



COMPOSITIO MATHEMATICA

Nonabelian Hodge theory for klt spaces and descent theorems for vector bundles

Daniel Greb, Stefan Kebekus, Thomas Peternell and Behrouz Taji

Compositio Math. **155** (2019), 289–323.

[doi:10.1112/S0010437X18007923](https://doi.org/10.1112/S0010437X18007923)



FOUNDATION
COMPOSITIO
MATHEMATICA



LONDON
MATHEMATICAL
SOCIETY
EST. 1865



Nonabelian Hodge theory for klt spaces and descent theorems for vector bundles

Daniel Greb, Stefan Kebekus, Thomas Peternell and Behrouz Taji

ABSTRACT

We generalise Simpson’s nonabelian Hodge correspondence to the context of projective varieties with Kawamata log terminal (klt) singularities. The proof relies on a descent theorem for numerically flat vector bundles along birational morphisms. In its simplest form, this theorem asserts that given any klt variety X and any resolution of singularities, any vector bundle on the resolution that appears to come from X numerically, does indeed come from X . Furthermore, and of independent interest, a new restriction theorem for semistable Higgs sheaves defined on the smooth locus of a normal, projective variety is established.

Contents

1	Introduction	289
2	Notation and elementary facts	291
3	The nonabelian Hodge correspondence for locally free Higgs sheaves	298
4	Descent of vector bundles from klt spaces	301
5	Descent of vector bundles to klt spaces	304
6	The restriction theorem for semistable Higgs sheaves	306
7	Ascent of semistable Higgs bundles	310
8	Proof of the nonabelian Hodge correspondence for klt spaces	317
	Acknowledgements	320
	References	320

1. Introduction

1.1 Nonabelian Hodge theory for singular spaces

Given a projective manifold X , a seminal result of Simpson [Sim92] exhibits a natural equivalence between the category of local systems and the category of semistable, locally free Higgs sheaves with vanishing Chern classes. The first main result of this paper extends this correspondence to projective varieties with Kawamata log terminal (klt) singularities.

Received 24 November 2017, accepted in final form 2 November 2018, published online 7 February 2019.

2010 Mathematics Subject Classification 14E30, 53C07 (primary).

Keywords: nonabelian Hodge theory, klt singularities.

DG was partially supported by the DFG-Collaborative Research Center SFB/TR 45. SK gratefully acknowledges support through a joint fellowship of the Freiburg Institute of Advanced Studies (FRIAS) and the University of Strasbourg Institute for Advanced Study (USIAS). BT was partially supported by the DFG-Research Training Group GK1821. Research was partially completed while SK and TP were visiting the National University of Singapore in 2017. TP was supported by the DFG project ‘Zur Positivität in der komplexen Geometrie’.

This journal is © Foundation Compositio Mathematica 2019.

THEOREM 1.1 (Nonabelian Hodge correspondence for klt spaces, Theorem 3.4). *Let X be a projective, klt variety. Then there exists an equivalence between the category of local systems and the category of semistable, locally free Higgs sheaves with vanishing Chern classes.*

We refer the reader to [GKPT15, §5] or to the survey [GKT18, §6] for the (rather delicate) notions of Higgs sheaves, morphisms of Higgs sheaves and pull-back. Semistability is also discussed there.

In fact, there exists a unique way to choose the correspondences of Theorem 1.1 that makes them functorial in morphisms between klt spaces, and compatible with Simpson's construction wherever this makes sense; we refer to §3 for a precise formulation. In particular, functoriality applies to resolutions of singularities as well as morphisms whose images are entirely contained in the singular loci of the target varieties. Our results imply that the pull-back of semistable, locally free Higgs sheaves with vanishing Chern classes under any of these maps remains semistable.

1.2 Descent of vector bundles

The proof of Theorem 1.1 relies in part on a descent theorem for vector bundles, which is of independent interest. To put the result into perspective, consider a desingularisation $\pi : \tilde{X} \rightarrow X$ of a normal variety. It is well known that if X has rational singularities, then any line bundle on \tilde{X} that comes from X topologically does in fact come from X holomorphically. If X is klt, we generalise this result to vector bundles of arbitrary rank: we show that any vector bundle on \tilde{X} that appears to come from X numerically does indeed come from X .

THEOREM 1.2 (Descent of vector bundles to klt spaces, Theorem 5.1). *Let $f : X \rightarrow Y$ be a birational, projective morphism of normal, quasi-projective varieties. Assume that there exists a Weil \mathbb{Q} -divisor Δ_Y such that (Y, Δ_Y) is klt. If \mathcal{F}_X is any locally free, f -numerically flat sheaf on X , then there exists a locally free sheaf \mathcal{F}_Y on Y such that $\mathcal{F}_X \cong f^* \mathcal{F}_Y$.*

The notion of ‘numerical flatness for vector bundles’ generalises the notion of ‘numerical triviality’ for line bundles and is recalled in Definition 2.11. The importance of Theorem 1.2 in the current investigation stems from the fact that locally free Higgs sheaves coming from local systems on resolutions of klt spaces are numerically flat relative to the resolution morphism; see Proposition 7.9.

In addition to the descent result for vector bundles spelled out above, Theorem 5.1 also discusses descent of locally free Higgs sheaves. Theorem 4.1 contains a related result where X (rather than Y) is assumed to be klt.

1.3 Optimality of results

We expect that varieties with klt singularities form the largest natural class where our results can possibly hold in full generality. The construction of a pull-back functor for Higgs sheaves is rather delicate and hinges on the existence of a functorial pull-back for reflexive differentials, for morphisms between klt spaces. For classes of varieties with singularities that are just slightly more general than klt, there are elementary examples [Keb13, Example 1.9] which show that *functorial* pull-back for reflexive differentials does not exist, even though it is known that reflexive differentials still lift to resolutions of singularities in these cases [GK14, Theorem 1.4]. In particular, it is *not possible* to define functorial pull-back of Higgs sheaves for these spaces.

For spaces with arbitrary singularities, we do not expect that a correspondence between the two categories in Theorem 1.1 holds, even at the level of objects.

1.4 Applications

Theorem 1.1 has applications to the quasi-étale uniformisation problem for minimal varieties of general type. Eventually, we expect that all uniformisation theorems of nonabelian Hodge theory have analogues for klt varieties. In particular, we expect to generalise the uniformisation result [GKPT15, Theorem 1.2] to arbitrary klt varieties: if X is minimal, klt and of general type, and if equality holds in the \mathbb{Q} -Miyaoaka–Yau inequality, [GKPT15, Theorem 1.1], i.e.,

$$(2(n+1) \cdot \widehat{c}_2(\mathcal{T}_X) - n \cdot \widehat{c}_1(\mathcal{T}_X)^2) \cdot [K_X]^{n-2} = 0,$$

then the canonical model of X is a singular ball quotient.

To keep the current paper reasonably short, these applications will appear in a separate paper, cf. [GKPT18]. Please see the survey paper [GKT18] for related results in this direction.

One might expect similar uniformisation results in more general contexts, such as ‘pairs’ or ‘orbifolds’. In the setting of pairs, the Miyaoaka–Yau inequality has already been established in [GT16]. We refer to [GKPT15, § 10] for a more precise formulation.

1.5 Relation to the work of Mochizuki

In a large body of work, Mochizuki set up a complete theory of Higgs bundles on arbitrary smooth quasi-projective varieties; among others, see [Moc06, Moc07a, Moc07b]. Our study differs from Mochizuki’s in at least two aspects. First, while our main result, Theorem 1.1 above, traces a correspondence between topological and algebro-geometric properties of X , thereby taking the singularities of X into account, the correspondence established by Mochizuki in the present setup would focus on the connection between such properties for the smooth locus X_{reg} . Second, ultimately, our correspondence is induced geometrically from the nonabelian Hodge correspondence in the projective case, which in turn requires much less sophisticated analytic results than Mochizuki’s approach. However, Mochizuki’s theory will be used in the sequel paper [GKPT18].

1.6 Structure of the paper

Section 2 gathers notation, known results and global conventions that will be used throughout the paper. Section 3 formulates the nonabelian Hodge correspondence for klt spaces in detail, discusses functoriality and its consequence, and states a number of singular generalisations of Simpson’s classical results.

The results are then proven in the remaining sections. Sections 4 and 5 prepare for the proof, establishing the descent theorems for vector bundles mentioned in § 1.2 above. Section 6 establishes a restriction theorem of Mehta–Ramanathan type for Higgs sheaves that is slightly more general than the versions found in the literature. This restriction theorem is then used in § 7 to prove that Higgs bundles with vanishing Chern classes on projective, klt varieties that are semistable with respect to an ample divisor remain semistable with respect to *ample* divisors when pulled back to a resolution of singularities. With these preparations in place, the nonabelian Hodge correspondences can then be constructed in § 8 without much effort.

2. Notation and elementary facts

2.1 Global conventions

Throughout the present paper, all varieties and schemes will be defined over the complex numbers. We will freely switch between the algebraic and analytic context if no confusion is likely to arise. Apart from that, we follow the notation used in the standard reference books

[Har77, KM98], with the exception that for klt pairs (X, Δ) , the boundary divisor Δ is always assumed to be effective. Varieties are always assumed to be irreducible and reduced.

2.2 Varieties, sets and morphisms

Normal varieties are S_2 , which implies that regular functions can be extended across sets of codimension two. The following notation will be used.

Notation 2.1 (Big and small subsets). Let X be a normal, quasi-projective variety. A Zariski-closed subset $Z \subset X$ is said to be *small* if $\text{codim}_X Z \geq 2$. A Zariski-open subset $U \subseteq X$ is said to be *big* if $X \setminus U$ is small. A birational morphism $\varphi : X \rightarrow Y$ of normal, projective varieties is called a *small morphism* if there exists a big open set $X^\circ \subseteq X$ such that $\varphi|_{X^\circ}$ is an open immersion.

Notation 2.2 (Set-theoretic fibre). Given a morphism of varieties $\varphi : X \rightarrow Y$ and a point $y \in Y$, we call the reduced scheme $\varphi^{-1}(y)_{\text{red}}$ the *set-theoretic fibre* of φ over y .

DEFINITION 2.3 (Covers and covering maps, Galois morphisms). A *cover* or *covering map* is a finite, surjective morphism $\gamma : Y \rightarrow X$ of normal, quasi-projective varieties or complex spaces. The covering map γ is said to be *Galois* if there exists a finite group $G \subseteq \text{Aut}(Y)$ such that X is isomorphic to the quotient map $Y \rightarrow Y/G$.

DEFINITION 2.4 (Quasi-étale morphisms). A morphism $f : X \rightarrow Y$ between normal varieties is said to be *quasi-étale* if f is of relative dimension zero and étale in codimension one. In other words, f is quasi-étale if $\dim X = \dim Y$ and there exists a closed subset $Z \subseteq X$ of codimension $\text{codim}_X Z \geq 2$ such that $f|_{X \setminus Z} : X \setminus Z \rightarrow Y$ is étale.

Note. Combining Definitions 2.3 and 2.4, a quasi-étale cover is finite, surjective, and étale in codimension one.

2.3 Line bundles and sheaves

Reflexive sheaves are in many ways easier to handle than arbitrary coherent sheaves, and we will therefore frequently take reflexive hulls. The following notation will be used.

Notation 2.5 (Reflexive hull). Given a normal, quasi-projective variety X and a coherent sheaf \mathcal{E} on X of rank r , write

$$\Omega_X^{[p]} := (\Omega_X^p)^{**}, \quad \mathcal{E}^{[m]} := (\mathcal{E}^{\otimes m})^{**}, \quad \text{Sym}^{[m]} \mathcal{E} := (\text{Sym}^m \mathcal{E})^{**}$$

and $\det \mathcal{E} := (\wedge^r \mathcal{E})^{**}$. Given any morphism $f : Y \rightarrow X$, write $f^{[*]} \mathcal{E} := (f^* \mathcal{E})^{**}$.

DEFINITION 2.6 (Relative Picard number). Given a projective surjection $f : X \rightarrow Y$ of normal, quasi-projective varieties, let $N_1(X/Y)$ be the \mathbb{R} -vector space generated by irreducible curves $C \subseteq X$ such that $f(C)$ is a point, modulo numerical equivalence. The dimension of $N_1(X/Y)$ is called the relative Picard number of X/Y and is denoted by $\rho(X/Y)$. Let $\overline{NE}(X/Y) \subseteq N_1(X/Y)$ be the closed cone generated by classes of effective curves which are contracted by f .

2.4 Cycles

The Chow variety represents a functor. The associated notion of ‘families of cycles’ is however somewhat awkward to formulate. For the sake of simplicity, we restrict ourselves to families over normal base varieties where the definition becomes somewhat simpler. The book [BM14] discusses these matters in detail.

Notation 2.7 (Families of cycles). Let X be a quasi-projective, n -dimensional variety, not necessarily normal. Given any subscheme $Y \subseteq X$, we denote the associated cycle by $[Y]$. Let $f : X \rightarrow Y$ be an equidimensional morphism of normal, algebraic varieties, of relative dimension d . Recall from [Kol96, I Theorem 3.17] or [BM14, Theorem 3.4.1 on p. 449] that f is then a well-defined family of d -dimensional proper algebraic cycles over Y , in the sense of [Kol96, I Definition 3.10]. In particular, if $y \in Y$ is any closed point, Kollár defines in [Kol96, I Definition 3.10.4] the associated cycle-theoretic fibre, which we denote by $f^{[-1]}(y)$.

Warning 2.8. In the setting of Notation 2.7, it is generally *not* true that the cycle-theoretic fibre $f^{[-1]}(y)$ is the cycle associated with the scheme-theoretic fibre $f^{-1}(y)$. Using the notation introduced above, the cycles $f^{[-1]}(y)$ and $[f^{-1}(y)]$ do not agree in general. An example is discussed in the preprint version of this paper.

Reminder 2.9 (Pull-back of Weil divisors). For arbitrary morphisms $f : X \rightarrow Y$ normal, projective varieties, there is generally no good notion of pull-back for Weil divisors. If f is finite, or more generally equidimensional, then a good pull-back map exists. We refer to [CKT16, § 2.4.1] for a discussion.

2.5 Numerical classes

We briefly fix our notation for numerical classes and intersection numbers. The following definition allows us to discuss slope and stability for arbitrary sheaves on arbitrarily singular spaces. We refer to [GKP16, § 4.1] for a more detailed discussion.

Notation 2.10 (Numerical classes and intersection numbers). Let X be a projective variety. If $\mathcal{L} \in \text{Pic}(X)$ is invertible and $D \in \text{Div}(X)$ is a Cartier divisor, we denote the numerical classes in $N^1(X)_{\mathbb{R}}$ by $[\mathcal{L}]$ and $[D]$; see [Kol96, § II.4] for a brief definition and discussion of the space $N^1(X)_{\mathbb{R}}$ of numerical Cartier divisor classes. If A is any purely d -dimensional cycle on X , we denote the intersection number with numerical classes of invertible sheaves \mathcal{L}_i by $[\mathcal{L}_1] \cdots [\mathcal{L}_d] \cdot A \in \mathbb{Z}$. For brevity of notation we will often write $[\mathcal{L}_1] \cdots [\mathcal{L}_n] \in \mathbb{Z}$ instead of the more correct $[\mathcal{L}_1] \cdots [\mathcal{L}_n] \cdot [X]$. Ditto for intersection with Cartier divisors.

For the reader’s convenience, we recall the notion of ‘numerical flatness for vector bundles’, which generalises the notion of ‘numerical triviality’ for line bundles.

DEFINITION 2.11 (Nefness and numerical flatness [DPS94, Definition 1.17]). Let $\varphi : X \rightarrow Y$ be a projective morphism of quasi-projective varieties. Given a locally free sheaf \mathcal{F} on X , consider the composed morphism

$$\mathbb{P}(\mathcal{F}) \xrightarrow{\delta} X \xrightarrow{\varphi} Y.$$

The sheaf \mathcal{F} is said to be φ -nef if $[\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)]$ is δ -nef, that is, $[\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)] \cdot C \geq 0$, for every irreducible curve $C \subset \mathbb{P}(\mathcal{F})$ such that $\delta(C)$ is a point. The sheaf \mathcal{F} is said to be φ -numerically flat if \mathcal{F} and its dual \mathcal{F}^* are both φ -nef. If Y is a point, we simply say that \mathcal{F} is numerically flat.

Remark 2.12 (Alternate formulations of numerical flatness). Setting as in Definition 2.11. The following conditions are equivalent.

- (2.12.1) The bundle \mathcal{F} and its dual \mathcal{F}^* are both φ -nef.
- (2.12.2) The bundle \mathcal{F} is φ -nef and the invertible sheaf $(\det \mathcal{F})^*$ is φ -nef.
- (2.12.3) The bundle \mathcal{F} is φ -nef and the invertible sheaf $\det \mathcal{F}$ is φ -numerically trivial.

Chern classes of numerically flat bundles vanish. The proof is based on two deep facts: the Uhlenbeck–Yau theorem asserting the existence of Hermite–Einstein metrics on stable vector bundles, and the resulting Kobayashi–Lübke flatness criterion derived from Lübke’s inequality on Chern classes of Hermite–Einstein vector bundles.

THEOREM 2.13 (Chern classes of numerically flat bundles [DPS94, Corollary 1.19]). *Let X be a smooth, projective variety and \mathcal{F} a numerically flat, locally free sheaf on X . Then all Chern classes $c_i(\mathcal{F}) \in H^{2i}(X, \mathbb{R})$ vanish.*

2.6 klt spaces, the basepoint-free theorem and contractions

We will mostly work with klt pairs (X, Δ) , but the boundary divisor Δ is usually irrelevant in our discussion. We will use the following shorthand notation throughout. We refer to [KM98] for a standard reference concerning klt pairs but recall the global convention that the boundary divisor is always assumed to be effective in this paper.

DEFINITION 2.14 (klt spaces). A normal, quasi-projective variety is called a *klt space* if there exists an effective Weil \mathbb{Q} -divisor Δ such that (X, Δ) is klt.

Note. Recall that a pair (X, Δ) as above is said to be klt if the Weil \mathbb{Q} -divisor $K_X + \Delta$ is \mathbb{Q} -Cartier, $[\Delta] = 0$, and if for one (equivalently: for every) resolution of singularities, $\pi : \tilde{X} \rightarrow X$, there exists a \mathbb{Q} -linear equivalence of the form

$$K_{\tilde{X}} + \pi_*^{-1}\Delta \sim_{\mathbb{Q}} \pi^*(K_X + \Delta) + \sum_i a_i \cdot E_i,$$

where the $E_i \subset \tilde{X}$ are π -exceptional and the numbers $a_i \in \mathbb{Q}$ satisfy $a_i > -1$ for all i .

