3 Kinematics of Rotational Motion

> But though the professed aim of all scientific work is to unravel the secrets of nature, it has another effect, not less valuable, on the mind of the worker. It leaves him in possession of methods which nothing but scientific work could have led him to invent.
>
> James Clerk Maxwell, *The Theory of Molecules*

The dynamics of rigid bodies is a chapter of classical mechanics that deserves to be highlighted not only owing to its intrinsic physical interest but also because it involves important mathematical techniques. Before, however, embarking on the study of dynamics, it is necessary to formulate efficacious methods to describe the motion of rigid bodies. A considerable space will be dedicated to the study of rotational kinematics in the perspective that several of the mathematical tools to be developed are of great generality, finding wide application in other domains of theoretical physics.

3.1 Orthogonal Transformations

A rigid body has, in general, six degrees of freedom. Obviously, three of them correspond to translations of the body as a whole, whereas the other three degrees of freedom describe the orientations of the body relative to a system of axes fixed in space. A simple way to specify the orientation of the rigid body consists in setting up a Cartesian system of axes *fixed in the body*, which move along with it, and consider the angles that these axes make with axes parallel to those that remain fixed in space, represented by dashed lines in Fig. 3.1.

Direction Cosines

Let Σ be a Cartesian coordinate system (x_1, x_2, x_3) with corresponding unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ representing axes fixed in space, and let Σ' be a Cartesian coordinate system (x'_1, x'_2, x'_3) with unit vectors $\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3$ whose axes remain attached to the rigid body, as in Fig. 3.2. An arbitrary vector $\boldsymbol{g}$ can be expressed in terms of the basis Σ or in terms of the basis Σ':

$$\boldsymbol{g} = g_1\hat{\mathbf{e}}_1 + g_2\hat{\mathbf{e}}_2 + g_3\hat{\mathbf{e}}_3 = \sum_{j=1}^{3} g_j\hat{\mathbf{e}}_j \,, \tag{3.1}$$

$$\boldsymbol{g} = g'_1\hat{\mathbf{e}}'_1 + g'_2\hat{\mathbf{e}}'_2 + g'_3\hat{\mathbf{e}}'_3 = \sum_{j=1}^{3} g'_j\hat{\mathbf{e}}'_j \,. \tag{3.2}$$

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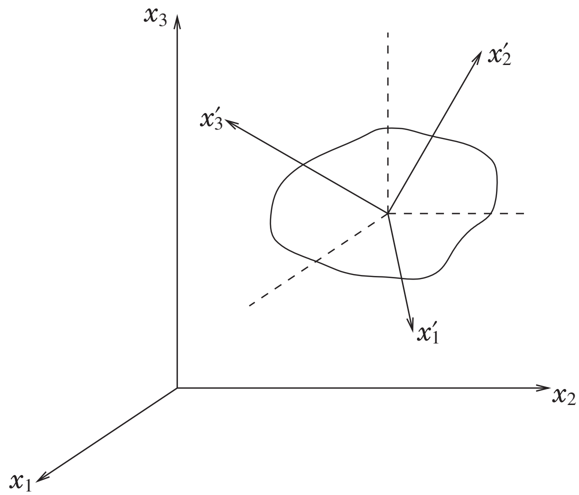

Fig. 3.1 Axes fixed in space and primed axes attached to a rigid body.

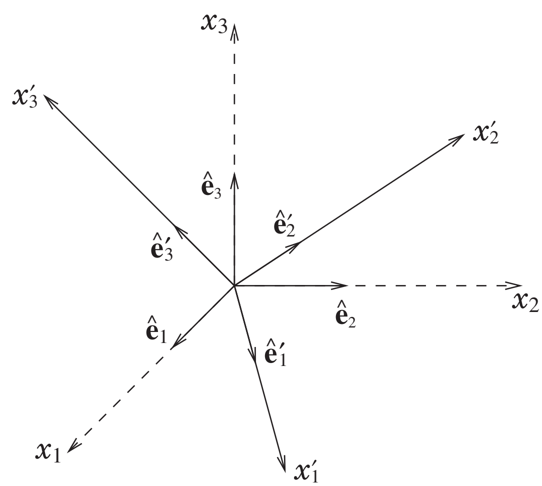

Fig. 3.2 Unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ fixed in space and unit vectors $\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3$ attached to the rigid body.

Of course, the components of $\boldsymbol{g}$ in the bases Σ and Σ' are related to one another. Indeed,

$$g'_i = \hat{\mathbf{e}}'_i \cdot \boldsymbol{g} = \sum_{j=1}^{3} g_j \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j \equiv \sum_{j=1}^{3} a_{ij}\, g_j \,, \qquad i = 1, 2, 3\,, \tag{3.3}$$

where

$$a_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j = \text{cosine of the angle between axes } x'_i \text{ and } x_j\,. \tag{3.4}$$

The quantities a_{ij} – nine in all – are called **direction cosines** of Σ' relative to Σ. Therefore, the components of $\boldsymbol{g}$ in Σ' are expressed in terms of its components in Σ by means of a *linear transformation* whose coefficients are the direction cosines.

