Canad. Math. Bull. Vol. **42** (3), 1999 pp. 285–290

On Kloosterman Sums with Oscillating Coefficients

Peiming Deng

Abstract. In this paper we prove: for any positive integers a and q with $(a, q) = 1$, we have uniformly

$$\sum_{\substack{n\le N\\(n,q)=1,\ n\bar n\equiv 1 (\mathrm{mod}\ q)}} \mu(n) e\left(\frac{a\bar n}{q}\right) \ll Nd(q)\left\{\frac{\log^{\frac52} N}{q^{\frac12}} + \frac{q^{\frac15}\log^{\frac{13}{5}} N}{N^{\frac15}}\right\}.$$

This improves the previous bound obtained by D. Hajela, A. Pollington and B. Smith [5].

1 Introduction

For any positive integers a and q with $(a, q) = 1$, we write

$$S(N, a, q) = \sum_{\substack{n\le N\\ n\bar n\equiv 1 (\mathrm{mod}\ q)}} \mu(n)\delta_q(n) e\left(\frac{a\bar n}{q}\right), \tag{1}$$

where $\mu(n)$ is the Möbius function, $\delta_q(n) = 1$ when $(n, q) = 1$ and 0 otherwise, and $e(x) = e^{2xix}$ for the real x.

In [5], Hajela, Pollington and Smith considered Kloosterman sums with the above type of oscillating coefficients. They showed that

$$S(N, a, q) \ll_\varepsilon Nq^\varepsilon \left\{\frac{\log^{\frac52} N}{q^{\frac12}} + \frac{q^{\frac{3}{10}}(\log N)^{\frac{11}{5}}}{N^{\frac15}}\right\}, \tag{2}$$

which is valid for any positive integers a and q with $(a, q) = 1$, and $1 \le q \le N^{\frac23-\varepsilon}$. Interest in estimating Kloosterman sums of this and similar types stem from applications to additive problems with smooth coefficients; we refer to [3] for some examples. The purpose of this paper is to sharpen (2) by proving the following theorem.

Theorem *For any positive integers a and q with $(a, q) = 1$, and $1 \le q \le N/\log^{\frac34} N$, we have uniformly*

$$S(N, a, q) \ll Nd(q)\left\{\frac{\log^{\frac52} N}{q^{\frac12}} + \frac{q^{\frac15}\log^{\frac{13}{5}} N}{N^{\frac15}}\right\}, \tag{3}$$

where $d(q)$ is the number of positive divisors of q.

Received by the editors September 17, 1997; revised July 23, 1998.
AMS subject classification: 10G10.
Keywords: Kloosterman sums, oscillating coefficients, estimate.

Under the generalized Riemann hypothesis, we show that

$$S(N,a,q) \ll_\varepsilon q^{\frac{1}{2}} N^{\frac{1}{2}+\varepsilon}, \tag{4}$$

in the range $1 \ll q \ll N^{1-\varepsilon}$, which can be compared to our theorem.

From (1) we have

$$\begin{aligned} S(N,a,q) &= \sum_{m=1}^{q} \sum_{\substack{n\le N \\ nm\equiv 1 (\bmod q)}} \mu(n)\delta_q(n) e\left(\frac{am}{q}\right) \\ &= \frac{1}{\varphi(q)} \sum_{\chi} \Big\{ \sum_{n\le N} \chi(n)\mu(n) \Big\} \Big\{ \sum_{m=1}^{q} \chi(m) e\left(\frac{am}{q}\right) \Big\} \\ &= \frac{1}{\varphi(q)} \sum_{\chi} G(a,\chi) \sum_{n\le N} \chi(n)\mu(n), \end{aligned}$$

where $G(a,\chi)$ is the Gauss sum defined by

$$G(a,\chi) = \sum_{m=1}^{q} \chi(m) e\left(\frac{am}{q}\right).$$

It is known that

$$S(N,a,q) \ll q^{\frac{1}{2}} \max \Big| \sum_{n\le N} \chi(n)\mu(n) \Big|.$$

We conclude that (4) is true for any $\varepsilon > 0$ under the generalized Riemann hypothesis.