The results of § 4 rely in part on the basepoint-free theorem and its immediate Corollary 2.16, which proves Theorem 4.1 for line bundles. We recall the statement for the reader’s convenience. Full details are found in the standard references, cf. including [KMM87, Theorem 3-1-1 and Remark 3-1-2(i)], [KM98, Theorem 3.24] and [Laz04, Theorem 2.1.27].

THEOREM 2.15 (Basepoint-free theorem). *Let $\varphi : X \rightarrow Y$ be a projective surjection of normal, quasi-projective varieties. Assume that there exists an effective Weil \mathbb{Q} -divisor Δ on X such that (X, Δ) is klt. If L is any φ -nef Cartier divisor on X and $m \in \mathbb{N}^+$ any number such that $m \cdot L - (K_X + \Delta)$ is φ -nef and φ -big, then there exists a unique factorisation via a normal variety Z ,*

$$X \begin{array}{c} \xrightarrow{\varphi} \\ \xrightarrow{\alpha} Z \xrightarrow{\beta} Y \end{array} \quad (2.15.1)$$

such that the following holds.

- (2.15.2) *The morphisms α and β are surjective. The morphism α has connected fibres.*
- (2.15.3) *The Cartier divisor L is the pull-back of a β -ample Cartier divisor L_Z on Z .*
- (2.15.4) *If $Y^\circ \subseteq Y$ is open with preimages X° and Z° , and $L|_{X^\circ}$ is $\varphi|_{X^\circ}$ -ample, then $\alpha|_{X^\circ} : X^\circ \rightarrow Z^\circ$ is isomorphic.*

Note. Item (2.15.3) implies that the morphism α contracts exactly those irreducible curves C with $[L] \cdot C = 0$ and $\dim \varphi(C) = 0$.

COROLLARY 2.16 (Descent of invertible sheaves). *Let $\varphi : X \rightarrow Y$ be a birational, projective morphism of normal, quasi-projective varieties, and let L be any φ -numerically trivial Cartier divisor on X . If there exists a Weil \mathbb{Q} -divisor Δ on X such that (X, Δ) is klt and $-(K_X + \Delta)$ is φ -nef, then L is linearly equivalent to the pull-back of a Cartier divisor on Y .*

Proof. We claim that L is φ -nef and that $D := L - (K_X + \Delta)$ is φ -nef and φ -big. Relative nefness of L and D holds by assumption. The condition that D is φ -big is void since φ is assumed to be birational. Theorem 2.15 thus gives a factorisation of φ via a normal variety Z as in (2.15.1) and a β -ample Cartier divisor L_Z on Z such that $L \sim \alpha^*L_Z$.

As a next step, we claim that β is finite. If not, we would find a curve $C \subseteq X$ which is mapped to a point by φ but not by α . The image curve $\alpha(C)$ would then be contained in a β -fibre and would thus have positive intersection with the β -ample divisor L_Z . In particular, $\deg L|_C = \deg \alpha^*(L_Z)|_C > 0$, contradicting the assumption that L is φ -numerically trivial.

In summary, see that β is both birational and finite. Zariski’s main theorem [GW10, Corollary 12.88] applies to show that β is isomorphic. Corollary 2.16 follows. \square

We list some basic properties of the contraction morphism associated with an extremal face and refer to [KMM87, Definition 3-2-3] for terminology.

PROPOSITION 2.17 (Relative contractions of extremal faces). *Assume we are given a sequence of projective surjections between normal, quasi-projective varieties $f : X \rightarrow Y$ and $Y \rightarrow Z$. Assume also that there exists an effective Weil \mathbb{Q} -divisor Δ on X such that (X, Δ) is klt.*

- (2.17.1) *If f has only connected fibres and $-(K_X + \Delta)$ is f -ample, then there exists a $(K_X + \Delta)$ -negative extremal face F of $\overline{NE}(X/Z)$ such that f is the contraction morphism of F .*
- (2.17.2) *If there exists a $(K_X + \Delta)$ -negative extremal face F of $\overline{NE}(X/Z)$ such that f is the contraction morphism of F , then relative Picard numbers are additive:*

$$\rho(X/Z) = \rho(X/Y) + \rho(Y/Z).$$

The extremal face F is an extremal ray if and only if $\rho(X/Y) = 1$.

Proof. Item (2.17.1) is [KMM87, Lemma 3-2-5(1)]. The additivity of Picard numbers in Item (2.17.2) is [KMM87, Lemma 3-2-5(3)]. It remains to consider the relation between Picard numbers and the dimension of F in Item (2.17.2).

If the dimension of F is one, it follows from the definition that $\rho(X/Y) = 1$ once we know that there is a curve in X that is mapped to a point by f . This is shown in [KMM87, Lemma 3-2-4]. As for the converse direction, if $\rho(X/Y) = 1$, the definition implies that there must be curves that are contracted. The face F cannot be empty, and it is necessarily of dimension one. \square

Warning 2.18. Picard numbers are generally not additive for compositions of arbitrary morphisms. See [KM98, §2.2] for a discussion and [KMM87, Remark 3-2-6] for an explicit example.

2.7 Sheaves with operators and Higgs sheaves

Higgs sheaves and sheaves with operators on singular varieties were defined and discussed in detail in [GKPT15, §§4 and 5]. We briefly recall the main definitions here and then discuss pull-back of Higgs sheaves.

2.7.1 Fundamental definitions.

DEFINITION 2.19 (Sheaf with an operator, invariant subsheaves [GKPT15, Definition 4.1]). Let X be a normal, quasi-projective variety and \mathcal{W} be a coherent sheaf of \mathcal{O}_X -modules. A *sheaf with a \mathcal{W} -valued operator* is a pair (\mathcal{E}, θ) where \mathcal{E} is a coherent sheaf and $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{W}$ is an \mathcal{O}_X -linear sheaf morphism.

DEFINITION 2.20 (Invariant subsheaf [GKPT15, Definition 4.8]). Setting as in Definition 2.19. A coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$ is said to be *θ -invariant* if $\theta(\mathcal{F})$ is contained in the image of the natural map $\mathcal{F} \otimes \mathcal{W} \rightarrow \mathcal{E} \otimes \mathcal{W}$. We say \mathcal{F} is *generically invariant* if the restriction $\mathcal{F}|_U$ is invariant with respect to $\theta|_U$, where $U \subseteq X$ is the maximal, dense, open subset where \mathcal{W} is locally free.

DEFINITION 2.21 (Stability of sheaves with operator [GKPT15, Definition 4.13]). Let X be a normal, projective variety and H be any nef, \mathbb{Q} -Cartier \mathbb{Q} -divisor on X . Let (\mathcal{E}, θ) be a sheaf with an operator, as in Definition 2.19, where \mathcal{E} is torsion free. We say that (\mathcal{E}, θ) is *semistable with respect to H* if the inequality $\mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E})$ holds for all generically θ -invariant subsheaves $\mathcal{F} \subseteq \mathcal{E}$ with $0 < \text{rank } \mathcal{F} < \text{rank } \mathcal{E}$. The pair (\mathcal{E}, θ) is said to be *stable with respect to H* if strict inequality holds. Direct sums of stable sheaves with operator are said to be *polystable*.

On a singular variety, some attention has to be paid concerning the definition of ‘Higgs sheaf’ at singular points. Again, we recall the relevant definitions here.

DEFINITION 2.22 (Higgs sheaf, stability, morphisms [GKPT15, Definitions 5.1 and 5.2, §5.6]). Let X be a normal variety. A *Higgs sheaf* is a pair (\mathcal{E}, θ) of a coherent sheaf \mathcal{E} of \mathcal{O}_X -modules, together with an $\Omega_X^{[1]}$ -valued operator $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^{[1]}$, called a *Higgs field*, such that the composed morphism

$$\mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \Omega_X^{[1]} \xrightarrow{\theta \otimes \text{Id}} \mathcal{E} \otimes \Omega_X^{[1]} \otimes \Omega_X^{[1]} \xrightarrow{\text{Id} \otimes [\wedge]} \mathcal{E} \otimes \Omega_X^{[2]}$$

vanishes. A Higgs sheaf is said to be *stable* if it is stable as a sheaf with an $\Omega_X^{[1]}$ -valued operator; ditto for *semistable* and *polystable*. A *morphism of Higgs sheaves*, written $f : (\mathcal{E}_1, \theta_1) \rightarrow (\mathcal{E}_2, \theta_2)$, is a morphism $f : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ of coherent sheaves that commutes with the Higgs fields, $(f \otimes \text{Id}_{\Omega_X^{[1]}}) \circ \theta_1 = \theta_2 \circ f$.

2.7.2 *Pull-back.* If $f : Y \rightarrow X$ is a morphism of normal varieties, and if (\mathcal{E}, θ) is a Higgs sheaf on X , there is generally no way to equip the pull-back sheaf $f^*\mathcal{E}$ with a Higgs field, even if \mathcal{E} is locally free. It is a non-trivial fact that pull-backs do exist for klt spaces. We refer to [GKPT15, §§5.3 and 5.4] for details. In brief, if X is klt, recall from [Keb13, Theorems 1.3 and 5.2] that there exists a natural pull-back functor for reflexive differentials on klt pairs that is compatible with the usual pull-back of Kähler differentials and gives rise to a sheaf morphism

$$d_{\text{ref}} f : f^*\Omega_X^{[1]} \rightarrow \Omega_Y^{[1]}.$$

One can then define a Higgs field on $f^*\mathcal{E}$ as the composition of the following morphisms:

$$f^*\mathcal{E} \xrightarrow{f^*\theta} f^*(\mathcal{E} \otimes \Omega_X^{[1]}) = f^*\mathcal{E} \otimes f^*\Omega_X^{[1]} \xrightarrow{\text{Id}_{f^*\mathcal{E}} \otimes d_{\text{ref}} f} f^*\mathcal{E} \otimes \Omega_Y^{[1]}. \tag{2.22.1}$$

2.8 Stability for sheaves on the smooth locus

In the situation discussed in the present paper, it makes sense to generalise the stability notions of § 2.7 to the case where the (Higgs-) sheaves are defined on the smooth locus of a normal variety only.

DEFINITION 2.23 (Slope and stability for sheaves on the smooth locus). Let X be a normal, projective variety and let \mathcal{E}° be a torsion-free, coherent sheaf on X_{reg} of positive rank. If $H \in \text{Div}(X)$ is nef, define the *slope of \mathcal{E}° with respect to H* as

$$\mu_H(\mathcal{E}^\circ) := \frac{c_1(\iota_*\mathcal{E}^\circ) \cdot [H]^{\dim X - 1}}{\text{rank } \mathcal{E}^\circ},$$

where $\iota : X_{\text{reg}} \rightarrow X$ is the inclusion.

Remark 2.24 (Algebraicity assumption). We underline that \mathcal{E}° is assumed to be algebraic in Definition 2.23. For coherent *analytic* sheaves on $X_{\text{reg}}^{\text{an}}$, the push-forward $\iota_*\mathcal{E}^\circ$ need not be coherent in general.

DEFINITION 2.25 (Slope and stability for sheaves on the smooth locus). Setting as in Definition 2.23. If \mathcal{W}° is coherent on X_{reg} and if $\theta^\circ : \mathcal{E}^\circ \rightarrow \mathcal{E}^\circ \otimes \mathcal{W}^\circ$ is a \mathcal{W}° -valued operator, we say that $(\mathcal{E}^\circ, \theta^\circ)$ is *stable with respect to H* if the inequality $\mu_H(\mathcal{F}^\circ) < \mu_H(\mathcal{E}^\circ)$ holds for all generically θ° -invariant subsheaves $\mathcal{F}^\circ \subseteq \mathcal{E}^\circ$ with $0 < \text{rank } \mathcal{F}^\circ < \text{rank } \mathcal{E}^\circ$. Analogously, define notions of semistable and polystable for sheaves with operators on X_{reg} ; ditto for Higgs sheaves.

The following two lemmas summarise properties of the generalised stability notions that will be used later. Proofs are elementary and are therefore omitted.

LEMMA 2.26 (Restriction of stable sheaves to X_{reg}). *Let X be a normal, projective variety, and let (\mathcal{E}, θ) be a torsion-free sheaf with a \mathcal{W} -valued operator. If $H \in \text{Div}(X)$ is nef, then (\mathcal{E}, θ) is semistable (respectively stable) with respect to H as a sheaf with a \mathcal{W} -valued operator if and only if $(\mathcal{E}, \theta)|_{X_{\text{reg}}}$ is semistable (respectively stable) with respect to H as a sheaf with a $\mathcal{W}|_{X_{\text{reg}}}$ -valued operator.*

LEMMA 2.27 (Extensions of operators from subsheaves). *Let X be a normal, projective variety and let $(\mathcal{E}^\circ, \theta^\circ)$ be a torsion-free sheaf on X_{reg} with a \mathcal{W}° -valued operator. Let $\iota^\circ : \mathcal{W}^\circ \hookrightarrow \mathcal{V}^\circ$ be an inclusion of coherent sheaves on X_{reg} and consider the natural \mathcal{V}° -valued operator τ° that is defined as the composition*

$$\mathcal{E}^\circ \begin{array}{c} \xrightarrow{\theta^\circ} \\ \xrightarrow{\tau^\circ} \\ \xrightarrow{\text{Id} \otimes \iota^\circ} \end{array} \mathcal{E}^\circ \otimes \mathcal{W}^\circ \rightarrow \mathcal{E}^\circ \otimes \mathcal{V}^\circ.$$

If $H \in \text{Div}(X)$ is nef, then $(\mathcal{E}^\circ, \theta^\circ)$ is semistable (respectively stable) with respect to H as a sheaf with a \mathcal{W}° -valued operator if and only if $(\mathcal{E}^\circ, \tau^\circ)$ is semistable (respectively stable) with respect to H as a sheaf with a \mathcal{V}° -valued operator.

3. The nonabelian Hodge correspondence for locally free Higgs sheaves

3.1 Main result

Generalising work of Simpson [Sim92, Corollary 3.10], in this section we formulate a nonabelian Hodge correspondence for locally free Higgs sheaves on klt spaces, relating such locally free Higgs sheaves to representations of the fundamental group. The formulation uses the fact that every algebraic variety admits a distinguished, *canonical* resolution of singularities, see [BM97, Remark 1.16ff and § 13]. The following additional notation will be used.

Notation 3.1 (Categories in the nonabelian Hodge correspondence). Given a normal, projective variety X , consider the following categories.

Higgs $_X$ Locally free Higgs sheaves (\mathcal{E}, θ) on X having the property that there exists an ample divisor $H \in \text{Div}(X)$ such that (\mathcal{E}, θ) is semistable with respect to H and additionally satisfies $ch_1(\mathcal{E}) \cdot [H]^{n-1} = ch_2(\mathcal{E}) \cdot [H]^{n-2} = 0$.

LSys $_x$ Local systems on X .

Notation 3.2 (Higgs sheaves). There is no uniform definition of Higgs sheaves on singular spaces in the literature. Throughout the present paper, we use the definition given in § 2.7.1. (Semi)stability of Higgs sheaves is defined and discussed in [GKPT15, § 5.6]. We refer to [Del70, § I.1] for a discussion of the basic properties of local systems.

Notation 3.3 (Nonabelian Hodge correspondence for manifolds). If X is smooth, Simpson’s nonabelian Hodge correspondence gives an equivalence between the categories **Higgs $_X$** and **LSys $_X$** . We denote the relevant functors by $\eta_X : \text{LSys}_X \rightarrow \text{Higgs}_X$ and $\mu_X : \text{Higgs}_X \rightarrow \text{LSys}_X$.

The following is now the main result. As we will see in § 3.2 below, the properties spelled out in (3.4.2) and (3.4.3) immediately imply that the nonabelian Hodge correspondence presented here is in fact fully functorial in morphisms of klt spaces.

THEOREM 3.4 (Nonabelian Hodge correspondence for klt spaces). *For every projective klt space X , there exists an equivalence of categories, given by functors $\eta_X : \text{LSys}_X \rightarrow \text{Higgs}_X$ and $\mu_X : \text{Higgs}_X \rightarrow \text{LSys}_X$, such that the following additional properties hold.*

(3.4.1) *If X is smooth, then η_X and μ_X equal the functors from Simpson’s nonabelian Hodge correspondence.*

(3.4.2) *If $(\mathcal{E}, \theta) \in \text{Higgs}_X$ and $\pi : \tilde{X} \rightarrow X$ is the canonical resolution of singularities, then $\pi^*(\mathcal{E}, \theta) \in \text{Higgs}_{\tilde{X}}$, and there exists a canonical isomorphism of local systems,*

$$M_{\pi, (\mathcal{E}, \theta)} : \pi^* \mu_X(\mathcal{E}, \theta) \rightarrow \mu_{\tilde{X}}(\pi^*(\mathcal{E}, \theta)).$$

(3.4.3) *If $E \in \text{LSys}_X$ is any local system and $\pi : \tilde{X} \rightarrow X$ is the canonical resolution of singularities, then there exists a canonical isomorphism of Higgs sheaves,*

$$N_{\pi, E} : \pi^* \eta_X(E) \rightarrow \eta_{\tilde{X}}(\pi^*E).$$

Note (Pull-back of Higgs sheaves). Item (3.4.2) discusses the pull-back of the Higgs sheaf $\eta_X(E)$ from X to the resolution of singularities, \tilde{X} , as discussed in § 2.7.2 above.

Theorem 3.4 is shown in § 8.1. Section 7 prepares for the proof.

3.2 Functoriality

Items (3.4.2) and (3.4.3) of Theorem 3.4 allow us to describe the functors η_X and μ_X on any given klt space X in terms of the classical nonabelian Hodge correspondence that exists on the canonical resolution of singularities. In fact, a much more general functoriality holds true.

THEOREM 3.5 (Functoriality in morphisms). *The correspondence of Theorem 3.4 is functorial in morphisms. More precisely, for every morphism $f : Y \rightarrow X$ of projective klt spaces, every $E \in \text{LSys}_X$ and every $(\mathcal{E}, \theta) \in \text{Higgs}_X$, there exist canonical isomorphisms*

$$M_{f,(\mathcal{E},\theta)} : f^* \mu_X(\mathcal{E}, \theta) \rightarrow \mu_Y(f^*(\mathcal{E}, \theta)) \quad \text{and} \quad N_{f,E} : f^* \eta_X(E) \rightarrow \eta_Y(f^*E).$$

The collection of these isomorphisms satisfies the following properties.

Functoriality: Given morphisms $g : Z \rightarrow Y$ and $f : Y \rightarrow X$ between projective klt spaces and $E \in \text{LSys}_X$, the following diagram commutes:

$$g^* f^* \eta_X(E) \begin{array}{c} \xrightarrow{g^* N_{f,E}} \\ \xrightarrow{N_{g \circ f, E}} \\ \xrightarrow{N_{g, f^*E}} \end{array} g^* \eta_Y(f^*E) \xrightarrow{N_{g, f^*E}} \eta_Z(g^* f^*E).$$

Ditto for the functor μ_\bullet and the isomorphisms $M_{\bullet,\bullet}$.