The nine quantities a_{ij} cannot be mutually independent, for three quantities suffice to define the orientation of Σ' relative to Σ. In order to obtain the conditions to which the direction cosines must be subject, all one has to do is take into account that the magnitude of a vector does not depend on the basis used to express it:

$$\boldsymbol{g} \cdot \boldsymbol{g} = \sum_{i=1}^{3} g'_i\, g'_i = \sum_{i=1}^{3} g_i\, g_i \,. \tag{3.5}$$

Making use of Eq. (3.3) we find

$$\sum_{i=1}^{3} g'_i\, g'_i = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} a_{ij}\, g_j \right) \left(\sum_{k=1}^{3} a_{ik}\, g_k \right) = \sum_{k,j=1}^{3} \left(\sum_{i=1}^{3} a_{ij} a_{ik} \right) g_j\, g_k \,. \tag{3.6}$$

With the help of the Kronecker delta symbol δ_{jk} introduced in Appendix A, it is possible to write

$$\sum_{i=1}^{3} g_i g_i \equiv g_1^2 + g_2^2 + g_3^2 = \sum_{k,j=1}^{3} \delta_{jk} g_j g_k \,. \tag{3.7}$$

Comparing equations (3.6) and (3.7), using (3.5) and the arbitrariness of g_1, g_2, g_3, one infers

$$\sum_{i=1}^{3} a_{ij} a_{ik} = \delta_{jk} \,. \tag{3.8}$$

These are the conditions obeyed by the direction cosines, showing that not all of them are independent.

Orthogonal Matrices

Equation (3.3) is a compact representation of the three equations

$$\begin{aligned} g'_1 &= a_{11} g_1 + a_{12} g_2 + a_{13} g_3 \,, \\ g'_2 &= a_{21} g_1 + a_{22} g_2 + a_{23} g_3 \,, \\ g'_3 &= a_{31} g_1 + a_{32} g_2 + a_{33} g_3 \,. \end{aligned} \tag{3.9}$$

Equations of this type can be made much more concise by means of matrix notation. Definining the matrices[1]

$$\boldsymbol{g}_\Sigma = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}, \quad \boldsymbol{g}_{\Sigma'} = \begin{pmatrix} g'_1 \\ g'_2 \\ g'_3 \end{pmatrix}, \quad \boldsymbol{\mathcal{A}} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{3.10}$$

Eqs. (3.9) can be written in the form

$$\boldsymbol{g}_{\Sigma'} = \boldsymbol{\mathcal{A}} \boldsymbol{g}_\Sigma \,. \tag{3.11}$$

[1] In this chapter square matrices will be typically denoted by bold capital calligraphic letters such as $\boldsymbol{\mathcal{A}}$. A vector written as a column matrix will be denoted by a bold small letter. Whenever necessary, a subscript will be added to indicate the basis in which the vector is expressed, such as in $\boldsymbol{g}_\Sigma$.

On the other hand, noting that $a_{ij} = (\mathcal{A}^T)_{ji}$, where $\mathcal{A}^T$ denotes the transpose of $\mathcal{A}$, Eq. (3.8) is equivalent to $\sum_{i=1}^{3}(\mathcal{A}^T)_{ji}(\mathcal{A})_{ik} = \delta_{jk}$. Thus, taking into account the definition of matrix product, Eq. (3.8) can be rewritten in the matrix form

$$\mathcal{A}^T \mathcal{A} = \boldsymbol{I}\,, \tag{3.12}$$

where

$$\boldsymbol{I} = (\delta_{jk}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.13}$$

is the 3×3 identity matrix.

Exercise 3.1.1 Prove that the inverse transformation of (3.3) is

$$g_i = \sum_{j=1}^{3} a_{ji}\, g'_j \tag{3.14}$$

and, using the invariance of $\boldsymbol{g} \cdot \boldsymbol{g}$, prove that

$$\sum_{i=1}^{3} a_{ji}\, a_{ki} = \delta_{jk}\,. \tag{3.15}$$

Show that, in matrix notation, this equations takes the form[2]

$$\mathcal{A}\mathcal{A}^T = \boldsymbol{I}\,. \tag{3.16}$$

Prove, further, the direct and inverse transformation laws

$$\hat{\mathbf{e}}'_j = \sum_{l=1}^{3} a_{jl}\, \hat{\mathbf{e}}_l\,, \qquad \hat{\mathbf{e}}_j = \sum_{l=1}^{3} a_{lj}\, \hat{\mathbf{e}}'_l\,. \tag{3.17}$$

Bases $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ and $\{\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3\}$ are made up of mutually orthogonal unit vectors, so the following equations hold:

$$\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k = \hat{\mathbf{e}}'_j \cdot \hat{\mathbf{e}}'_k = \delta_{jk}\,. \tag{3.18}$$

Derive (3.8) and (3.15) from (3.17) and (3.18).