2 Proof of the Theorem

The technique that we use to prove our theorem is an application of Vaughan's identity [2], [7] along with an estimate for incomplete Kloosterman sums [4], [6] which follows from Weil's estimate for Kloosterman sums. We first establish a suitable version of Vaughan's inequality.

Lemma 1 *Let N, U, V be real numbers with $1 \le U$, $1 \le V$ and $UV \le N$, let $f(n)$ be an arithmetic function such that $|f(n)| \le 1$ for all integers n. Then we have*

$$\begin{aligned} \sum_{n\le N} \mu(n) f(n) &\ll U + V + \sum_{m\le UV} d(m) \Big| \sum_{r\le N/m} f(mr) \Big| \\ &\quad + \max_{U<Y\le N/V} Y^{\frac{1}{2}} \log^{\frac{5}{2}} N \Big\{ \sum_{Y<s\le 2Y} \Big| \sum_{V<t\le N/s} \mu(t) f(ts) \Big|^2 \Big\}^{\frac{1}{2}} \end{aligned} \tag{5}$$

Proof By Vaughan's identity (see [2, p. 138]), we have

$$\sum_{n\le N} \mu(n) f(n) = S_1 + S_2 - S_3 - S_4, \tag{6}$$

https://doi.org/10.4153/CMB-1999-034-8 Published online by Cambridge University Press

where

$$S_1 = \sum_{n \le U} \mu(n) f(n), \qquad S_2 = \sum_{n \le V} \mu(n) f(n),$$

$$S_3 = \sum_{\substack{str \le N \\ s \le U \\ t \le V}} \mu(s)\mu(t) f(str), \qquad S_4 = \sum_{\substack{str \le N \\ s > U \\ t > V}} \mu(t) \Big\{ \sum_{\substack{d \mid s \\ d < U}} \mu(d) \Big\} f(st).$$

The trivial estimate yields $|S_1| \le U$, $|S_2| \le V$, and

$$S_3 = \sum_{m \le UV} \Big\{ \sum_{\substack{st = m \\ s \le U \\ t \le V}} \mu(s)\mu(t) \Big\} \sum_{r \le N/m} f(mr),$$

so that

$$|S_3| \le \sum_{m \le UV} d(m) \Big| \sum_{r \le N/m} f(mr) \Big|. \tag{7}$$

To estimate S_4, we use Cauchy's inequality,

$$\begin{aligned} S_4 &= \sum_{U < s \le N/V} \Big(\sum_{\substack{d \mid s \\ d < U}} \mu(d) \Big) \sum_{V < t \le N/s} \mu(t) f(st) \\ &\ll \sum_{U < s \le N/V} d(s) \Big| \sum_{V < t \le N/s} \mu(t) f(st) \Big| \\ &\ll \log N \max_{U < Y \le N/V} \sum_{Y < s \le 2Y} d(s) \Big| \sum_{V < t \le N/s} \mu(t) f(st) \Big| \\ &\ll \log N \max_{U < Y \le N/V} \Big(\sum_{Y < s \le 2Y} d^2(s) \Big)^{\frac{1}{2}} \Big\{ \sum_{Y < s \le 2Y} \Big| \sum_{V < t \le N/s} \mu(t) f(st) \Big|^2 \Big\}^{\frac{1}{2}} \\ &\ll \log^{\frac{5}{2}} N \max_{U < Y \le N/V} Y^{\frac{1}{2}} \Big\{ \sum_{Y < s \le 2Y} \Big| \sum_{V < t \le N/s} \mu(t) f(st) \Big|^2 \Big\}^{\frac{1}{2}}. \end{aligned}$$

The lemma follows from (6).