Behaviour under canonical resolution: For $\pi : \tilde{X} \rightarrow X$ the canonical resolution of a projective klt space, the morphisms $M_{\pi,\bullet}$ and $N_{\pi,\bullet}$ equal the isomorphisms given in Items (3.4.2) and (3.4.3) of Theorem 3.4.

Compatibility: If $f : Y \rightarrow X$ is a morphism between smooth projective varieties, then $M_{f,\bullet}$ and $N_{f,\bullet}$ are the standard isomorphisms given by functoriality of Simpson’s nonabelian Hodge correspondence.

Theorem 3.5 is shown in § 8.2 below.

Remark 3.6 (Uniqueness). One verifies in the blink of an eye that functoriality, behaviour under canonical resolution and compatibility determine the isomorphisms $M_{\bullet,\bullet}$ and $N_{\bullet,\bullet}$ uniquely.

Remark 3.7 (General resolutions). Theorem 3.5 implies in particular that the statement of Theorem 3.4 holds for any resolution of singularities, not just the canonical resolution. Taken together with [Tak03], for any klt space this establishes an equivalence of categories of Higgs_X with $\text{Higgs}_{\tilde{X}}$, where \tilde{X} is any resolution of singularities of X .

As a further consequence of functoriality, we observe that the nonabelian Hodge correspondence respects group actions and relates G -linearised local systems to Higgs G -sheaves in the sense of [GKPT15, Definition 5.1].

COROLLARY 3.8 (G -linearised local systems and Higgs G -sheaves). *Let X be a projective klt space, and let G be a group acting on X via a group morphism $G \rightarrow \text{Aut}(X)$.*

(3.8.1) *If $(\mathcal{E}, \theta) \in \text{Higgs}_X$ carries the structure of a Higgs G -sheaf, given by isomorphisms $\varphi_g : g^*(\mathcal{E}, \theta) \rightarrow (\mathcal{E}, \theta)$, then the following composed maps endow the local system $\mu_X(\mathcal{E}, \theta)$ with a G -linearisation:*

$$g^* \mu_X(\mathcal{E}, \theta) \xrightarrow{M_{g,(\mathcal{E},\theta)}} \mu_X(g^*(\mathcal{E}, \theta)) \xrightarrow{\mu_X(\varphi_g)} \mu_X(\mathcal{E}, \theta).$$

(3.8.2) *If $E \in \text{LSys}_X$ carries a G -linearisation given by isomorphisms $\varphi_g : g^*E \rightarrow E$, then the following composed maps endow $\eta_X(E)$ with the structure of a Higgs G -sheaf:*

$$g^* \eta_X(E) \xrightarrow{N_{g,E}} \eta_X(g^*E) \xrightarrow{\eta_X(\varphi_g)} \eta_X(E).$$

3.3 Independence of polarisation

As in Simpson’s original setup, a Higgs bundle on a klt space X is in Higgs_X if and only if it satisfies the conditions of Notation 3.1 with respect to any ample class. The following proposition makes this assertion precise.

THEOREM 3.9 (Independence of polarisation). *Let X be a projective klt space of dimension n . Given any locally free Higgs sheaf (\mathcal{E}, θ) on X , the following statements are equivalent.*

- (3.9.1) *There exists an ample divisor $H \in \text{Div}(X)$ such that $ch_1(\mathcal{E}) \cdot [H]^{n-1} = ch_2(\mathcal{E}) \cdot [H]^{n-2} = 0$ and such that (\mathcal{E}, θ) is semistable with respect to H .*
- (3.9.2) *For all ample divisors $H \in \text{Div}(X)$, we have $ch_1(\mathcal{E}) \cdot [H]^{n-1} = ch_2(\mathcal{E}) \cdot [H]^{n-2} = 0$ and (\mathcal{E}, θ) is semistable with respect to H .*
- (3.9.3) *All Chern classes $c_i(\mathcal{E}) \in H^{2i}(X, \mathbb{Q})$ vanish and (\mathcal{E}, θ) is semistable with respect to any ample divisor on X .*
- (3.9.4) *There exists a resolution of singularities, $\pi : \tilde{X} \rightarrow X$, and an ample divisor $\tilde{H} \in \text{Div}(\tilde{X})$ such that $ch_1(\pi^* \mathcal{E}) \cdot [\tilde{H}]^{n-1} = ch_2(\pi^* \mathcal{E}) \cdot [\tilde{H}]^{n-2} = 0$ and such that $\pi^*(\mathcal{E}, \theta)$ is semistable with respect to \tilde{H} .*
- (3.9.5) *For any resolution of singularities $\pi : \tilde{X} \rightarrow X$ and any ample divisor $\tilde{H} \in \text{Div}(\tilde{X})$, we have intersection numbers $ch_1(\pi^* \mathcal{E}) \cdot [\tilde{H}]^{n-1} = ch_2(\pi^* \mathcal{E}) \cdot [\tilde{H}]^{n-2} = 0$, and $\pi^*(\mathcal{E}, \theta)$ is semistable with respect to \tilde{H} .*

The analogous equivalences hold when ‘semistable’ is replaced by ‘stable’ or ‘polystable’.

Theorem 3.9 is shown in § 8.3 below.

3.4 Harmonic bundles and differential graded categories

Simpson constructs his nonabelian Hodge correspondence first in the case of polystable Higgs bundles and semisimple local systems. In this setup, the correspondence is a consequence of existence theorems for pluri-harmonic metrics on the underlying bundles. Given their central role in the theory, we remark that the nonabelian Hodge correspondence for klt spaces, Theorem 3.4, also has a description in terms of harmonic structures, although in our case the harmonic metric exists on the smooth locus of the underlying space only. The proof of the following proposition is simple and therefore omitted. Item (3.4.3) of Theorem 3.4 allows us to relate the correspondence on X to that on a resolution.

PROPOSITION 3.10 (Hodge correspondence for klt spaces via harmonic bundles). *Let X be a projective, klt space and $\mathbf{E} \in \text{LSys}_X$ a semisimple local system on X , with underlying C^∞ -bundle E . We claim that there exists a tame and purely imaginary harmonic bundle $(E^\circ, \bar{\partial}_{E^\circ}, \theta^\circ, h^\circ)$ on X_{reg} with the following two properties.*

- (3.10.1) *The induced flat bundle $(E^\circ, \nabla_{h^\circ})$ corresponds to the local system $\mathbf{E}|_{X_{\text{reg}}}$.*
- (3.10.2) *Writing \mathcal{E}° for the sheaf of holomorphic sections in $(E^\circ, \bar{\partial}_{E^\circ})$, the functor η_X of the nonabelian Hodge correspondence satisfies $(\mathcal{E}^\circ, \theta^\circ) \cong \eta_X(\mathbf{E})|_{X_{\text{reg}}}$.*

Remark 3.11. We refer to [Sab13, § 1] for a discussion of harmonic bundles, and to [Sim90, p. 723] and [Moc07b, § 22.1] for the notions of ‘tame’ and ‘purely imaginary’.

As a second point, we note that Theorem 3.4 includes equivalences of differential graded categories (DGCs) that appear in Simpson’s nonabelian Hodge theory. For detailed discussion

of DGCs and their relation to this theory we refer the reader to [Sim92, § 3] and the references therein. Given a resolution $\pi : \tilde{X} \rightarrow X$, Theorem 3.4 implies that there is an equivalence of DGCs between the category of extensions of stable Higgs bundles with vanishing Chern classes on X and the category of flat connections on \tilde{X} . The same conclusions as in the smooth projective setting then follow.

4. Descent of vector bundles from klt spaces

The proof of Theorem 5.1, our main result concerning descent of vector bundles to klt spaces, relies on the following auxiliary statement, which we prove in this section.

THEOREM 4.1 (Descent of vector bundles from klt spaces). *Let $\varphi : X \rightarrow Y$ be a projective, birational morphism of normal, quasi-projective varieties. Assume that there exists a Weil \mathbb{Q} -divisor Δ_X on X such that the pair (X, Δ_X) is klt and $-(K_X + \Delta_X)$ is φ -nef. If \mathcal{F}_X is any locally free, φ -numerically flat sheaf on X , then there exists a locally free sheaf \mathcal{F}_Y on Y such that $\mathcal{F}_X \cong \varphi^* \mathcal{F}_Y$.*

4.1 Proof of Theorem 4.1: setup and notation

We maintain the notation and assumptions of Theorem 4.1 throughout § 4. To avoid trivial cases, we may assume throughout the proof that the rank of \mathcal{F}_X is positive. Set $r := (\text{rank } \mathcal{F}_X) - 1$. Using Grothendieck’s terminology, we consider the associated \mathbb{P}^r -bundle $\mathbb{P}_X := \mathbb{P}_X(\mathcal{F}_X)$. The following diagram summarises the situation:

$$\begin{array}{ccc}
 \mathbb{P}_X & & \\
 \rho_X \downarrow & \searrow \delta & \\
 X & \xrightarrow{\varphi, \text{birational}} & Y.
 \end{array} \tag{4.1.1}$$

Since ρ_X is a locally trivial \mathbb{P}^r -bundle, the variety \mathbb{P}_X is normal, and the pair $(\mathbb{P}_X, \Delta_{\mathbb{P}_X})$ is klt, where $\Delta_{\mathbb{P}_X} := \rho_X^* \Delta_X$. Let $Y^\circ \subseteq Y$ be the maximal open set over which φ is isomorphic, and write $X^\circ := \varphi^{-1}(Y^\circ)$. As Y is normal, the subset Y° is big.

We will also consider the invertible sheaves $\mathcal{L}_X := \mathcal{O}_{\mathbb{P}_X(\mathcal{F}_X)}(1)$ and $\mathcal{M}_X := \det \mathcal{F}_X$. The assumption that \mathcal{F}_X is φ -numerically flat has immediate consequences for \mathcal{L}_X and \mathcal{M}_X , which we state for later reference.

Observation 4.2.

(4.2.1) The sheaf \mathcal{L}_X is δ -nef.

(4.2.2) The sheaf \mathcal{M}_X is φ -numerically trivial. In particular, Corollary 2.16 implies the existence of an invertible sheaf $\mathcal{M}_Y \in \text{Pic}(Y)$ such that $\mathcal{M}_X \cong \varphi^* \mathcal{M}_Y$.

For convenience of notation, choose Cartier divisors L_X and M_Y representing the bundles \mathcal{L}_X and \mathcal{M}_Y . The Cartier divisor $M_X := \varphi^* M_Y$ will then represent \mathcal{M}_X .

4.2 Proof of Theorem 4.1: factorisation of δ

We aim to construct a locally free sheaf \mathcal{F}_Y on Y . Rather than doing so directly, we will first construct a factorisation of δ via a morphism $\rho_Y : \mathbb{P}_Y \rightarrow Y$ that agrees with \mathbb{P}_X over the big open set where Y° . Later, we will show that ρ_Y is equidimensional, and has in fact the structure of a linear \mathbb{P}^r -bundle. The sheaf \mathcal{F}_Y will then be constructed as the push-forward of the relative hyperplane bundle on \mathbb{P}_Y .

CLAIM 4.3 (Construction of \mathbb{P}_Y). *There exists a commutative diagram of surjective projective morphisms with connected fibres extending Diagram (4.1.1) as follows:*

$$\begin{array}{ccc}
 \mathbb{P}_X & \xrightarrow{\Phi} & \mathbb{P}_Y \\
 \rho_X \downarrow & \searrow \delta & \downarrow \rho_Y \\
 X & \xrightarrow{\varphi, \text{birational}} & Y
 \end{array} \tag{4.3.1}$$

as well as a ρ_Y -ample sheaf $\mathcal{L}_Y \in \text{Pic}(\mathbb{P}_Y)$ such that $\mathcal{L}_X \cong \Phi^* \mathcal{L}_Y$. The variety \mathbb{P}_Y and the morphisms of Diagram (4.3.1) are unique up to isomorphism, and the following holds in addition.

(4.3.2) *The restricted morphism $\Phi|_{\rho_X^{-1}(X^\circ)} : \rho_X^{-1}(X^\circ) \rightarrow \rho_Y^{-1}(Y^\circ)$ is isomorphic. In particular, Φ is birational.*

(4.3.3) *We have an isomorphism of sheaves $\mathcal{M}_Y \cong \det((\rho_Y)_* \mathcal{L}_Y)$.*

Proof of Claim 4.3. The factorisation of δ via an intermediate variety \mathbb{P}_Y will be constructed using Theorem 2.15 (‘Basepoint-free theorem’). To apply the theorem, it suffices to show that L_X is δ -nef and that

$$D := L_X - (K_{\mathbb{P}_X} + \Delta_{\mathbb{P}_X})$$

is δ -nef and δ -big. Relative nefness of L_X is clear from Observation (4.2.1). To analyse D , we use the standard formula for the canonical bundle of a projectivised vector bundle to obtain a \mathbb{Q} -linear equivalence of \mathbb{Q} -divisors,

$$D \sim_{\mathbb{Q}} (r + 2) \cdot L_X - \rho_X^*(K_X + \Delta_X + M_X). \tag{4.3.4}$$

The divisor L_X is ρ_X -ample and therefore ample on the general fibre of δ . Since φ is birational, (4.3.4) implies that D is δ -big. Relative nefness of D also follows from (4.3.4), using Observations (4.2.1) and (4.2.2) as well as the assumption that $-(K_X + \Delta_X)$ is φ -nef. Theorem 2.15 thus applies and yields a unique factorisation as in Diagram (4.3.1) as well as a ρ_Y -ample sheaf \mathcal{L}_Y such that $\mathcal{L}_X \cong \Phi^* \mathcal{L}_Y$.

Item (4.3.2) follows from Item (2.15.4) of Theorem 2.15, using the fact that D is ρ_X -ample over X° and therefore δ -ample over Y° . It remains to show that $\mathcal{M}_Y \cong \det((\rho_Y)_* \mathcal{L}_Y)$. By Item (4.3.2), an isomorphism exists at least over the big open set Y° . But since both sides of the equation are reflexive, the isomorphism exists globally. This ends the proof of Claim 4.3. \square

4.3 Proof of Theorem 4.1: equidimensionality

As indicated above, we will now show that ρ_Y is equidimensional. This is the point where the assumption that \mathcal{F}_X is φ -numerically flat is used in a crucial way.

CLAIM 4.4 (Equidimensionality of ρ_Y). *The morphism ρ_Y is equidimensional, of relative dimension r .*

Proof of Claim 4.4. We argue by contradiction and assume that there exists a point $y \in Y$ whose fibre $\rho_Y^{-1}(y)$ contains a subvariety Z of dimension $r + 1$. Recalling that \mathcal{L}_Y is ρ_Y -ample, we have a positive intersection number

$$[\mathcal{L}_Y]^{r+1} \cdot [Z] > 0. \tag{4.4.1}$$

We will see that this is absurd. To this end, let $F \subseteq \varphi^{-1}(y)$ be any irreducible component. Choose a desingularisation $\pi : \tilde{F} \rightarrow F$ and extend Diagram (4.3.1) to the left by taking fibre products as follows:

$$\begin{array}{ccccccc}
 \mathbb{P}_{\tilde{F}} & \xrightarrow{\Pi} & \mathbb{P}_F & \hookrightarrow & \mathbb{P}_X & \xrightarrow{\Phi} & \mathbb{P}_Y \\
 \rho_{\tilde{F}} \downarrow & & \rho_F \downarrow & & \rho_X \downarrow & & \rho_Y \downarrow \\
 \tilde{F} & \xrightarrow{\pi} & F & \hookrightarrow & X & \xrightarrow{\varphi} & Y
 \end{array}
 \qquad
 \begin{array}{ccccccc}
 \mathbb{P}_{\tilde{F}} & \xrightarrow{\Pi} & \mathbb{P}_F & \xrightarrow{\Phi|_{\mathbb{P}_F}} & \rho_Y^{-1}(y) & \hookrightarrow & \mathbb{P}_Y \\
 \rho_{\tilde{F}} \downarrow & & \rho_F \downarrow & & \downarrow & & \downarrow \rho_Y \\
 \tilde{F} & \xrightarrow{\pi} & F & \xrightarrow{\varphi|_F} & \{y\} & \hookrightarrow & Y
 \end{array}$$

Setting $\mathcal{F}_{\tilde{F}} := \pi^* \mathcal{F}_X$ and $\mathcal{L}_{\tilde{F}} := \Pi^* \mathcal{L}_X$, we obtain identifications

$$\mathbb{P}_{\tilde{F}} \cong \mathbb{P}_{\tilde{F}}(\mathcal{F}_{\tilde{F}}) \quad \text{and} \quad \mathcal{L}_{\tilde{F}} \cong \mathcal{O}_{\mathbb{P}_{\tilde{F}}}(\mathcal{F}_{\tilde{F}})(1).$$

If $Z_{\tilde{F}} \subseteq \mathbb{P}_{\tilde{F}}$ is any $(r + 1)$ -dimensional subvariety that dominates Z , (4.4.1) immediately implies that

$$[\mathcal{L}_{\tilde{F}}]^{r+1} \cdot [Z_{\tilde{F}}] = [\Pi^* \Phi^* \mathcal{L}_Y]^{r+1} \cdot [Z_{\tilde{F}}] > 0. \tag{4.4.2}$$

On the other hand, as the pull-back of a numerically flat bundle, $\mathcal{F}_{\tilde{F}}$ is numerically flat. Theorem 2.13 hence implies that all Chern classes $c_i(\mathcal{F}_{\tilde{F}})$ of $\mathcal{F}_{\tilde{F}}$ vanish. A standard Chern class computation on projectivised vector bundles [Ful98, Remark 3.2.4 on p. 55] thus gives

$$c_1(\mathcal{L}_{\tilde{F}})^{r+1} = - \sum_{i=1}^r \underbrace{c_i(\rho_{\tilde{F}}^* \mathcal{F}_{\tilde{F}})}_{=0} \cdot c_1(\mathcal{L}_{\tilde{F}})^{r+1-i} = 0. \tag{4.4.3}$$

Items (4.4.2) and (4.4.3) are obviously in contradiction. The assumption that $\rho_Y^{-1}(y)$ contains an $(r + 1)$ -dimensional subvariety is thus absurd. In summary, we obtain that ρ_Y is equidimensional, thus finishing the proof of Claim 4.4. \square

Building on work of Kollár and Höring-Novelli, it has been shown by Araujo and Druel [AD14, Proposition 4.10] that equidimensionality and the existence of the relatively ample sheaf \mathcal{L}_Y whose restriction to general ρ_Y -fibres is the hyperplane bundle implies that ρ_Y has the structure of a linear bundle. We briefly recall the argument.

