–315.

12. G. Lunardon, Flocks, ovoids of Q(4, q) and designs, *Geom. Dedicata* **66** (1997), 163–173.

13. S. E. Payne, Generalized quadrangles as group coset geometries, *Congr. Numer.* **29** (1980), 717–734.

14. S. E. Payne, A garden of generalized quadrangles, *Algebras, Groups, Geom.* **3** (1985), 323–354.

15. S. E. Payne, An essay on skew translation generalized quadrangles, *Geom. Dedicata* **32** (1989), 93–118.

16. S. E. Payne, Collineations of the generalized quadrangles associated with q-clans, *Ann. Discrete Math.* **52** (1992), 449–461.

17. S. E. Payne, Flocks and generalized quadrangles: an update, in *Proceedings of the Academy Contact Forum "Generalized Polygons" 20 October, Palace of the Academies* (Brussels, Belgium, 2001), 61–98.

18. S. E. Payne and J. A. Thas, *Finite generalized quadrangles*, Research Notes in Mathematics No. 110 (Pitman, 1984).

19. S. E. Payne and J. A. Thas, Conical flocks, partial flocks, derivation, and generalized quadrangles, *Geom. Dedicata* **38** (1991), 229–243.

20. S. E. Payne and J. A. Thas, Generalized quadrangles, BLT-sets, and Fisher flocks, *Congr. Numer.* **84** (1991), 161–192.

21. T. Penttila and B. Williams, Ovoids of parabolic spaces, *Geom. Dedicata* **82** (2000), 1–19.

22. L. A. Rogers, Characterization of the kernel of certain translation generalized quadrangles, *Simon Stevin* **64** (1990), 319–328.

23. J. A. Thas, The m-dimensional projective space $S_m(M_n(\mathbf{GF}(q)))$ over the total matrix algebra $M_n(\mathbf{GF}(q))$ of the $n \times n$-matrices with elements in the Galois field $\mathbf{GF}(q)$, *Rend. Mat. (6)* **4** (1971), 459–532.

24. J. A. Thas, Translation 4-gonal configurations, *Rend. Accad. Naz. Lincei* **56** (1974), 303–314.

25. J. A. Thas, Generalized quadrangles and flocks of cones, *European J. Combin.* **8** (1987), 441–452.

26. J. A. Thas, Generalized quadrangles of order (s, s^2), I, *J. Combin. Theory Ser. A* **67** (1994), 140–160.

27. J. A. Thas, Symplectic spreads in $\mathbf{PG}(3, q)$, inversive planes and projective planes, *Discrete Math.* **174** (1997), 329–336.

28. J. A. Thas, Generalized quadrangles of order (s, s^2), III, *J. Combin. Theory Ser. A* **87** (1999), 247–272.

29. J. A. Thas, Characterizations of translation generalized quadrangles, *Des. Codes Cryptogr.* **23** (2001), 249–257.

30. J. A. Thas and F. De Clerck, Partial geometries satisfying the axiom of Pasch, *Simon Stevin* **51** (1977), 123–137.

31. J. A. Thas and K. Thas, On translation generalized quadrangles and translation duals, I, *Discrete Math.*, submitted.

32. J. A. Thas and H. Van Maldeghem, Generalized quadrangles and the axiom of Veblen, in *Geometry, combinatorial designs and related structures*, London Math. Soc. Lecture Note Ser. No. 245 (Cambridge University Press, 1997), 241–253.

33. K. Thas, *Symmetrieën in Eindige Veralgemeende Vierhoeken*, Master Thesis, (Ghent University, Ghent, 1999).

34. K. Thas, On symmetries and translation generalized quadrangles, in *Finite geometries, Proceedings of the Fourth Isle of Thorns conference, 16–21 July 2000*, Developments in Mathematics No. 3 (2001) (Kluwer Academic Publishers), 333–345.

35. K. Thas, Automorphisms and characterizations of finite generalized quadrangles, in *Proceedings of the Academy Contact Forum "Generalized Polygons" 20 October, Palace of the Academies* (Brussels, Belgium, 2001), 111–172.

36. K. Thas, Classification of span-symmetric generalized quadrangles of order s, *Adv. Geom.* **2** (2002), 189–196.

37. K. Thas, The classification of generalized quadrangles with two translation points, *Beiträge Algebra Geom.* **43** (2002), 365–398.

38. K. Thas, *A Lenz-Barlotti classification for finite generalized quadrangles* (Research Monograph, submitted).

39. J. Tits, Sur la trialité et certains groupes qui s'en déduisent, *Inst. Hautes Etudes Sci. Publ. Math.* **2** (1959), 13–60.

40. M. Walker, A class of translation planes, *Geom. Dedicata* **5** (1976), 135–146.