The estimate for incomplete Kloosterman sums that we shall need is the following (see [4, p. 36]):

Lemma 2 *For any positive number N, we have*

$$\sum_{n \le N} \delta_q(n) e\Big(\frac{b\bar{n}}{q}\Big) \ll \Big[\frac{N}{q}\Big](b, q) + q^{\frac{1}{2}} d(q)(b, q)^{\frac{1}{2}} \log q. \tag{8}$$

https://doi.org/10.4153/CMB-1999-034-8 Published online by Cambridge University Press

Now, we can prove the theorem by using the above lemmas. Taking $f(n) = \delta_q(n)e(\frac{a\bar{n}}{q})$ in Lemma 1, if U, V are two parameters such that $1 \le U$, $1 \le V$, $UV \le N$, then

$$S(N,a,q) \ll U + V + \sum_{m\le UV} d(m)\Big|\sum_{r\le N/m} \delta_q(mr)e\Big(\frac{a\bar{m}\bar{r}}{q}\Big)\Big| \tag{9}$$
$$+ \log^{\frac{5}{2}} N \max_{U<y\le N/V} y^{\frac{1}{2}}\Big(\sum_{y<s\le 2y}\Big|\sum_{V<t\le N/s}\mu(t)\delta_q(st)e\Big(\frac{a\bar{s}\bar{t}}{q}\Big)\Big|^2\Big)^{\frac{1}{2}}.$$

It is known that

$$\sum_{n\le x}\frac{d(n)}{n} \ll \log^2 x.$$

By Lemma 2,

$$\sum_{m\le UV} d(m)\Big|\sum_{r\le N/m}\delta_q(mr)e\Big(\frac{a\bar{m}\bar{r}}{q}\Big)\Big| \ll \sum_{m\le UV} d(m)\Big\{\frac{N}{mq} + q^{\frac{1}{2}}d(q)\log q\Big\} \tag{10}$$
$$\ll \frac{N}{q}\log^2 N + UVq^{\frac{1}{2}}d(q)\log^2 N.$$

To estimate the last term of right hand side of (9), we have

$$\begin{aligned}
&\sum_{y<s\le 2y}\Big|\sum_{V<t\le N/s}\mu(t)\delta_q(st)e\Big(\frac{a\bar{s}\bar{t}}{q}\Big)\Big|^2\\
&\quad= \sum_{y<s\le 2y}\delta_q(s)\sum_{\substack{V<t_1\le N/s\\ V<t_2\le N/s}}\mu(t_1)\mu(t_2)\delta_q(t_1)\delta_q(t_2)e\Big(\frac{a\bar{s}(\bar{t}_1-\bar{t}_2)}{q}\Big)\\
&\quad\ll \sum_{\substack{V<t_1\le N/y\\ V<t_2\le N/y}}\Big|\sum_{y<s\le 2y}\delta_q(s)e\Big(\frac{a\bar{s}(\bar{t}_1-\bar{t}_2)}{q}\Big)\Big|\\
&\quad\ll \sum_{\substack{V<t_1\le N/y\\ V<t_2\le N/y}}\Big\{\frac{y(\bar{t}_1-\bar{t}_2,q)}{q} + q^{\frac{1}{2}}d(q)(\bar{t}_1-\bar{t}_2,q)^{\frac{1}{2}}\log q\Big\}\\
&\quad\ll \frac{y}{q}\sum_{d|q} d\sum_{V<t_1\le N/y}\Big\{\frac{N}{yd}+1\Big\} + q^{\frac{1}{2}}d(q)\log q\sum_{d|q}d^{\frac{1}{2}}\sum_{V<t_1\le N/y}\Big\{\frac{N}{yd}+1\Big\}\\
&\quad\ll \frac{N^2d(q)}{qy} + Nd(q) + \frac{N^2q^{\frac{1}{2}}d^2(q)\log q}{y^2} + \frac{Nqd^2(q)\log q}{y}.
\end{aligned}$$

https://doi.org/10.4153/CMB-1999-034-8 Published online by Cambridge University Press