CLAIM 4.5 (Linear bundle structure of ρ_Y). *The sheaf $\mathcal{F}_Y := (\rho_Y)_* \mathcal{L}_Y$ is locally free on Y . The morphism $\rho_Y : \mathbb{P}_Y \rightarrow Y$ can be identified as the projection $\mathbb{P}_Y(\mathcal{F}_Y) \rightarrow Y$. The invertible sheaf \mathcal{L}_Y becomes $\mathcal{O}_{\mathbb{P}_Y(\mathcal{F}_Y)}(1)$ under this identification.*

Proof. By (4.3.2), we know that the general fibre $\mathbb{P}_{Y,y}$ of ρ_Y is isomorphic to \mathbb{P}^r , with $\mathcal{L}_Y|_{\mathbb{P}_{Y,y}} \cong \mathcal{O}_{\mathbb{P}^r}(1)$. Since \mathcal{L}_Y is ρ_Y -ample, [HN13, Proposition 3.1] applies to guarantee that in fact all fibres of ρ_Y are irreducible and generically reduced, and that the normalisation of any fibre is isomorphic to \mathbb{P}^r . In particular, the Hilbert polynomial of the normalisation of the fibres is constant, and [Kol11, Theorem 12] applies to show that the variety \mathbb{P}_Y admits a simultaneous normalisation, which is a finite, birational morphism $\eta : \hat{\mathbb{P}}_Y \rightarrow \mathbb{P}_Y$ with the property that the morphism $\rho_Y \circ \eta$ is flat and that all its fibres are normal. But since \mathbb{P}_Y already is normal, Zariski’s main theorem [Har77, V Theorem 5.2] applies to show that η is isomorphic, and all fibres of ρ_Y are therefore smooth, and isomorphic to \mathbb{P}^r . Grauert’s theorem [Har77, III Corollary 12.9] thus shows that \mathcal{F}_Y is locally free and that the natural morphism $(\rho_Y)^* \mathcal{F}_Y = (\rho_Y)^*(\rho_Y)_* \mathcal{L}_Y \rightarrow \mathcal{L}_Y$ is surjective. The universal property of projectivisation [Har77, II Proposition 7.12] thus gives a morphism $\alpha : \mathbb{P}_Y \rightarrow \mathbb{P}_Y(\mathcal{F}_Y)$ that identifies \mathcal{L}_Y with the pull-back of $\mathcal{O}_{\mathbb{P}_Y(\mathcal{F}_Y)}(1)$. But since the identification $\mathcal{L}_Y|_{(\rho_Y)^{-1}(y)} \cong \mathcal{O}_{\mathbb{P}^r}(1)$ now holds for every $y \in Y$, the morphism α is clearly bijective, and hence isomorphic by Zariski’s main theorem. \square

4.4 Proof of Theorem 4.1: end of proof

It remains to show that $\mathcal{F}_X \cong \varphi^* \mathcal{F}_Y$. Recalling that

$$\mathcal{F}_Y = (\rho_Y)_* \mathcal{L}_Y \quad \text{and} \quad \mathcal{F}_X = (\rho_X)_* \mathcal{L}_X = (\rho_X)_* \Phi^* \mathcal{L}_Y,$$

there exists a natural morphism $\alpha : \varphi^* \mathcal{F}_Y \rightarrow \mathcal{F}_X$, cf. [Har77, III Remark 9.3.1]. The restriction of the morphism α to the open set X° is clearly isomorphic. We claim that it is isomorphic everywhere. To this end, consider its determinant $\det \alpha : \varphi^* \mathcal{M}_Y \rightarrow \mathcal{M}_X$. Using that $\varphi^* \mathcal{M}_Y \cong \mathcal{M}_X$, the determinant can be seen as a section in $\mathcal{H}om(\mathcal{M}_X, \mathcal{M}_X) \cong \mathcal{O}_X$, and hence as a function on X that does not vanish on X° . But since the φ -exceptional set $E := X \setminus X^\circ$ is contracted to the small subvariety $Y \setminus Y^\circ$ of Y , the function in fact cannot vanish anywhere. It follows that the morphism $\det \alpha$ is isomorphic, and hence so is α . This ends the proof of Theorem 4.1. \square

5. Descent of vector bundles to klt spaces

In this section, we will prove the following theorem, a simplified form of which appeared as Theorem 1.2 in the introduction.

THEOREM 5.1 (Descent of vector bundles to klt spaces). *Let $f : X \rightarrow Y$ be a birational, projective morphism of normal, quasi-projective varieties. Assume Y to be a klt space. Then the following holds.*

- (5.1.1) *If \mathcal{F}_X is any locally free, f -numerically flat sheaf on X , then there exists a locally free sheaf \mathcal{F}_Y on Y such that $\mathcal{F}_X \cong f^* \mathcal{F}_Y$.*
- (5.1.2) *If $(\mathcal{F}_X, \theta_X)$ is any locally free Higgs sheaf on X , where \mathcal{F}_X is f -numerically flat, then there exists a locally free Higgs sheaf $(\mathcal{F}_Y, \theta_Y)$ on Y such that $(\mathcal{F}_X, \theta_X) \cong f^*(\mathcal{F}_Y, \theta_Y)$.*

5.1 Proof of Theorem 5.1: setup and notation

We maintain the notation and assumptions of Theorem 5.1 throughout §5. Choose an effective Weil \mathbb{Q} -divisor Δ_Y such that (Y, Δ_Y) is klt. We denote by $\Delta_X := f_*^{-1} \Delta_Y$ the strict transform of Δ_Y and by $E_X := \text{Exc}(f)$ the divisorial part of the f -exceptional locus $\text{Exc}(f)$.

5.2 Proof of Statement (5.1.1)

Consider a resolution of singularities $\pi : \tilde{X} \rightarrow X$. If we can show that $\pi^* \mathcal{F}_X$ is of the form $(f \circ \pi)^* \mathcal{F}_Y$ for a suitable sheaf \mathcal{F}_Y on Y , then \mathcal{F}_X will be isomorphic to $f^* \mathcal{F}_Y$ by the projection formula. We are therefore free to replace X by \tilde{X} and assume without loss of generality that the following holds.

ASSUMPTION W.L.O.G. 5.2. The variety X is smooth, the f -exceptional set equals E_X , and $\text{supp}(\Delta_X + E_X)$ is a simple normal crossing divisor in X .

Step 1: Factorisation via an f -relative MMP. We will factor the resolution $f : X \rightarrow Y$ via a relative minimal model program (MMP) of X over Y , cf. the discussion in [GKKP11, §23], whose organisation we will follow closely. By the definition of ‘klt pair’, there exist effective f -exceptional divisors F and G without common components such that $[F] = 0$ and such that the following \mathbb{Q} -linear equivalence holds:

$$K_X + \Delta_X + F \sim_{\mathbb{Q}} f^*(K_Y + \Delta_Y) + G.$$

For $\varepsilon \in (0, 1) \cap \mathbb{Q}$ we let $\Delta_\varepsilon := \Delta_X + F + \varepsilon \cdot E$. For $0 < \varepsilon \ll 1$ small enough, the pair (X, Δ) is klt. Fix one such ε and let $H \in \mathbb{Q} \operatorname{Div}(X)$ be an f -ample divisor such that $(X, \Delta_\varepsilon + H)$ is still klt and $K_X + \Delta_\varepsilon + H$ is f -nef. We may then run the f -relative (X, Δ_ε) MMP with scaling of H , cf. [BCHM10, Corollary 1.4.2] to obtain a diagram

$$\begin{array}{ccccccc}
 X & \xlongequal{\quad} & X_n & \xrightarrow{\varphi_n} & X_{n-1} & \xrightarrow{\varphi_{n-1}} & \cdots & \xrightarrow{\varphi_1} & X_0 \\
 f \downarrow & & f_n \downarrow & & f_{n-1} \downarrow & & & & f_0 \downarrow \\
 Y & \xlongequal{\quad} & Y & \xlongequal{\quad} & Y & \xlongequal{\quad} & \cdots & \xlongequal{\quad} & Y
 \end{array}$$

with the following properties.

- (5.2.1) The spaces X_i are \mathbb{Q} -factorial. Writing Δ_{X_i} for the cycle-theoretic image of Δ_ε , the pairs (X_i, Δ_{X_i}) are klt.
- (5.2.2) The maps φ_i are either divisorial contractions of $(K_{X_i} + \Delta_{X_i})$ -negative extremal rays in $R_i \subseteq \overline{NE}(X_i/Y)$ or flips associated to small contractions of such rays.
- (5.2.3) The log-canonical divisor $K_{X_0} + \Delta_{X_0}$ is f_0 -nef, and it hence follows from the negativity lemma [GKKP11, Lemma 2.16.2] that the morphism f_0 is small and crepant, cf. [GKKP11, Claim 23.4]. In other words, $\Delta_{X_0} = (f_0)_*^{-1}(\Delta_Y)$ and $K_{X_0} + \Delta_{X_0} \sim_{\mathbb{Q}} (f_0)^*(K_Y + \Delta_Y)$. As a consequence, also $-(K_{X_0} + \Delta_{X_0})$ is f_0 -nef.

Step 2: Construction of bundles. Next, we construct vector bundles on the X_i .

CLAIM 5.3. *There exist locally free sheaves \mathcal{F}_{X_i} on the varieties X_i such that the following holds.*

- (5.3.1) *The sheaf \mathcal{F}_{X_n} equals \mathcal{F}_X .*
- (5.3.2) *Given any index i , the sheaf \mathcal{F}_{X_i} is f_i -numerically flat.*
- (5.3.3) *If $i > 0$ is any index such that $\mathcal{F}_{X_{i-1}}$ is isomorphic to $(f_{i-1})^*\mathcal{G}$ for a locally free sheaf \mathcal{G} on Y , then $\mathcal{F}_{X_i} \cong (f_i)^*\mathcal{G}$.*

Proof of Claim 5.3. We construct the vector bundles inductively. Start by setting $\mathcal{F}_{X_n} := \mathcal{F}_X$. Next, assume that we are given an index $i > 0$ for which vector bundles $\mathcal{F}_{X_n}, \dots, \mathcal{F}_{X_i}$ have already been constructed. We consider separately the cases where φ_i is a divisorial contraction and where it is a flip.

Divisorial contraction. If $\varphi_i : X_i \rightarrow X_{i-1}$ is a divisorial contraction, then $-(K_{X_i} + \Delta_{X_i})$ is φ_i -nef. Theorem 4.1 (‘Descent of vector bundles from klt spaces’) hence proves the existence of a locally free sheaf $\mathcal{F}_{X_{i-1}}$ on X_{i-1} such that $\mathcal{F}_{X_i} \cong \varphi_i^*\mathcal{F}_{X_{i-1}}$. This isomorphism guarantees that Properties (5.3.2) and (5.3.3) both hold.

Flip. If $\varphi_i : X_i \dashrightarrow X_{i-1}$ is a flip, consider the associated ‘flipping diagram’

$$\begin{array}{ccccc}
 & & \varphi_i & & \\
 & & \text{---} & & \\
 X_i & \xrightarrow{\alpha} & Z & \xleftarrow{\beta} & X_{i-1} \\
 & \searrow f_i & \downarrow \gamma & \swarrow f_{i-1} & \\
 & & Y & &
 \end{array}$$

where α is obtained by contracting a $(K_{X_i} + \Delta_{X_i})$ -extremal ray, which implies as above that $-(K_{X_i} + \Delta_{X_i})$ is α -nef. In this setting, Theorem 4.1 again proves the existence of a locally free

sheaf \mathcal{F}_Z on Z such that $\mathcal{F}_{X_i} \cong \alpha^* \mathcal{F}_Z$. Properties (5.3.2) and (5.3.3) will hold once we set $\mathcal{F}_{X_{i-1}} := \beta^* \mathcal{F}_Z$. This finishes the proof of Claim 5.3. \square

Step 3: End of proof. To end the proof of Statement (5.1.1), recall from (5.2.3) that (X_0, Δ_{X_0}) is klt and that $-(K_{X_0} + \Delta_{X_0})$ is f_0 -nef. Since \mathcal{F}_{X_0} is f_0 -numerically flat by (5.3.2), we may therefore apply Theorem 4.1 to obtain a locally free sheaf \mathcal{F}_Y on Y such that $\mathcal{F}_{X_0} \cong (f_0)^* \mathcal{F}_Y$. A repeated application of Property (5.3.3) then shows that the sheaves $\mathcal{F}_{X_1}, \mathcal{F}_{X_2}, \dots, \mathcal{F}_{X_n} = \mathcal{F}$ are all pull-backs of \mathcal{F}_Y . Statement (5.1.1) follows.

5.3 Proof of Statement (5.1.2)

For $\bullet \in \{X, Y\}$, set

$$\mathcal{A}_\bullet^i := \mathcal{H}om(\mathcal{F}_\bullet, \mathcal{F}_\bullet \otimes \Omega_\bullet^{[i]}).$$

A Higgs field on \mathcal{F}_Y is, by definition, a section of the sheaf \mathcal{A}_Y^1 such that the induced section in \mathcal{A}_Y^2 vanishes. In our case, the sheaf \mathcal{F}_Y is locally free, which implies that both \mathcal{A}_Y^1 and \mathcal{A}_Y^2 are reflexive. To give a Higgs field on \mathcal{F}_Y it is therefore equivalent to give a Higgs field on the restriction of \mathcal{F}_Y to any big open subset of Y . The Higgs field θ_X , however, clearly induces a Higgs field on the restriction of \mathcal{F}_Y to the big open set where f^{-1} is well defined and isomorphic. The existence of θ_Y follows.

It remains to show that $f^*(\mathcal{F}_Y, \theta_Y) \cong (\mathcal{F}_X, \theta_X)$. In other words, we need to show that the two sections θ_X and $f^*\theta_Y \in H^0(X, \mathcal{A}_X^1)$ agree. They will clearly agree over the open set $f^{-1}(Y^\circ)$. Since \mathcal{F}_X is locally free and \mathcal{A}_X^1 therefore reflexive, this suffices to show that they are the same. This finishes the proof of Theorem 5.1. \square

6. The restriction theorem for semistable Higgs sheaves

The proof of the nonabelian Hodge correspondence uses the following restriction theorem for (semi)stable Higgs sheaves, which generalises a number of earlier results including [GKPT15, Theorem 5.22].

THEOREM 6.1 (Restriction of (semi)stable Higgs sheaves). *Let X be a normal, projective variety, $\dim X \geq 2$, and let $H \in \text{Div}(X)$ be big and semiample. Given any torsion-free Higgs sheaf $(\mathcal{E}^\circ, \theta^\circ)$ on X_{reg} that is semistable (respectively stable) with respect to H in the sense of Definition 2.25, there exists an integer $M \in \mathbb{N}^+$ satisfying the following conditions. If $\mathbb{B} \subseteq |m \cdot H|$ is any basepoint-free linear system with $m > M$, then there exists a dense, open subset $\mathbb{B}^\circ \subseteq \mathbb{B}$ such that the following properties hold for all $D \in \mathbb{B}^\circ$.*

(6.1.1) *The hypersurface D is irreducible and normal, and $D_{\text{reg}} = D \cap X_{\text{reg}}$.*

(6.1.2) *The Higgs sheaf $(\mathcal{E}^\circ, \theta^\circ)|_{D_{\text{reg}}}$ is torsion-free and semistable (respectively stable) with respect to $H|_D$.*

Remark 6.2 (Algebraicity assumption). We stress that the Higgs sheaf $(\mathcal{E}^\circ, \theta^\circ)$ of Theorem 6.1 is assumed to be algebraic.

Remark 6.3 (Restriction theorem for sheaves on X). Recalling from Lemma 2.26 that a Higgs sheaf on X is semistable (respectively stable) if and only if its restriction to X_{reg} is semistable (respectively stable), Theorem 6.1 immediately implies a restriction theorem for torsion-free Higgs sheaves on X that is more general than the results found in the literature. In practical applications, the variety X might admit a finite group action, and the linear system $\mathbb{B} \subseteq |m \cdot H|$ might be chosen to contain invariant divisors only.

Theorem 6.1 is shown in § 6.2 below. The following corollary discusses the behaviour of semistability under pull-back. It complements [GKPT15, § 5.6], where the (G -)stable case was discussed. Its proof, spelled out in the arXiv version of this paper, arXiv:1711.08159, applies Theorem 6.1 repeatedly to cut down to a curve, where the result is classically known.

COROLLARY 6.4 (Semistability under generically finite morphisms). *Let X and Y be two projective, klt spaces. Let $H \in \text{Div}(X)$ be big and semiample, and let $f : Y \rightarrow X$ be a surjective and generically finite morphism. Let (\mathcal{E}, θ) be a reflexive Higgs sheaf on X .*

- *If \mathcal{E} is locally free, then the following are equivalent.*

(6.4.1) *The Higgs bundle (\mathcal{E}, θ) is semistable with respect to H .*

(6.4.2) *The Higgs bundle $f^*(\mathcal{E}, \theta)$ is semistable with respect to f^*H .*

- *If Y is smooth, then the following are equivalent.*

(6.4.3) *The Higgs sheaf (\mathcal{E}, θ) is semistable with respect to H .*

(6.4.4) *The Higgs sheaf $f^{[*]}(\mathcal{E}, \theta)$ is semistable with respect to f^*H .*

Remark 6.5. One might wonder why the assumptions in Corollary 6.4 are so much more restrictive compared to Theorem 6.1. If X is not klt, then pull-back of Higgs sheaves does not exist in general. If the sheaf \mathcal{E} is not locally free and Y is not smooth, its pull-back is generally neither reflexive nor torsion-free, and no good notion of ‘stability’ is defined in this case. Also, we do not know whether the reflexive pull-back $f^{[*]}\mathcal{E}$ carries a natural Higgs field in this case.

6.1 Restriction theorem for sheaves with operators

The following restriction theorem for sheaves with operators is a generalisation of [GKPT15, Theorem A.3 in the arXiv version, arXiv:1511.08822]. It serves as the main technical tool used in the proof of the restriction theorem for Higgs sheaves, Theorem 6.1.

THEOREM 6.6 (Restriction theorem for sheaves with operators). *Let X be a normal, projective variety, $\dim X \geq 2$, let H be a big and semiample divisor on X , and let \mathcal{W}° be a reflexive sheaf on X_{reg} . Let $(\mathcal{E}^\circ, \theta^\circ)$ be a torsion free sheaf on X_{reg} with a \mathcal{W}° -valued operator, and assume that $(\mathcal{E}^\circ, \theta^\circ)$ is semistable (respectively stable) with respect to H . Then there exists $M \in \mathbb{N}^+$ satisfying the following conditions. If $\mathbb{B} \subseteq |m \cdot H|$ is any basepoint-free linear system with $m > M$, then there exists a dense, open subset $\mathbb{B}^\circ \subseteq \mathbb{B}$ such that the following properties hold for all $D \in \mathbb{B}^\circ$.*

(6.6.1) *The hypersurface D is irreducible and normal, and $D_{\text{reg}} = X_{\text{reg}} \cap D$.*

(6.6.2) *The sheaf $\mathcal{E}^\circ|_{D_{\text{reg}}}$ is torsion-free, and $(\mathcal{E}^\circ|_{D_{\text{reg}}}, \theta^\circ|_{D_{\text{reg}}})$ is semistable (respectively stable) with respect to $H|_D$, as a sheaf with a $\mathcal{W}|_{D_{\text{reg}}}$ -valued operator.*

Remark 6.7 (Algebraicity and restriction theorem for sheaves on X). As before, we underline that $(\mathcal{E}^\circ, \theta^\circ)$ is assumed to be algebraic. Also as before, Theorem 6.6 implies a restriction theorem for sheaves \mathcal{E} with operators that are defined on all of X . In particular, if \mathcal{E} is a torsion-free sheaf on X (sheaf equipped with the zero operator) that is not necessarily semistable, then $\mu_H^{\max}(\mathcal{E}) = \mu_{H|_D}^{\max}(\mathcal{E}|_D)$ for all $D \in \mathbb{B}^\circ$, and the Harder–Narasimhan filtration of $\mathcal{E}|_D$ equals the restriction of the Harder–Narasimhan filtration of \mathcal{E} ; in symbols: $\text{HN}_H^\bullet(\mathcal{E})|_D = \text{HN}_{H|_D}^\bullet(\mathcal{E}|_D)$.

Proof of Theorem 6.6. Let $\iota : X^\circ \rightarrow X$ be the inclusion map. Set $\mathcal{E} = \iota_*(\mathcal{E}^\circ)$ and $\mathcal{W} = \iota_*(\mathcal{W}^\circ)$. The sheaf \mathcal{E} is torsion-free; furthermore, \mathcal{W} is reflexive and therefore embeds into a locally free sheaf \mathcal{V} . Composing morphisms as in Lemma 2.27, the operator θ° induces an operator

$\tau^\circ : \mathcal{E}^\circ \rightarrow \mathcal{E}^\circ \otimes \mathcal{V}|_{X_{\text{reg}}}$, and Lemma 2.27 shows that $(\mathcal{E}^\circ, \tau^\circ)$, considered as a sheaf with a $\mathcal{V}|_{X_{\text{reg}}}$ -valued operator, is again semistable (respectively stable) with respect to H . Pushing forward, we can extend the operator τ° , which *a priori* is only defined on X_{reg} , to an operator $\tau : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{V}$. Lemma 2.26 then shows that (\mathcal{E}, τ) is again semistable (respectively stable) with respect to H , when considered as a sheaf with a \mathcal{V} -valued operator. We claim the following.

(6.8.1) There exist a number $M \in \mathbb{N}^+$, and for every $m > M$ and basepoint-free linear system $\mathbf{B} \subseteq |m \cdot B|$, a dense open subset \mathbf{B}° such that (6.6.1) holds, $\mathcal{E}|_D$ is torsion free and $(\mathcal{E}|_D, \theta|_D)$ is semistable (respectively stable) with respect to $H|_D$, as a sheaf with a $\mathcal{V}|_D$ -valued operator.

As soon as (6.8.1) is true, Lemma 2.26 asserts that $(\mathcal{E}|_{D_{\text{reg}}}, \tau|_{D_{\text{reg}}}) = (\mathcal{E}^\circ|_{D_{\text{reg}}}, \tau^\circ|_{D_{\text{reg}}})$ is semistable (respectively stable) with respect to H as a sheaf with an $\mathcal{V}|_{D_{\text{reg}}}$ -valued operator. A final application of Lemma 2.27 then shows that $(\mathcal{E}^\circ|_{D_{\text{reg}}}, \theta^\circ|_{D_{\text{reg}}})$ is semistable (respectively stable) with respect to H , which would end the proof.

It remains to show (6.8.1). To this end, observe that for any resolution of singularities, $\pi : \tilde{X} \rightarrow X$, the operator τ induces an operator on the torsion-free pull-back of \mathcal{E} to \tilde{X} ,

$$\tilde{\tau} : \pi^* \mathcal{E} / \text{tor} \rightarrow \pi^*(\mathcal{E} \otimes \mathcal{V}) / \text{tor} = \pi^* \mathcal{E} / \text{tor} \otimes \pi^* \mathcal{V}.$$

Replacing X by a suitable resolution and replacing (\mathcal{E}, τ) by $(\pi^* \mathcal{E} / \text{tor}, \tilde{\tau})$, we may assume without loss of generality that X is smooth and that \mathcal{E} is locally free. Now, if (\mathcal{E}, τ) is stable with respect to H , the stability claim of Theorem 6.6 has been shown already for every $D \in \mathbf{B}$ fulfilling (6.6.1) and having the property that $\mathcal{E}|_D$ is torsion-free, see [Lan15, Theorem 9] and [GKPT15, Theorem A.3 in the arXiv version]. These two assumptions can be guaranteed for D belonging to an open subsystem \mathbf{B}° by the classical Bertini theorem and by [EGAIV, Theorem 12.2.1], respectively. In the case where (\mathcal{E}, τ) is only semistable, the assumption that \mathcal{V} is locally free guarantees the existence of a Jordan–Hölder filtration, which presents (\mathcal{E}, τ) as a repeated extension of stable sheaves with \mathcal{V} -valued operator of equal slope. The claim then follows by induction on the length of the filtration. \square

6.2 Proof of Theorem 6.1

For convenience of notation, write $n := \dim X$, let $\iota : X_{\text{reg}} \rightarrow X$ be the inclusion map, and let $\mathcal{E} := \iota_* \mathcal{E}^\circ$ be the torsion-free extension of \mathcal{E}° to X . Twisting the Higgs sheaf $(\mathcal{E}^\circ, \theta^\circ)$ with a sufficiently positive line bundle and noticing that semistability considerations are unaffected by this operation, we may also assume that the following holds.

ASSUMPTION W.L.O.G. 6.9. The slope of \mathcal{E} is positive, $\mu_H(\mathcal{E}) > 0$.

Choice 6.10. Choose $M \gg 0$ large enough so that the following holds.

(6.10.1) If $m > M$ and $\mathbf{B} \subset |mH|$ is basepoint-free, then Theorem 6.6 and Remark 6.7 apply to yield a dense, open set $\mathbf{B}' \subset \mathbf{B}$ such that for every $D \in \mathbf{B}'$, we have $\mu_H^{\max}(\mathcal{E}) = \mu_{H|_D}^{\max}(\mathcal{E}|_D)$.

(6.10.2) The number M is larger than the number given by Theorem 6.6 for $(\mathcal{E}^\circ, \theta^\circ)$ as a sheaf with an $\Omega_{X_{\text{reg}}}^1$ -valued operator.

(6.10.3) We have $[M \cdot H]^n > nr \cdot \mu_H^{\max}(\mathcal{E})$.

Step 1: Argument by contradiction. Assume for the remainder of the proof that we are given a number $m > M$ and a basepoint-free linear system $\mathbf{B} \subseteq |m \cdot H|$. Let \mathbf{B}° be the intersection of the open subsets given by the two applications of (6.10.1) and (6.10.2) above, and let $D \in \mathbf{B}^\circ$ be

any element. We aim to show that $(\mathcal{E}^\circ, \theta^\circ)|_{D_{\text{reg}}}$ is semistable (respectively stable) with respect to H . We argue by contradiction and assume that this is not the case.

ASSUMPTION 6.11. There exists a generically Higgs-invariant, saturated subsheaf $0 \subsetneq \mathcal{F}_{D_{\text{reg}}} \subsetneq \mathcal{E}^\circ|_{D_{\text{reg}}}$ with $\mu_{H|_D}(\mathcal{F}_{D_{\text{reg}}}) > \mu_H(\mathcal{E})$ (respectively \geq instead of $>$).

Step 2: Cutting down. Repeated applications of Theorem 6.6 (and Remark 6.7) allow to find an increasing sequence of numbers $M < m \leq m_2 \leq \dots \leq m_{n-1}$ and hyperplanes $D_i \in |m_i \cdot H|$ such that the associated intersection $C := D \cap D_2 \cap \dots \cap D_{n-1}$ has the following properties.

(6.12.1) The scheme C is a smooth curve, entirely contained in X_{reg} . The sheaves $\mathcal{E}|_C$ and $\mathcal{F}_{D_{\text{reg}}}|_C$ are torsion-free and hence locally free. The natural morphism $\mathcal{F}_{D_{\text{reg}}}|_C \rightarrow \mathcal{E}|_C$ is an injection.

(6.12.2) We have $\mu_H^{\max}(\mathcal{E}) = \mu_{H|_C}^{\max}(\mathcal{E}|_C)$.

(6.12.3) Because of (6.10.2), the restriction $(\mathcal{E}|_C, \theta^\circ|_C)$ is semistable (respectively stable) as a sheaf with an $\Omega_X^1|_C$ -valued operator.

Step 3: Computation. In the following, write $\mathcal{F}_C := \mathcal{F}_{D_{\text{reg}}}|_C$ and consider the associated sequence

$$0 \longrightarrow \mathcal{F}_C \longrightarrow \mathcal{E}|_C \xrightarrow{q} \mathcal{Q} \longrightarrow 0. \tag{6.13.1}$$

Let $N_{C/X}^*$ denote the conormal bundle of C in X . If $\mathcal{B} \subseteq \mathcal{Q} \otimes N_{C/X}^*$ is any coherent subsheaf of positive rank, then we will show in this step that

$$\deg_C(\mathcal{B}) \leq (n-1)r \cdot \mu_H^{\max}(\mathcal{E}) - [m \cdot H]^n. \tag{6.13.2}$$

To prove (6.13.2), consider first any coherent subsheaf $\mathcal{A} \subseteq \mathcal{Q}$ of positive rank. We obtain from (6.13.1) an exact sequence $0 \rightarrow \mathcal{F}_C \rightarrow q^{-1}\mathcal{A} \xrightarrow{q} \mathcal{A} \rightarrow 0$, which allows us to estimate the degree of \mathcal{A} as follows:

$$\begin{aligned} \deg_C(\mathcal{A}) &= \deg_C(q^{-1}\mathcal{A}) - \deg_C(\mathcal{F}_C) \leq \deg_C(q^{-1}\mathcal{A}) && \text{since } \mu_{H|_C}(\mathcal{F}_C) \geq \mu_H(\mathcal{E}) > 0 \\ &\leq \text{rank}(q^{-1}\mathcal{A}) \cdot \mu_{H|_C}(q^{-1}\mathcal{A}) && \text{definition of slope} \\ &\leq \text{rank}(\mathcal{E}) \cdot \mu_H^{\max}(\mathcal{E}) && \text{Item (6.12.2)}. \end{aligned}$$

In order to apply this inequality to the problem at hand, recall that C is constructed as a complete intersection. The normal bundle of C in X is hence described as

$$N_{C/X} = \bigoplus_i N_{D_i/X}|_C \quad \text{where } N_{D_i/X}|_C \cong \mathcal{O}_C(m \cdot H|_C) \text{ for all } i.$$

We can therefore view $\mathcal{A} := \mathcal{B}(m \cdot H|_C)$ as a subsheaf of $\mathcal{Q}^{\oplus n-1}$. An induction using the inequality obtained above then shows the following, which immediately implies (6.13.2):

$$\deg_C(\mathcal{B}) + \text{rank}(\mathcal{B}) \cdot m \cdot \deg_C(H|_C) = \deg_C \mathcal{B}(m \cdot H|_C) \stackrel{\text{Induction}}{\leq} (n-1)r \cdot \mu_H^{\max}(\mathcal{E}).$$

Step 4: End of proof. Consider the operator $\theta^\circ|_C$ and its restriction $\theta_{\mathcal{F}} : \mathcal{F}_C \rightarrow \mathcal{E}|_C \otimes \Omega_X^1|_C$. The target of $\theta_{\mathcal{F}}$ appears in the following commutative square:

$$\begin{array}{ccccccc} & & & & \mathcal{E}|_C \otimes \Omega_X^1|_C & \longrightarrow & \mathcal{E}|_C \otimes \Omega_C^1 \\ & & & & \downarrow & & \downarrow \\ & & & & q \otimes \text{Id} & \downarrow & \\ 0 & \longrightarrow & \mathcal{Q} \otimes N_{C/X}^* & \xrightarrow{\alpha} & \mathcal{Q} \otimes \Omega_X^1|_C & \xrightarrow{\beta} & \mathcal{Q} \otimes \Omega_C^1 \longrightarrow 0. \end{array}$$

Assumption 6.11 and Item (6.12.3) guarantee that \mathcal{F}_C is not invariant with respect to $\theta^\circ|_C$. The composed map $(q \otimes \text{Id}) \circ \theta_{\mathcal{F}}$ is therefore *not* the zero morphism. But since \mathcal{F}_C is a Higgs-invariant subsheaf of $(\mathcal{E}^\circ, \theta^\circ)|_C$ by assumption, the morphism $\beta \circ (q \otimes \text{Id}) \circ \theta_{\mathcal{F}}$ is zero. In summary, we obtain a non-trivial morphism $\tau : \mathcal{F}_C \rightarrow \mathcal{L} \otimes N_{C/X}^*$. But then, we compute degrees as follows:

$$\begin{aligned}
 (n-1)r \cdot \mu_H^{\max}(\mathcal{E}) - [m \cdot H]^n &\geq \deg(\text{img } \tau) && \text{Inequality (6.13.2)} \\
 &= \deg_C(\mathcal{F}_C) - \deg_C(\ker \tau) \\
 &\geq -\deg_C(\ker \tau) && \text{Assumptions 6.9, 6.11} \\
 &\geq -\text{rank}(\ker \tau) \cdot \mu_H^{\max}(\mathcal{E}) && \text{Item (6.12.2)} \\
 &\geq -r \cdot \mu_H^{\max}(\mathcal{E}) && \text{Assumption 6.9.}
 \end{aligned}$$

We obtain a contradiction to (6.10.3), which finishes the proof of Theorem 6.1. □

7. Ascent of semistable Higgs bundles

The proof of the nonabelian Hodge correspondence on klt spaces relies on the following result, which can be seen as an inverse to the descent results obtained in the first part of this paper. It asserts that the pull-back of a Higgs sheaf in Higgs_X to any resolution of singularities is again in Higgs_X .

THEOREM 7.1 (Ascent of semistable Higgs bundles). *Let X be a projective, klt space and let $\pi : \tilde{X} \rightarrow X$ be a resolution of singularities. If $(\mathcal{E}, \theta) \in \text{Higgs}_X$, then $\pi^*(\mathcal{E}, \theta) \in \text{Higgs}_{\tilde{X}}$.*

The following is an almost immediate consequence of Theorems 7.1 and 5.1 and Item (5.1.2); see also Fact 7.7 below.

COROLLARY 7.2 (Boundedness of Higgs sheaves). *Let X be a projective, klt space and let $r \in \mathbb{N}^+$ be any number. Let \mathbf{F} be the family of locally free Higgs sheaves $(\mathcal{F}, \Theta) \in \text{Higgs}_X$ with $\text{rank } \mathcal{F} = r$. Then the family \mathbf{F} is bounded.*

Theorem 7.1 will be shown in §§ 7.1–7.6 below. The main difficulty here is that it is not clear from the outset that a pull-back of a semistable Higgs bundle via the resolution map is again semistable with respect to an *ample* divisor. It turns out that the assumption of vanishing Chern classes is sufficient to resolve this problem in dimension two. The higher-dimensional case will be reduced to the surface case by restriction techniques.

7.1 Preparation for the proof of Theorem 7.1: boundedness

We will use the following iterated Bertini-type theorem for bounded families of Higgs sheaves, generalising [GKP16, Corollary 5.3], where the same result was shown for reflexive sheaves without a Higgs field that are defined on a projective variety and not just on some big open subset.

PROPOSITION 7.3 (Iterated Bertini-type theorem for bounded families on X_{reg}). *Let X be a normal, projective variety of dimension $n \geq 2$, let $H \in \text{Div}(X)$ be ample and let $X^\circ \subseteq X_{\text{reg}}$ be a big open subset. Let $(\mathcal{E}_{X^\circ}, \theta_{\mathcal{E}_{X^\circ}})$ be a locally free Higgs sheaf on X° , and let \mathbf{F}° be a bounded family of locally free Higgs sheaves on X° . If $m \gg 0$ is large enough and if $(D_1, D_2, \dots, D_{n-1}) \in |m \cdot H|^{n-1}$ is a general tuple with associated complete intersection curve*

$C := D_1 \cap \dots \cap D_{n-1}$, then $C \subset X^\circ$ is smooth and the following holds for all Higgs sheaves $(\mathcal{F}_{X^\circ}, \theta_{\mathcal{F}_{X^\circ}})$ in F° :

$$(\mathcal{F}_{X^\circ}, \theta_{\mathcal{F}_{X^\circ}}) \cong (\mathcal{E}_{X^\circ}, \theta_{\mathcal{E}_{X^\circ}}) \iff (\mathcal{F}_{X^\circ}, \theta_{\mathcal{F}_{X^\circ}})|_C \cong (\mathcal{E}_{X^\circ}, \theta_{\mathcal{E}_{X^\circ}})|_C.$$

Proof. As a first step in the proof, we extend all relevant sheaves from X° to X . To this end, let \mathcal{E}_X be the unique reflexive sheaf on X whose restriction to X° is \mathcal{E}_{X° . Doing the same with the sheaves that appear in \mathcal{F}° , observe that the family

$$F := \{\text{isomorphism classes of reflexive } \mathcal{F}_X \text{ on } X \text{ such that there exists a member } (\mathcal{F}_{X^\circ}, \theta_{\mathcal{F}_{X^\circ}}) \text{ in } F^\circ \text{ with } \mathcal{F}_X|_{X^\circ} \cong \mathcal{F}_{X^\circ}\}$$

is likewise bounded. This fact is crucial in the proof of the following two claims.

CLAIM 7.4. Setting $D_1^\circ := D_1 \cap X^\circ$, the restriction maps

$$\text{Hom}_{X^\circ}(\mathcal{F}_{X^\circ}, \mathcal{E}_{X^\circ}) \rightarrow \text{Hom}_{D_1^\circ}(\mathcal{F}_{X^\circ}|_{D_1^\circ}, \mathcal{E}_{X^\circ}|_{D_1^\circ}) \tag{7.4.1}$$

$$\text{Hom}_{X^\circ}(\mathcal{F}_{X^\circ}, \mathcal{E}_{X^\circ}) \rightarrow \text{Hom}_C(\mathcal{F}_{X^\circ}|_C, \mathcal{E}_{X^\circ}|_C) \tag{7.4.2}$$

$$\text{Hom}_{X^\circ}(\mathcal{F}_{X^\circ}, \mathcal{F}_{X^\circ} \otimes \Omega_{X^\circ}^1) \rightarrow \text{Hom}_C(\mathcal{F}_{X^\circ}|_C, \mathcal{F}_{X^\circ}|_C \otimes \Omega_{X^\circ}^1|_C) \tag{7.4.3}$$

are isomorphic, for all $(\mathcal{F}_{X^\circ}, \theta_{\mathcal{F}_{X^\circ}})$ in F° .