By (9), we have

$$S(N,a,q) \ll U + V + \frac{N}{q}\log^2 N + UVq^{\frac{1}{2}}d(q)\log^2 N + \log^{\frac{5}{2}} N \frac{Nd(q)}{q^{\frac{1}{2}}} + \frac{Nd(q)\log^{\frac{5}{2}} N}{V^{\frac{1}{2}}} + \frac{Nq^{\frac{1}{4}}d(q)\log^3 N}{U^{\frac{1}{2}}} + N^{\frac{1}{2}}q^{\frac{1}{2}}d(q)\log^3 N. \tag{11}$$

Let

$$U = N^{\frac{2}{3}}q^{-\frac{1}{6}}V^{-\frac{2}{3}}\log^{\frac{2}{3}} N, \tag{12}$$

then

$$UVq^{\frac{1}{2}}d(q)\log^2 N + \frac{Nq^{\frac{1}{4}}d(q)\log^3 N}{U^{\frac{1}{2}}} \ll N^{\frac{2}{3}}q^{\frac{1}{3}}V^{\frac{1}{3}}d(q)\log^{\frac{8}{3}} N. \tag{13}$$

Let

$$V = N^{\frac{2}{5}}q^{-\frac{2}{5}}\log^{-\frac{1}{5}} N, \tag{14}$$

then

$$N^{\frac{2}{3}}q^{\frac{1}{3}}V^{\frac{1}{3}}d(q)\log^{\frac{8}{3}} N + \frac{Nd(q)\log^{\frac{5}{2}} N}{V^{\frac{1}{2}}} \ll N^{\frac{4}{5}}q^{\frac{1}{5}}d(q)\log^{\frac{13}{5}} N. \tag{15}$$

It follows that

$$S(N,a,q) \ll \frac{Nd(q)\log^{\frac{5}{2}} N}{q^{\frac{1}{2}}} + N^{\frac{4}{5}}q^{\frac{1}{5}}d(q)\log^{\frac{13}{5}} N + N^{\frac{1}{2}}q^{\frac{1}{2}}d(q)\log^3 N. \tag{16}$$

Since $q \le N/\log^{\frac{4}{3}} N$, we have $N^{\frac{1}{2}}q^{\frac{1}{2}}\log^3 N \le N^{\frac{4}{5}}q^{\frac{1}{5}}\log^{\frac{13}{5}} N$, moreover, $1 \le V$, $1 \le U$, and $UV \le N$ by (14) and (12). Thus (16) becomes

$$S(N,a,q) \ll Nd(q)\left\{\frac{\log^{\frac{5}{2}} N}{q^{\frac{1}{2}}} + \frac{q^{\frac{1}{5}}\log^{\frac{13}{5}} N}{N^{\frac{1}{5}}}\right\}, \tag{17}$$

which completes the proof of the theorem.

Acknowledgement I am grateful to Professor A. Granville for advice and the referee for his kind help and suggestion.

References

[1] T. Apostol, *Introduction to Analytic Number Theory.* Springer-Verlag, 1976.
[2] H. Davenport, *Multiplicative Number Theory.* Springer-Verlag, 1980.
[3] J. M. Deshouillers and H. Iwaniec, *Kloosterman Sums and Fourier Coefficients of Cusp Forms.* Invent. Math. **70**(1982), 219–288.
[4] P. D. T. A. Elliott, *Arithmetic Functions and Integer Products.* Springer-Verlag, 1984.
[5] D. Hajela, A. Pollington and B. Smith, *On Kloosterman Sums with Oscillating Coefficients.* Canad. Math. Bull. (1) **31**(1988), 32–36.
[6] C. Hooley, *On the Brun-Titchmarsh Theorem.* J. Reine Angew. Math. **225**(1972), 60–79.
[7] R. C. Vaughan, *An Elementary Method in Prime Number Theory.* Recent Progress in Analytic Number Theory, Academic Press, 1981.

Department of Mathematics
The Normal University of Guangxi
Guilin, Guangxi 541004
P.R. China

https://doi.org/10.4153/CMB-1999-034-8 Published online by Cambridge University Press