Proof of Claim 7.4. For brevity, we consider (7.4.1) only and leave the rest to the reader. Since $m \gg 0$ is assumed to be large, we have vanishing

$$h^0(X, \mathcal{H}om(\mathcal{F}_X, \mathcal{E}_X) \otimes \mathcal{I}_D) = h^1(X, \mathcal{H}om(\mathcal{F}_X, \mathcal{E}_X) \otimes \mathcal{I}_D) = 0$$

for all members \mathcal{F}_X of the bounded family F . In fact, vanishing for h^1 follows from [SGA2, Exp. XII, Proposition 1.5] since X is normal, which implies by [Har80, Proposition 1.3] that the sheaves $\mathcal{H}om(\mathcal{F}_X, \mathcal{E}_X) \otimes \mathcal{I}_D$ have depth ≥ 2 at every point of X . As a consequence, we obtain that the natural restriction maps

$$\text{Hom}_X(\mathcal{F}_X, \mathcal{E}_X) \rightarrow H^0(D_1, \mathcal{H}om(\mathcal{F}_X, \mathcal{E}_X)|_{D_1}) \tag{7.4.4}$$

are isomorphic for all \mathcal{F}_X in F . In order to relate (7.4.4) to the problem at hand, use boundedness of F again and recall from [HL10, Corollary 1.1.14] that the sheaves $\mathcal{F}_X|_{D_1}$, $\mathcal{E}_X|_{D_1}$ and $\mathcal{H}om(\mathcal{F}_X, \mathcal{E}_X)|_{D_1}$ are reflexive on the normal variety D_1 , for all \mathcal{F}_X in F . As a consequence, we find that the natural restriction morphisms

$$\begin{aligned} \text{Hom}_X(\mathcal{F}_X, \mathcal{E}_X) &\rightarrow \text{Hom}_{X^\circ}(\mathcal{F}_{X^\circ}, \mathcal{E}_{X^\circ}) \\ H^0(D_1, \mathcal{H}om(\mathcal{F}_X, \mathcal{E}_X)|_{D_1}) &\rightarrow \text{Hom}_{D_1^\circ}(\mathcal{F}_{X^\circ}|_{D_1^\circ}, \mathcal{E}_{X^\circ}|_{D_1^\circ}) \end{aligned}$$

are all isomorphic. □ (Claim 7.4)

CLAIM 7.5. If $\iota_C : C \rightarrow X^\circ$ denotes the inclusion, then the composed morphisms $\delta_{\mathcal{F}_{X^\circ}}$ of the diagrams

$$\begin{array}{ccc} & \text{Hom}_C(\mathcal{F}_{X^\circ}|_C, \mathcal{F}_{X^\circ}|_C \otimes N_{C/X}^*) & \\ & \downarrow & \\ \text{Hom}_{X^\circ}(\mathcal{F}_{X^\circ}, \mathcal{F}_{X^\circ} \otimes \Omega_{X^\circ}^1) & \xrightarrow[\text{isom. by (7.4.3)}]{\text{restriction}} \text{Hom}_C(\mathcal{F}_{X^\circ}|_C, \mathcal{F}_{X^\circ}|_C \otimes \Omega_{X^\circ}^1|_C) & (7.5.1) \\ & \searrow \delta_{\mathcal{F}_{X^\circ}} & \downarrow \text{Id} \otimes d\iota_C \\ & & \text{Hom}_C(\mathcal{F}_{X^\circ}|_C, \mathcal{F}_{X^\circ}|_C \otimes \Omega_C^1) \end{array}$$

are injective, for all $(\mathcal{F}_{X^\circ}, \theta_{\mathcal{F}_{X^\circ}})$ in F° .

Proof of Claim 7.5. By exactness of the vertical sequence, it suffices to show that

$$\mathrm{Hom}_C(\mathcal{F}_{X^\circ}|_C, \mathcal{F}_{X^\circ}|_C \otimes N_{C/X}^*) = \{0\} \quad \text{for all } \mathcal{F}_{X^\circ} \text{ in } \mathbb{F}^\circ. \tag{7.5.2}$$

To this end, observe that the normal bundle of C in X is very positive. In fact, we have $N_{C/X} \cong \mathcal{O}_X(m \cdot H)^{\oplus n-1}|_C$ by construction. Since $m \gg 0$ is assumed to be large, it follows from semicontinuity of the Harder–Narasimhan filtration [HL10, Theorem 2.3.2] that

$$\mu_X^{\max}((\mathcal{F}_X^* \otimes \mathcal{F}_X \otimes \mathcal{O}_X(-m \cdot H))^{**}) < 0 \quad \text{for all } \mathcal{F}_X \text{ in } \mathbb{F}.$$

In particular, Flenner’s version of the Mehta–Ramanathan theorem [HL10, Theorem 7.1.1] implies that the sheaf

$$\mathcal{H}om_C(\mathcal{F}_{X^\circ}|_C, \mathcal{F}_{X^\circ}|_C \otimes N_{C/X}^*)$$

has no section. □ (Claim 7.5)

Coming back to the proof of Proposition 7.3, we need to show the implication ‘ \Leftarrow ’ only. Assume that an element $(\mathcal{F}_{X^\circ}, \theta_{\mathcal{F}_{X^\circ}})$ in \mathbb{F}° and an isomorphism of Higgs sheaves

$$\varphi_C : (\mathcal{F}_{X^\circ}, \theta_{\mathcal{F}_{X^\circ}})|_C \rightarrow (\mathcal{E}_{X^\circ}, \theta_{\mathcal{E}_{X^\circ}})|_C$$

are given. In other words, denoting the obvious inclusion map by $\iota_C : C \rightarrow X^\circ$, we are given an isomorphism of sheaves $\varphi_C : \mathcal{F}_{X^\circ}|_C \rightarrow \mathcal{E}_{X^\circ}|_C$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_C \xrightarrow{(\mathrm{Id}_{\mathcal{F}_C} \otimes d_C) \circ (\theta_{\mathcal{F}_{X^\circ}}|_C)} \mathcal{F}_C \otimes \Omega_C^1 & & \\ \varphi_C \Big\| \cong & & \Big\| \varphi_C \otimes \mathrm{Id}_{\Omega_C^1} \\ \mathcal{E}_C \xrightarrow{(\mathrm{Id}_{\mathcal{E}_C} \otimes d_C) \circ (\theta_{\mathcal{E}_{X^\circ}}|_C)} \mathcal{E}_C \otimes \Omega_C^1 & & \end{array} \quad \text{where } \begin{array}{l} \mathcal{F}_C := \mathcal{F}_{X^\circ}|_C \\ \mathcal{E}_C := \mathcal{E}_{X^\circ}|_C. \end{array} \tag{7.6.1}$$

Item (7.4.2) guarantees that the sheaf morphism φ_C extends in a unique manner to an isomorphism $\varphi_{X^\circ} : \mathcal{F}_{X^\circ} \rightarrow \mathcal{E}_{X^\circ}$. This way, we obtain two Higgs fields on \mathcal{F}_{X° , namely,

$$\theta_{\mathcal{F}_{X^\circ}} \quad \text{and} \quad \theta'_{\mathcal{F}_{X^\circ}} := (\varphi_{X^\circ} \otimes \mathrm{Id}_{\Omega_{X^\circ}^1})^{-1} \circ \theta_{\mathcal{E}_{X^\circ}} \circ \varphi_{X^\circ}.$$

Diagram (7.6.1) implies that both Higgs fields $\theta_{\mathcal{F}_{X^\circ}}$ and $\theta'_{\mathcal{F}_{X^\circ}}$ restrict to the same Higgs field on $\mathcal{F}_{X^\circ}|_C$. But then Claim 7.5 guarantees that the two Higgs fields $\theta_{\mathcal{F}_{X^\circ}}$ and $\theta'_{\mathcal{F}_{X^\circ}}$ are in fact equal. In other words, the sheaf isomorphism φ_{X° is an isomorphism of Higgs sheaves. □

We use the following boundedness result for families of Higgs sheaves. Its proof is very similar to [GKPT15, Claim 8.5 and proof] and is therefore omitted. We emphasise that, unlike Corollary 7.2, the result formulated here does not depend on Theorem 7.1 and can therefore be used in the proof of that theorem.

Fact 7.7 (Boundedness of Higgs sheaves). Let X be a klt space, let $r \in \mathbb{N}^+$ be any number, and let $\pi : \tilde{X} \rightarrow X$ be a resolution of singularities. Let \mathbb{F} be the family of locally free Higgs sheaves (\mathcal{F}, Θ) on X that satisfy $\mathrm{rank} \mathcal{F} = r$ and have the additional property that $\pi^*(\mathcal{F}, \Theta) \in \mathrm{Higgs}_{\tilde{X}}$. Then the family \mathbb{F} is bounded.

7.2 Preparation for the proof of Theorem 7.1: vanishing of Chern classes

The following observation is rather standard but is used several times in this section.

LEMMA 7.8 (Description of bundles with pull-back in $\text{Higgs}_{\tilde{X}}$). *Let X be a projective, klt space and let $\pi : \tilde{X} \rightarrow X$ be a resolution of singularities. If (\mathcal{E}, θ) is any locally free Higgs sheaf on X such that $\pi^*(\mathcal{E}, \theta) \in \text{Higgs}_{\tilde{X}}$, then all Chern classes $c_i(\mathcal{E}) \in H^{2i}(X, \mathbb{Q})$ vanish and (\mathcal{E}, θ) is semistable with respect to any nef divisor $N \in \text{Div}(X)$.*

Proof. To prove that (\mathcal{E}, θ) is semistable with respect to any nef divisor, it is equivalent to show that $\pi^*(\mathcal{E}, \theta)$ is semistable with respect to any nef divisor on \tilde{X} , [GKPT15, Proposition 5.20]. Argue by contradiction and assume that $\pi^*(\mathcal{E}, \theta)$ is not semistable with respect some nef $N \in \text{Div}(\tilde{X})$. As non-semistability is an open condition, if $A \in \text{Div}(\tilde{X})$ is ample and $\delta \in \mathbb{Q}^+$ sufficiently small, then $\pi^*(\mathcal{E}, \theta)$ fails to be semistable with respect to the ample divisor $N + \delta \cdot A$. This contradicts the fact that $\pi^*(\mathcal{E}, \theta)$ is semistable with respect to any ample divisor on \tilde{X} , cf. [BR06, Theorem 1.3]. Semistability of (\mathcal{E}, θ) with respect to any nef divisor follows as desired.

Concerning Chern classes, recall from [Sim92, Remark on p. 36] that the C^∞ -bundle \tilde{E} on \tilde{X}^{an} underlying $\tilde{\mathcal{E}}$ is induced by a local system. Takayama has shown in [Tak03, Proposition 2.1] that X^{an} can be covered by contractible, open subsets U_i with simply connected inverse images $(\pi^{\text{an}})^{-1}(U_i)$. We infer that the C^∞ -bundle E on X^{an} underlying $\mathcal{E} \cong \pi_* \tilde{\mathcal{E}}$ is again induced by a local system. But then, a classical result of Deligne and Sullivan [DS75] implies that there exists a finite, étale cover $\gamma : \hat{X} \rightarrow X$ where $(\gamma^{\text{an}})^* \mathcal{E}$ is trivial. Vanishing of Chern classes downstairs then follows from the Leray spectral sequence and from the splitting of the natural map $\mathbb{Q}_X \rightarrow (\gamma^{\text{an}})_* \mathbb{Q}_{\hat{X}}$. □

7.3 Preparation for the proof of Theorem 7.1: numerical flatness of Higgs sheaves

The following observation allows to apply the results obtained in §5 above to Higgs sheaves on resolutions of klt spaces.

PROPOSITION 7.9 (Numerical flatness of Higgs sheaves). *Let $\pi : \tilde{X} \rightarrow X$ be a resolution of a klt space, and let \mathbf{E} be any local system on \tilde{X} . Then the corresponding locally free Higgs sheaf (\mathcal{E}, θ) is π -numerically flat.*

Proof. Given any smooth, projective curve C and any morphism $\gamma : C \rightarrow \tilde{X}$ such that $\pi \circ \gamma$ is constant, we need to show that both $\gamma^* \mathcal{E}$ and its dual are nef. To this end, recall from [Tak03, p. 827] that every fibre of π admits a small, simply connected neighbourhood U , open in the Euclidean topology. The local system $\gamma^*(\pi^* \mathbf{E})$ is therefore trivial. As Simpson’s nonabelian Hodge correspondence is functorial in morphisms between manifolds [Sim92, Remark on p. 36], it follows that the pull-back $\gamma^*(\mathcal{E}, \theta)$ corresponds to the trivial local system and is therefore trivial itself; in particular, the pull-back $\gamma^* \tilde{\mathcal{E}}$ and its dual are both nef, as desired. □

7.4 Proof of Theorem 7.1 if X is a surface

The following proposition immediately implies Theorem 7.1 (‘Ascent of semistable Higgs bundles’) in the case where X is a surface.

PROPOSITION 7.10 (Independence of the polarisation for surfaces). *Let X be a smooth, projective surface. Given a locally free Higgs sheaf (\mathcal{E}, θ) on X , the following statements are equivalent.*

(7.10.1) *There exists a big and nef divisor N such that $ch_1(\mathcal{E}) \cdot N = ch_2(\mathcal{E}) = 0$ and such that (\mathcal{E}, θ) is semistable with respect to N .*

(7.10.2) *All Chern classes of $c_i(\mathcal{E}) \in H^{2i}(X, \mathbb{Q})$ vanish and (\mathcal{E}, θ) is semistable with respect any ample divisor on X .*

Proof. Using the fact that big and nef divisors are limits of ample divisors, the direction (7.10.2) \Rightarrow (7.10.1) is immediate. We will therefore consider the converse direction (7.10.1) \Rightarrow (7.10.2) for the remainder of the proof. Set $r := \text{rank } \mathcal{E}$.

Step 1. Proof in the case where (\mathcal{E}, θ) is stable with respect to N . Using the assumption on stability with respect to a big and nef class, recall from [Lan02, Theorem 2.1] or [GKPT15, Theorem 6.1] that \mathcal{E} satisfies the Bogomolov–Gieseker inequality,

$$2r \cdot c_2(\mathcal{E}) - (r - 1) \cdot c_1(\mathcal{E})^2 \geq 0. \tag{7.10.3}$$

Substituting the assumption $ch_2(\mathcal{E}) = 0$ into (7.10.3), we obtain that $c_1(\mathcal{E})^2 \geq 0$. On the other hand, the assumption $ch_1(\mathcal{E}) \cdot N = 0$ together with the Hodge index theorem implies that $c_1(\mathcal{E})^2 \leq 0$, with equality if and only if $c_1(\mathcal{E}) = 0$. We obtain that both $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$ vanish, as desired.

Next, let $H \subset S$ be any ample divisor. By openness of stability [GKPT15, Proposition 4.17] there exists $\varepsilon \in \mathbb{Q}^+$ such that (\mathcal{E}, θ) is stable with respect to the ample class $(N + \varepsilon H)$. Lemma 7.8 then asserts that (\mathcal{E}, θ) is stable with respect to any ample divisor. This finishes the proof in the stable case.

Step 2. Proof in general: setup. Since X is smooth, there exists a Jordan–Hölder filtration for the Higgs bundle (\mathcal{E}, θ) and the nef polarisation N . More precisely, we obtain a filtration

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_{i-1} \subsetneq \mathcal{E}_i = \mathcal{E}$$

with the following properties.

(7.10.4) Each of the sheaves \mathcal{E}_i is saturated in \mathcal{E}_{i+1} , and hence reflexive, and hence locally free since X is a surface.

(7.10.5) The torsion-free quotients $\mathcal{Q}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ satisfy $c_1(\mathcal{Q}_i) \cdot N = 0$ and inherit Higgs fields τ_i making (\mathcal{Q}_i, τ_i) stable with respect to N .

Moreover, Ω_X^1 being locally free, the reflexive hulls \mathcal{Q}_i^{**} , which are automatically locally free, also inherit Higgs fields, say τ_i^{**} , which make $(\mathcal{Q}_i^{**}, \tau_i^{**})$ stable with respect to N . Since \mathcal{Q}_i and \mathcal{Q}_i^{**} agree in codimension one, $c_1(\mathcal{Q}_i) = c_1(\mathcal{Q}_i^{**})$ and $c_1(\mathcal{Q}_i^{**}) \cdot N = 0$.

*Step 3. Proof in general: the Chern character of \mathcal{Q}_i^{**} .* We will prove in this step that

$$ch_2(\mathcal{Q}_i^{**}) = ch_2(\mathcal{Q}_i) = 0 \quad \text{for all } i, \tag{7.10.6}$$

following [Sim92, proof of Theorem 2] closely. In fact, since $c_2(\mathcal{Q}_i^{**}) \leq c_2(\mathcal{Q}_i)$, cf. [HL10, p. 80], we have the inverse inequality for the second Chern character, $ch_2(\mathcal{Q}_i) \leq ch_2(\mathcal{Q}_i^{**})$. Additivity of Chern characters then implies that

$$0 = ch_2(\mathcal{E}) = \sum ch_2(\mathcal{Q}_i) \leq \sum ch_2(\mathcal{Q}_i^{**}). \tag{7.10.7}$$

On the other hand, the Bogomolov–Gieseker inequality, [Lan02, Theorem 2.1] or [GKPT15, Theorem 6.1], gives that

$$2 \cdot ch_2(\mathcal{Q}_i^{**}) \leq \frac{1}{\text{rank } \mathcal{Q}_i} \cdot c_1(\mathcal{Q}_i^{**})^2. \tag{7.10.8}$$

As before, equality $c_1(\mathcal{Q}_i^{**}) \cdot N = 0$ together with the Hodge index theorem implies that $c_1(\mathcal{Q}_i^{**})^2 \leq 0$, so (7.10.8) reduces to $ch_2(\mathcal{Q}_i^{**}) \leq 0$. Hence, (7.10.7) implies (7.10.6).

Step 4. Proof in general: end of proof. Equation (7.10.6) allows us to apply the results of Step 1 to the Higgs bundles $(\mathcal{Q}_i^{**}, \tau_i^{**})$. This implies in particular that all Chern classes of \mathcal{Q}_i^{**} vanish, so that the sheaves \mathcal{Q}_i and \mathcal{Q}_i^{**} agree to start with. The Higgs sheaf (\mathcal{E}, θ) is thus an (iterated) extension of Higgs bundles with vanishing Chern classes that are stable with respect to any ample polarisation. Item (7.10.2) follows, which ends the proof of Proposition 7.10 and therefore the proof of Theorem 7.1 in dimension two. □

7.5 Proof of Theorem 7.1 if X is maximally quasi-étale

We will now prove Theorem 7.1 under the additional assumption that the natural push-forward morphism of algebraic fundamental groups, $\widehat{\pi}_1(X_{\text{reg}}) \rightarrow \widehat{\pi}_1(X)$, is isomorphic. Following [GKP16], we say that X is maximally quasi-étale. This assumption will be maintained throughout § 7.5. Choose a divisor Δ such that (X, Δ) is klt.

Notation 7.11. Choose an ample divisor $H \in \text{Div}(X)$ such that (\mathcal{E}, θ) is semistable with respect to H and satisfies $ch_1(\mathcal{E}) \cdot [H]^{n-1} = ch_2(\mathcal{E}) \cdot [H]^{n-2} = 0$. Set $(\widetilde{\mathcal{E}}, \widetilde{\theta}) := \pi^*(\mathcal{E}, \theta)$ and $\widetilde{H} := \pi^*H$. If A is a subvariety of B , we denote the obvious inclusion morphism by ι_A .

Finally, let F be the family of locally free Higgs sheaves (\mathcal{F}, Θ) on X that satisfy $\text{rank } \mathcal{F} = \text{rank } \mathcal{E}$ and have the additional property that $\pi^*(\mathcal{F}, \Theta) \in \text{Higgs}_{\widetilde{X}}$. Recall from Fact 7.7 that this family is bounded.

Step 1: Choice of a complete intersection surface. Our proof relies on the choice of a sufficiently general complete intersection surface $S \subset X$ to which we can restrict. To be precise, choosing a sufficiently increasing sequence of numbers $0 \ll m_1 \ll m_2 \ll \dots \ll m_{n-2}$ as well as a sufficiently general tuple of hyperplanes

$$(D_1, \dots, D_{n-2}) \in |m_1 \cdot H| \times \dots \times |m_{n-2} \cdot H|,$$

the following will hold.

- (7.12.1) The intersection $S := D_1 \cap \dots \cap D_{n-2}$ is an irreducible, normal surface, not contained in any component of Δ , and the pair $(S, \Delta|_S)$ is klt (Seidenberg’s theorem [BS95, Theorem 1.7.1] and [KM98, Lemma 5.17]).
- (7.12.2) The restriction $(\mathcal{E}, \theta)|_S$ is semistable with respect to the ample divisor $H|_S$ (Restriction theorem, Theorem 6.1).
- (7.12.3) The natural morphism $i_* : \pi_1(S_{\text{reg}}) \rightarrow \pi_1(X_{\text{reg}})$, induced by the inclusion $i : S_{\text{reg}} \rightarrow X_{\text{reg}}$, is isomorphic (Lefschetz hyperplane theorem for fundamental groups, [GM88, Theorem in § II.1.2]).
- (7.12.4) Given any Higgs sheaf $(\mathcal{F}, \Theta) \in F$, then $(\mathcal{E}, \theta) \cong (\mathcal{F}, \Theta)$ if and only if $(\mathcal{E}, \theta)|_S \cong (\mathcal{F}, \Theta)|_S$ (Iterated Bertini theorem, Proposition 7.3).

Let $\widetilde{S} \subset \widetilde{X}$ be the strict transform of S in \widetilde{X} . Then \widetilde{S} is a smooth surface and $\pi_S : \widetilde{S} \rightarrow S$ is a resolution. The following diagram summarises the situation.

$$\begin{array}{ccc}
 \widetilde{S} & \xrightarrow{\iota_{\widetilde{S}}} & \widetilde{X} \\
 \pi_S \downarrow & & \downarrow \pi \\
 S & \xrightarrow{\iota_S} & X
 \end{array}
 \tag{7.12.5}$$

CLAIM 7.13. *The natural morphism of étale fundamental groups, $\widehat{\pi}_1(\widetilde{S}) \rightarrow \widehat{\pi}_1(\widetilde{X})$ is isomorphic. In particular, every local system on \widetilde{S} is the restriction of a local system on \widetilde{X} .*

Proof. Consider the following diagram of push-forward morphisms of étale fundamental groups.

$$\begin{array}{ccc}
 \widehat{\pi}_1(\widetilde{S}) & \xrightarrow{(\iota_{\widetilde{S}})_*} & \widehat{\pi}_1(\widetilde{X}) \\
 \alpha_S \downarrow & & \downarrow \alpha_X \\
 \widehat{\pi}_1(S) & \xrightarrow{(\iota_S)_*} & \widehat{\pi}_1(X) \\
 \beta_S \uparrow & & \uparrow \beta_X \\
 \widehat{\pi}_1(S_{\text{reg}}) & \xrightarrow{(\iota_{S_{\text{reg}}})_*} & \widehat{\pi}_1(X_{\text{reg}})
 \end{array}$$

The group morphisms α_S and α_X are isomorphisms by Takayama [Tak03]. The morphism β_S is surjective by e.g. [FL81, 0.7.B on p. 33] and by the fact the profinite completion is right exact [RZ10, Lemma 3.2.3 and Proposition 3.2.5]. The morphism β_X is isomorphic by assumption, and $(\iota_{S_{\text{reg}}})_*$ is isomorphic by Item (7.12.3). The remaining arrows $(\iota_{\widetilde{S}})_*$ and $(\iota_S)_*$ must then also be isomorphic. The asserted extension of local systems comes from the fact that representation of $\pi_1(S)$ comes from a representation of $\pi_1(X)$, cf. [Gro70, Theorem 1.2b] or [GKP16, § 8.1] for a detailed pedestrian proof. \square

Step 2: End of proof. We consider the restriction of $(\widetilde{\mathcal{E}}, \widetilde{\theta})$ to \widetilde{S} . Commutativity of Diagram (7.12.5) and the fact that all spaces involved are klt allow us to apply [GKPT15, Lemma 5.9] to conclude that

$$(\widetilde{\mathcal{E}}, \widetilde{\theta})|_{\widetilde{S}} \cong \pi_S^*((\mathcal{E}, \theta)|_S), \tag{7.13.1}$$

which is hence semistable with respect to $\pi_S^*(H|_S)$, thanks to Item (7.12.2) above and [GKPT15, Proposition 5.19]. Proposition 7.10 therefore implies that $(\widetilde{\mathcal{E}}, \widetilde{\theta})|_{\widetilde{S}} \in \text{Higgs}_{\widetilde{S}}$. The classical nonabelian Hodge correspondence applies, and gives a local system $E_{\widetilde{S}} \in \text{LSys}_{\widetilde{S}}$. According to Claim 7.13, we find a local system $E_{\widetilde{X}} \in \text{LSys}_{\widetilde{X}}$ whose restriction $E_{\widetilde{X}}|_{\widetilde{S}}$ is isomorphic to $E_{\widetilde{S}}$. The classical nonabelian Hodge correspondence on \widetilde{X} thus yields a Higgs sheaf $(\widetilde{\mathcal{F}}, \widetilde{\Theta}) \in \text{Higgs}_{\widetilde{X}}$ with vanishing Chern classes, whose restriction $(\widetilde{\mathcal{F}}, \widetilde{\Theta})|_{\widetilde{S}}$ is isomorphic to $(\widetilde{\mathcal{E}}, \widetilde{\theta})|_{\widetilde{S}}$ by functoriality. It follows from Proposition 7.9 that $\widetilde{\mathcal{F}}$ is π -numerically flat. Item (5.1.2) of Theorem 5.1 therefore yields a locally free Higgs sheaf $(\mathcal{F}, \Theta) \in \text{F}$ whose restriction $(\mathcal{F}, \Theta)|_S$ is isomorphic to $(\mathcal{E}, \theta)|_S$ owing to (7.13.1). In particular, Item (7.12.4) applies to show that $(\mathcal{E}, \theta) \cong (\mathcal{F}, \Theta)$. By construction, $\pi^*(\mathcal{E}, \theta) \cong (\widetilde{\mathcal{F}}, \widetilde{\Theta}) \in \text{Higgs}_{\widetilde{X}}$, which concludes the proof of Theorem 7.1 in the case where X is maximally quasi-étale. \square

7.6 Proof of Theorem 7.1 in the general setting

Finally, we prove Theorem 7.1 (‘Ascent of semistable Higgs bundles’) without any additional assumptions. Choose an ample divisor $H \in \text{Div}(X)$ such that (\mathcal{E}, θ) is semistable with respect to H and satisfies $ch_1(\mathcal{E}) \cdot [H]^{n-1} = ch_2(\mathcal{E}) \cdot [H]^{n-2} = 0$, and write $(\widetilde{\mathcal{E}}, \widetilde{\theta}) := \pi^*(\mathcal{E}, \theta)$.

Recall from [GKP16, Theorem 1.5] that there exists a quasi-étale cover $f : Y \rightarrow X$ such that $\widehat{\pi}_1(Y_{\text{reg}}) \cong \widehat{\pi}_1(Y)$. Choose one such f , note that the corresponding Y is again klt, and let \widetilde{Y} be a desingularisation of the (unique) irreducible component of the fibre product $Y \times_X \widetilde{X}$ that

dominates both Y and \tilde{X} . We obtain a diagram of surjections as follows.

$$\begin{array}{ccc}
 \tilde{Y} & \xrightarrow{\tilde{f}, \text{ gen. finite}} & \tilde{X} \\
 \tilde{\pi}, \text{ resolution} \downarrow & & \downarrow \pi, \text{ resolution} \\
 Y & \xrightarrow{f, \text{ quasi-étale}} & X
 \end{array}$$

Recall from Corollary 6.4 that the Higgs sheaf $(\mathcal{E}_Y, \theta_Y) := f^*(\mathcal{E}, \theta)$ is semistable with respect to the ample divisor $H_Y := f^*H$. It clearly satisfies $ch_1(\mathcal{E}_Y) \cdot [H_Y]^{n-1} = ch_2(\mathcal{E}_Y) \cdot [H_Y]^{n-2} = 0$. In particular, $(\mathcal{E}_Y, \theta_Y) \in \text{Higgs}_Y$. We have seen in § 7.5 that this implies $\tilde{\pi}^*(\mathcal{E}_Y, \theta_Y) \in \text{Higgs}_{\tilde{Y}}$. Lemma 7.8 thus yields the following.

(7.13.2) All Chern classes $c_i(\tilde{\pi}^* \mathcal{E}_Y) = c_i(\tilde{f}^* \tilde{\mathcal{E}}) \in H^{2i}(\tilde{Y}, \mathbb{Q})$ vanish.

(7.13.3) The Higgs sheaf $\tilde{\pi}^*(\mathcal{E}_Y, \theta_Y)$ is semistable with respect to *any* nef divisor on \tilde{Y} .

Let $\tilde{H} \in \text{Div}(\tilde{X})$ be any ample divisor. Item (7.13.2) immediately implies that

$$ch_1(\tilde{\mathcal{E}}) \cdot [\tilde{H}]^{n-1} = ch_2(\tilde{\mathcal{E}}) \cdot [\tilde{H}]^{n-2} = 0.$$

In a similar vein, Item (7.13.3) and Corollary 6.4 imply that $(\tilde{\mathcal{E}}, \tilde{\theta})$ is semistable with respect to \tilde{H} . This concludes the proof of Theorem 7.1. □

8. Proof of the nonabelian Hodge correspondence for klt spaces

8.1 Proof of Theorem 3.4 (‘Nonabelian Hodge correspondence for klt spaces’)

Using the results on descent and ascent for Higgs bundles, Theorems 5.1 and 7.1, we construct the relevant functors between Higgs_X and LSys_X . Once the functors are constructed, we show that they indeed satisfy all the claims made in Theorem 3.4. We maintain the assumptions and notation of Theorem 3.4 throughout § 8.1. The canonical resolution of singularities is denoted by $\pi : \tilde{X} \rightarrow X$.

Step 1: From local systems to bundles. We establish the first half of Theorem 3.4, constructing a functor that maps local systems to Higgs bundles. Given a local system $E \in \text{LSys}_X$, consider the Higgs bundle $(\tilde{\mathcal{E}}, \tilde{\theta}) := \eta_{\tilde{X}}(\pi^*E)$ associated to π^*E via the nonabelian Hodge correspondence on the manifold \tilde{X} . Recall that $(\tilde{\mathcal{E}}, \tilde{\theta}) \in \text{Higgs}_{\tilde{X}}$ and set $\mathcal{E} := \pi_* \tilde{\mathcal{E}}$. Proposition 7.9 and Item (5.1.2) of Theorem 5.1 imply that \mathcal{E} is locally free and that it carries a unique Higgs field θ such that $(\tilde{\mathcal{E}}, \tilde{\theta}) \cong \pi^*(\mathcal{E}, \theta)$. Lemma 7.8 applies to show that $(\mathcal{E}, \theta) \in \text{Higgs}_X$. In summary, we have constructed a mapping $\eta_X : \text{LSys}_X \rightarrow \text{Higgs}_X$.

If $e : E_1 \rightarrow E_2$ is a morphism of local systems, we obtain a morphism $\tilde{e} : \pi^*(E_1) \rightarrow \pi^*(E_2)$ and hence a morphism between the associated Higgs sheaves, $\eta_{\tilde{X}}(\tilde{e}) : (\tilde{\mathcal{E}}_1, \tilde{\theta}_1) \rightarrow (\tilde{\mathcal{E}}_2, \tilde{\theta}_2)$. Denoting the associated locally free Higgs sheaves on X by $(\mathcal{E}_1, \theta_1)$ and $(\mathcal{E}_2, \theta_2)$, an elementary computation shows that $\eta_{\tilde{X}}(\tilde{e})$ descends to a morphism $\eta_X(e) : (\mathcal{E}_1, \theta_1) \rightarrow (\mathcal{E}_2, \theta_2)$. In other words, we have constructed a functor $\eta_X : \text{LSys}_X \rightarrow \text{Higgs}_X$.

Observation 8.1. If X is smooth, then the canonical resolution of singularities is the identity, and η_X equals the functor given by the nonabelian Hodge correspondence.

Observation 8.2. The natural map $\pi^* \mathcal{E} = \pi^* \pi_* \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$ induces an isomorphism of Higgs sheaves, $N_{\pi, E} : \pi^* \eta_X(E) \rightarrow \eta_{\tilde{X}}(\pi^*E)$.

Step 2: From bundles to local systems. Given a Higgs bundle $(\mathcal{E}, \theta) \in \mathbf{Higgs}_X$, consider the Higgs bundle $(\tilde{\mathcal{E}}, \tilde{\theta}) := \pi^*(\mathcal{E}, \theta)$. Theorem 7.1 asserts that $(\tilde{\mathcal{E}}, \tilde{\theta})$ is in $\mathbf{Higgs}_{\tilde{X}}$. Apply the classical nonabelian Hodge correspondence on the manifold \tilde{X} , in order to obtain a local system $E_{\tilde{X}} := \mu_{\tilde{X}}(\tilde{\mathcal{E}}, \tilde{\theta})$. As before, Takayama’s result [Tak03, p. 827] shows that $E := \pi_* E_{\tilde{X}}$ is a local system on X . We leave it to the reader to show that this construction does indeed give a functor $\mu_X : \mathbf{Higgs}_X \rightarrow \mathbf{LSys}_X$ and that the following observations hold.

Observation 8.3. If X is smooth, then the canonical resolution of singularities is the identity, and μ_X equals the functor given by the classical nonabelian Hodge correspondence.

Observation 8.4. The natural map $\pi^*E = \pi^*\pi_* E_{\tilde{X}} \rightarrow E_{\tilde{X}}$ induces an isomorphism of local systems, $M_{\pi, (\mathcal{E}, \theta)} : \pi^*\mu_X(E) \rightarrow \mu_{\tilde{X}}(\pi^*E)$.

Step 3: Equivalence of categories. Let X be a projective klt space, and let $\pi : \tilde{X} \rightarrow X$ be the canonical resolution. The functors $\eta_{\tilde{X}}$ and $\mu_{\tilde{X}}$ associated with the classical nonabelian Hodge correspondence on the manifold \tilde{X} form an equivalence of categories: the compositions $\eta_{\tilde{X}} \circ \mu_{\tilde{X}}$ and $\mu_{\tilde{X}} \circ \eta_{\tilde{X}}$ are naturally isomorphic to the identities on $\mathbf{Higgs}_{\tilde{X}}$ and $\mathbf{LSys}_{\tilde{X}}$, respectively. One checks immediately that these isomorphisms descend to X , showing that the functors η_X and μ_X constructed above do indeed give an equivalence of categories.

Step 4: End of proof. It remains to show Items (3.4.1)–(3.4.3) of Theorem 3.4. These, however, follow immediately from Observations 8.1–8.4. The proof of Theorem 3.4 is thus complete. \square

8.2 Proof of Theorem 3.5 (‘Functoriality in morphisms’)

To keep the paper reasonably short, we will only consider functoriality of the functors η_{\bullet} . Functoriality of μ_{\bullet} follows along the same lines of argument. We construct the isomorphisms $N_{\bullet, \bullet}$ in some detail, but leave the tedious and lengthy verification of the construction’s properties to the reader, as none of the required arguments is in any way surprising or holds a promise of new insight.

Step 1: Lifting morphisms. For morphism between manifolds, functoriality of the nonabelian Hodge correspondence is classically known [Sim92, Remark on p. 36]. If $f : X \rightarrow Y$ is a morphism between klt spaces that are not smooth, the following claim allows us to lift f to a morphism between spaces that are smooth, though possibly of higher dimension.

CLAIM 8.5. *Given a morphism $f : X \rightarrow Y$ of projective klt spaces, there exists a smooth, projective variety \tilde{W} , and a commutative diagram as follows:*

$$\begin{array}{ccccc}
 \tilde{W} & \xrightarrow{\tilde{f}} & W \subseteq \tilde{X} & & \\
 \Pi, \text{ conn. fibres} \downarrow & & \downarrow \pi_W, \text{ conn. fibres} & & \downarrow \pi, \text{ canon. resolution} \\
 \tilde{Y} & \xrightarrow{\quad} & Y & \xrightarrow{f} & Z \subseteq X
 \end{array} \tag{8.5.1}$$

where $Z = f(Y)$ and $W = \tilde{f}(\tilde{W})$.

Proof. Recall from [HM07, Corollary 1.7(1)] that there exists an irreducible variety $W \subseteq \pi^{-1}(Z) \subseteq \tilde{X}$ such that the induced morphism $\pi_W : W \rightarrow Z$ is surjective and such that its general fibre is rationally connected, and in particular irreducible. Choose one such W . Consider

the fibre product $W \times_Z \tilde{Y}$, choose the unique component that dominates both W and Y , and let \tilde{W} be its canonical resolution. The general fibre of Π is again rationally connected and irreducible, in particular connected. \square

Step 2: Construction of $N_{\bullet,\bullet}$. With the help of Claim 8.5, we construct the isomorphisms $N_{\bullet,\bullet}$ in this section.

CONSTRUCTION 8.6. Assume we are given a morphism $f : X \rightarrow Y$ of projective klt spaces and a local system $E \in \text{LSys}_X$. Choose a diagram (D) as in Claim 8.5 and construct an isomorphism

$$\alpha_D : \Pi^* \pi_Y^* f^* \eta_X(E) \rightarrow \Pi^* \pi_Y^* \eta_Y(f^*E)$$

as the composition of the following canonical isomorphisms:

$$\begin{aligned} \Pi^* \pi_Y^* f^* \eta_X(E) &\cong \tilde{f}^* \pi^* \eta_X(E) && \text{Commutativity of (D)} \\ &\cong \tilde{f}^* \eta_{\tilde{X}}(\pi^* E) && \text{Item (3.4.3) of Theorem 3.4} \\ &\cong \eta_{\tilde{W}}(\tilde{f}^* \pi^* E) && \text{Simpson's functoriality for } \tilde{f} : \tilde{W} \rightarrow \tilde{X} \\ &\cong \eta_{\tilde{W}}(\Pi^* \pi_Y^* f^* E) && \text{Commutativity of (8.5.1)} \\ &\cong \Pi^* \eta_{\tilde{Y}}(\pi_Y^* f^* E) && \text{Simpson's functoriality for } \Pi : \tilde{W} \rightarrow \tilde{Y} \\ &\cong \Pi^* \pi_Y^* \eta_Y(f^* E) && \text{Item (3.4.3) of Theorem 3.4.} \end{aligned}$$

Since $\pi_Y \circ \Pi$ has connected fibres, the isomorphism α_D descends to an isomorphism

$$\beta_D : f^* \eta_X(E) \rightarrow \eta_Y(f^*E).$$

CLAIM 8.7. *In the setting of Construction 8.6, the morphism β_D is independent of the choices made. More precisely, given two diagrams D_1 and D_2 as in Claim 8.5, we have $\beta_{D_1} = \beta_{D_2}$. We can therefore choose D arbitrarily and set $N_{f,E} := \beta_D$.*

Proof. Left to the reader. \square

Step 3: End of proof. The isomorphisms $N_{\bullet,\bullet}$ constructed in Step 2 clearly have the expected behaviour under canonical resolution and satisfy the compatibility conditions spelled out in Theorem 3.5. We leave it to the reader to verify functoriality and to write down the analogous construction of the morphisms $M_{\bullet,\bullet}$. \square

8.3 Proof of Theorem 3.9 ('Independence of polarisation')

We establish the following sequence of implications.

$$(3.9.5) \begin{array}{c} \xleftarrow{\text{trivial}} \\ \xrightarrow{\text{trivial}} \end{array} (3.9.4) \implies (3.9.3) \begin{array}{c} \xrightarrow{\text{trivial}} \\ \xrightarrow{\text{trivial}} \end{array} (3.9.2) \begin{array}{c} \xrightarrow{\text{trivial}} \\ \xrightarrow{\text{trivial}} \end{array} (3.9.1)$$

Implication (3.9.4) \Rightarrow (3.9.3) in the semistable case. This is given by Lemma 7.8 ('Description of bundles whose pull-back is in $\text{Higgs}_{\tilde{X}}$ ').

Implication (3.9.4) \Rightarrow (3.9.3) in the stable case. Choose $\pi : \tilde{X} \rightarrow X$ and $\tilde{H} \in \text{Div}(\tilde{X})$ as in (3.9.4). Vanishing of Chern classes has been shown in the semistable case. As for stability, let $H \in \text{Div}(X)$ be any ample divisor. Choose a sufficiently large number $m \gg 0$ such that $|m \cdot H|$ is basepoint-free, choose sufficiently general hyperplanes $H_1, \dots, H_{\dim X - 1} \in |m \cdot H|$, and consider

the associated complete intersection curve $C := H_1 \cap \dots \cap H_{\dim X - 1} \subset X$. Observe that C is entirely contained in X_{reg} and that π is isomorphic near C . Denote the preimage by $\tilde{C} := \pi^{-1}C$. We obtain the following obvious morphisms between fundamental groups.

$$\begin{array}{ccc}
 \pi_1(\tilde{C}) & \xrightarrow{\tilde{i}_*} & \pi_1(\tilde{X}) \\
 \downarrow \text{isomorphic} & & \downarrow \text{isomorphic} \\
 \pi_1(C) & \xrightarrow{\text{surjective}} & \pi_1(X) \\
 \text{[GM88, Theorem in § II.1.2]} & & \text{[FL81, 0.7.B on p. 33]}
 \end{array}$$

In particular, we see that the morphism \tilde{i}_* is surjective. Apply the nonabelian Hodge correspondence for klt spaces, Theorem 3.4, to obtain local systems E on X and $\pi^* E \cong \mu_{\tilde{X}}(\pi^*(\mathcal{E}, \theta))$ on \tilde{X} . The following will then end the proof:

$$\begin{aligned}
 \pi^*(\mathcal{E}, \theta) \text{ is stable w.r.t. } \tilde{H} &\Rightarrow \pi^* E \text{ is irreducible} && \text{[Sim92, p. 18ff]} \\
 &\Rightarrow (\pi^* E)|_{\tilde{C}} \text{ is irreducible} && \text{surjectivity of } \tilde{i}_* \\
 &\Rightarrow E|_C \text{ is irreducible} && \text{isomorphic} \\
 &\Rightarrow (\mathcal{E}, \theta)|_C \text{ is stable} && \text{[Sim92, p. 18ff]} \\
 &\Rightarrow (\mathcal{E}, \theta) \text{ is stable w.r.t. } H && \text{genl. choice.}
 \end{aligned}$$

Implication (3.9.4) \Rightarrow (3.9.3) in the polystable case. Analogous to the above, with ‘irreducible’ replaced by ‘semisimple’.

Implication (3.9.1) \Rightarrow (3.9.5) in the semistable case. This is Theorem 7.1 (‘Ascent of semistable Higgs bundles’) together with [BR06, Theorem 1.3].

Implication (3.9.1) \Rightarrow (3.9.5) in the stable case. The pull-back bundle $\pi^*(\mathcal{E}, \theta)$ is clearly stable with respect to the nef bundle π^*H , [GKPT15, Proposition 5.19]. But then openness of stability [GKPT15, Proposition 4.17] asserts that $\pi^*(\mathcal{E}, \theta)$ is stable with respect to one ample bundle, and hence any ample bundle.

Implication (3.9.1) \Rightarrow (3.9.5) in the polystable case. Same as the stable case. □

ACKNOWLEDGEMENTS

We would like to thank numerous colleagues for discussions, including Daniel Barlet, Oliver Bräunling, Andreas Höring, Annette Huber, Shane Kelly, Jong-Hae Keum, Adrian Langer and Jörg Schürmann. We thank the referee for carefully reading the paper and for her or his very valuable and interesting remarks.

REFERENCES

AD14 C. Araujo and S. Druel, *On codimension 1 del Pezzo foliations on varieties with mild singularities*, Math. Ann. **360** (2014), 769–798.

BM14 D. Barlet and J. Magnússon, *Cycles analytiques complexes. I. Théorèmes de préparation des cycles*, Cours Spécialisés [Specialized Courses], vol. 22 (Société Mathématique de France, Paris, 2014).

BS95 M. C. Beltrametti and A. J. Sommese, *The adjunction theory of complex projective varieties*, De Gruyter Expositions in Mathematics, vol. 16 (Walter de Gruyter, Berlin, 1995), doi:[10.1515/9783110871746](https://doi.org/10.1515/9783110871746).

- BM97 E. Bierstone and P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, *Invent. Math.* **128** (1997), 207–302, doi:[10.1007/s002220050141](https://doi.org/10.1007/s002220050141).
- BCHM10 C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, *J. Amer. Math. Soc.* **23** (2010), 405–468, doi:[10.1090/S0894-0347-09-00649-3](https://doi.org/10.1090/S0894-0347-09-00649-3).
- BR06 U. Bruzzo and D. H. Ruipérez, *Semistability vs. nefness for (Higgs) vector bundles*, *Differential Geom. Appl.* **24** (2006), 403–416, doi:[10.1016/j.difgeo.2005.12.007](https://doi.org/10.1016/j.difgeo.2005.12.007).
- CKT16 B. Claudon, S. Kebekus and B. Taji, *Generic positivity and applications to hyperbolicity of moduli spaces*, Preprint (2016), [arXiv:1610.09832](https://arxiv.org/abs/1610.09832). To appear as a chapter in a forthcoming book on hyperbolicity, October 2016.
- Del70 P. Deligne, *Équations différentielles à points singuliers réguliers*, *Lecture Notes in Mathematics*, vol. 163 (Springer, Berlin, 1970), doi:[10.1007/BFb0061194](https://doi.org/10.1007/BFb0061194).
- DS75 P. Deligne and D. Sullivan, *Fibrés vectoriels complexes à groupe structural discret*, *C. R. Math. Acad. Sci. Paris Sér. A-B* **281** (1975), A1, A1081–A1083.
- DPS94 J.-P. Demailly, T. Peternell and M. Schneider, *Compact complex manifolds with numerically effective tangent bundles*, *J. Algebraic Geom.* **3** (1994), 295–345.
- EGAIV A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. Troisième partie*, *Publ. Math. Inst. Hautes Études Sci.* **28** (1966), 5–255. Revised in collaboration with Jean Dieudonné, [numdam.PMIHES-1966-28-5-0](https://numdam.org/PMIHES-1966-28-5-0).
- Ful98 W. Fulton, *Intersection theory*, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, vol. 2, second edition (Springer, Berlin, 1998), doi:[10.1007/978-1-4612-1700-8](https://doi.org/10.1007/978-1-4612-1700-8).
- FL81 W. Fulton and R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*, in *Algebraic geometry (Chicago, IL, 1980)*, *Lecture Notes in Mathematics*, vol. 862 (Springer, Berlin, 1981), 26–92, doi:[10.1007/BFb0090889](https://doi.org/10.1007/BFb0090889).
- GM88 M. Goresky and R. D. MacPherson, *Stratified Morse theory*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, vol. 14 (Springer, Berlin, 1988), doi:[10.1007/978-3-642-71714-7](https://doi.org/10.1007/978-3-642-71714-7).
- GW10 U. Görtz and T. Wedhorn, *Algebraic geometry I, schemes, with examples and exercises*, *Advanced Lectures in Mathematics (Vieweg + Teubner, Wiesbaden, 2010)*, doi:[10.1007/978-3-8348-9722-0](https://doi.org/10.1007/978-3-8348-9722-0).
- GK14 P. Graf and S. J. Kovács, *An optimal extension theorem for 1-forms and the Lipman–Zariski conjecture*, *Doc. Math.* **19** (2014), 815–830.
- GKKP11 D. Greb, S. Kebekus, S. J. Kovács and T. Peternell, *Differential forms on log canonical spaces*, *Publ. Math. Inst. Hautes Études Sci.* **114** (2011), 87–169, doi:[10.1007/s10240-011-0036-0](https://doi.org/10.1007/s10240-011-0036-0). An extended version with additional graphics is available as [arXiv:1003.2913](https://arxiv.org/abs/1003.2913).
- GKP16 D. Greb, S. Kebekus and T. Peternell, *Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties*, *Duke Math. J.* **165** (2016), 1965–2004, doi:[10.1215/00127094-3450859](https://doi.org/10.1215/00127094-3450859).
- GKPT15 D. Greb, S. Kebekus, T. Peternell and B. Taji, *The Miyaoka–Yau inequality and uniformisation of canonical models*, *Ann. Sci. Éc. Norm. Supér. (4)*, to appear. Preprint (2015), [arXiv:1511.08822](https://arxiv.org/abs/1511.08822).
- GKPT18 D. Greb, S. Kebekus, T. Peternell and B. Taji, *Harmonic metrics on Higgs sheaves and uniformization of varieties of general type*, Preprint (2018), [arXiv:1804.01266](https://arxiv.org/abs/1804.01266).
- GKT18 D. Greb, S. Kebekus and B. Taji, *Uniformisation of higher-dimensional varieties*, in *Algebraic geometry: Salt Lake City 2015*, *Proceedings of Symposia in Pure Mathematics*, vol. 1, eds T. de Fernex, B. Hassett, M. Mustață, M. Olsson, M. Popa and R. Thomas (American Mathematical Society, Providence, RI, 2018), 277–308.

- Gro70 A. Grothendieck, *Représentations linéaires et compactification profinie des groupes discrets*, Manuscripta Math. **2** (1970), 375–396, doi:[10.1007/BF01719593](https://doi.org/10.1007/BF01719593).
- GT16 H. Guenancia and B. Taji, *Orbifold stability and Miyaoka–Yau inequality for minimal pairs*, Preprint (2016), [arXiv:1611.05981](https://arxiv.org/abs/1611.05981).
- HM07 C. D. Hacon and J. McKernan, *On Shokurov’s rational connectedness conjecture*, Duke Math. J. **138** (2007), 119–136; [euclid.dmj/1178738561](https://doi.org/10.1017/S0004972607005611).
- Har77 R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52 (Springer, New York, 1977), doi:[10.1007/978-1-4757-3849-0](https://doi.org/10.1007/978-1-4757-3849-0).
- Har80 R. Hartshorne, *Stable reflexive sheaves*, Math. Ann. **254** (1980), 121–176, doi:[10.1007/BF01467074](https://doi.org/10.1007/BF01467074).
- HN13 A. Höring and C. Novelli, *Mori contractions of maximal length*, Publ. Res. Inst. Math. Sci. **49** (2013), 215–228, doi:[10.4171/PRIMS/103](https://doi.org/10.4171/PRIMS/103).
- HL10 D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Cambridge Mathematical Library, second edition (Cambridge University Press, Cambridge, 2010), doi:[10.1017/CBO9780511711985](https://doi.org/10.1017/CBO9780511711985).
- KMM87 Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, in *Algebraic geometry, Sendai, 1985*, Advanced Studies in Pure Mathematics, vol. 10 (North-Holland, Amsterdam, 1987), 283–360.
- Keb13 S. Kebekus, *Pull-back morphisms for reflexive differential forms*, Adv. Math. **245** (2013), 78–112, doi:[10.1016/j.aim.2013.06.013](https://doi.org/10.1016/j.aim.2013.06.013).
- Kol96 J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32 (Springer, Berlin, 1996), doi:[10.1007/978-3-662-03276-3](https://doi.org/10.1007/978-3-662-03276-3).
- Kol11 J. Kollár, *Simultaneous normalization and algebra husks*, Asian J. Math. **15** (2011), 437–449, doi:[10.4310/AJM.2011.v15.n3.a6](https://doi.org/10.4310/AJM.2011.v15.n3.a6).
- KM98 J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134 (Cambridge University Press, Cambridge, 1998); with the collaboration of C. H. Clemens and A. Corti, translated from the 1998 Japanese original, doi:[10.1017/CBO9780511662560](https://doi.org/10.1017/CBO9780511662560).
- Lan02 A. Langer, *A note on Bogomolov’s instability and Higgs sheaves*, in *Algebraic geometry* (de Gruyter, Berlin, 2002), 237–256.
- Lan15 A. Langer, *Bogomolov’s inequality for Higgs sheaves in positive characteristic*, Invent. Math. **199** (2015), 889–920, doi:[10.1007/s00222-014-0534-z](https://doi.org/10.1007/s00222-014-0534-z).
- Laz04 R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 48 (Springer, Berlin, 2004), doi:[10.1007/978-3-642-18808-4](https://doi.org/10.1007/978-3-642-18808-4).
- Moc06 T. Mochizuki, *Kobayashi–Hitchin correspondence for tame harmonic bundles and an application*, Astérisque **309** (2006).
- Moc07a T. Mochizuki, *Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D -modules. I*, Mem. Amer. Math. Soc. **185** (2007), doi:[10.1090/memo/0869](https://doi.org/10.1090/memo/0869).
- Moc07b T. Mochizuki, *Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D -modules. II*, Mem. Amer. Math. Soc. **185** (2007), doi:[10.1090/memo/0870](https://doi.org/10.1090/memo/0870).
- RZ10 L. Ribes and P. Zalesskii, *Profinite groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 40, second edition (Springer, Berlin, 2010), doi:[10.1007/978-3-642-01642-4](https://doi.org/10.1007/978-3-642-01642-4).

- Sab13 C. Sabbah, *Théorie de Hodge et correspondance de Hitchin–Kobayashi sauvages (d’après T. Mochizuki)*, Astérisque (2013), (352):Exp. No. 1050, viii, 205–241. Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058.
- SGA2 A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)* (North-Holland Publishing Co., Amsterdam, 1968), Augmenté d’un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962, Advanced Studies in Pure Mathematics, vol. 2. Also available as [arXiv:math/0511279](https://arxiv.org/abs/math/0511279).
- Sim90 C. T. Simpson, *Harmonic bundles on noncompact curves*, J. Amer. Math. Soc. **3** (1990), 713–770, doi:[10.2307/1990935](https://doi.org/10.2307/1990935).
- Sim92 C. T. Simpson, *Higgs bundles and local systems*, Publ. Math. Inst. Hautes Études Sci. **75** (1992), 5–95, doi:[10.1007/BF02699491](https://doi.org/10.1007/BF02699491), [numdam.PMIHES-1992-75-5-0](https://numdam.org/item/PMIHES-1992-75-5-0).
- Tak03 S. Takayama, *Local simple connectedness of resolutions of log-terminal singularities*, Internat. J. Math. **14** (2003), 825–836, doi:[10.1142/S0129167X0300196X](https://doi.org/10.1142/S0129167X0300196X).

Daniel Greb daniel.greb@uni-due.de

Essener Seminar für Algebraische Geometrie und Arithmetik, Fakultät für Mathematik,
Universität Duisburg–Essen, 45117 Essen, Germany
<http://www.esaga.uni-due.de/daniel.greb>

Stefan Kebekus stefan.kebekus@math.uni-freiburg.de

Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Ernst-Zermelo-Strasse 1,
79104 Freiburg im Breisgau, Germany

and

Freiburg Institute for Advanced Studies (FRIAS), Freiburg im Breisgau, Germany
<https://cplx.vm.uni-freiburg.de>

Thomas Peternell thomas.peternell@uni-bayreuth.de

Mathematisches Institut, Universität Bayreuth, 95440 Bayreuth, Germany
<http://www.komplexe-analysis.uni-bayreuth.de>

Behrouz Taji btaji@nd.edu

University of Notre Dame, Department of Mathematics, 278 Hurley, Notre Dame,
IN 46556, USA
<http://sites.nd.edu/b-taji>