The transformation matrix between Cartesian systems obeys equations (3.12) and (3.16), which are equivalent to

$$\mathcal{A}^{-1} = \mathcal{A}^T\,. \tag{3.19}$$

Any matrix that satisfies this equation is said to be an **orthogonal matrix**, and the associated linear transformation (3.11) is called an **orthogonal transformation**.

[2] For finite matrices equations (3.12) and (3.16) are equivalent – that is, one is true if and only if the other is true (see Appendix D). This property does not extend to infinite matrices.

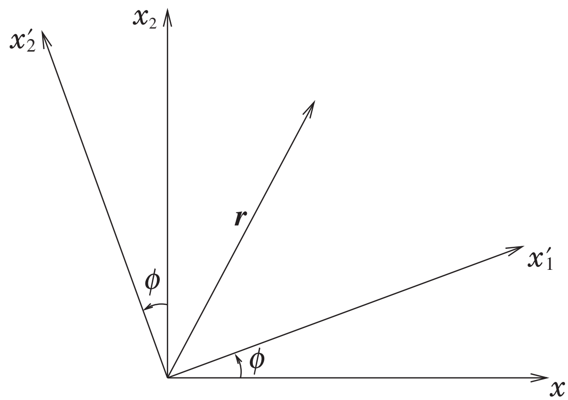

Fig. 3.3 Rotation in the plane.

Example 3.1 In the case of rotations in the plane (Fig. 3.3), we have

$$x_1' = x_1 \cos\phi + x_2 \sin\phi\,, \tag{3.20a}$$

$$x_2' = -x_1 \sin\phi + x_2 \cos\phi\,. \tag{3.20b}$$

The matrix associated with this orthogonal transformation is

$$\mathcal{A} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \tag{3.21}$$

and a single parameter – the angle ϕ – completely specifies the transformation.

Exercise 3.1.2 Using definition (3.4) and Eq. (3.3) with $\boldsymbol{g} = \mathbf{r}$, derive (3.20a) and (3.20b). By computing the product $\mathcal{A}\mathcal{A}^T$, check that the matrix $\mathcal{A}$ given by (3.21) is orthogonal.

The matrix $\mathcal{A}$ is associated with a linear transformation which, according to Eq. (3.11), acts on the components of a vector in Σ and transforms them into the components of the *same vector* in Σ'. This interpretation of Eq. (3.11), in which the vector stays the same and what changes is the coordinate system, is called the *passive point of view*. Alternatively, $\mathcal{A}$ may be thought of as an operator which transforms the vector $\boldsymbol{g}$ into *another vector* $\boldsymbol{g}'$, both considered expressed in the same coordinate system. This way of interpreting the transformation performed by matrix $\mathcal{A}$ is known as the *active point of view*, and in several occasions, as will be seen, turns out to be more convenient than the passive point of view. Of course, the specific operation represented by $\mathcal{A}$ will depend on the interpretation adopted. A rotation of the coordinate axes keeping the vector fixed has the same effect as keeping the axes fixed and rotating the vector by the same angle in the opposite sense. In Example 3.1, since $\mathcal{A}$ represents a counterclockwise rotation of the coordinate axes by angle ϕ, it will correspond to a clockwise rotation of any vector by the same angle ϕ.

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Group of Orthogonal Matrices

It is worthwhile to examine sucessive orthogonal transformations. Let $\mathcal{B} = (b_{ij})$ be the transformation matrix from Σ to Σ', and $\mathcal{A} = (a_{ij})$ be the transformation matrix from Σ' to Σ'', so that

$$g'_i = \sum_j b_{ij}\, g_j\,, \tag{3.22}$$

$$g''_k = \sum_i a_{ki}\, g'_i\,. \tag{3.23}$$

Consequently,

$$g''_k = \sum_{ij} a_{ki}\, b_{ij}\, g_j = \sum_j \left(\sum_i a_{ki}\, b_{ij}\right) g_j \equiv \sum_j c_{kj}\, g_j\,, \tag{3.24}$$

where the c_{kj} defined by

$$c_{kj} = \sum_i a_{ki}\, b_{ij} \tag{3.25}$$

are the elements of the matrix $\mathcal{C}$ which performs the direct transformation from Σ to Σ''. According to the definition of matrix product, Eq. (3.25) is equivalent to

$$\mathcal{C} = \mathcal{AB}\,. \tag{3.26}$$

It is essential to observe the order of factors in this last equation because matrix multiplication is not commutative. The matrix associated with the first transformation appears at the rightmost position in the product, immediately to the left comes the matrix of the second transformation and so on. It is intuitively clear that two successive rotations are equivalent to a single rotation. So, if $\mathcal{A}$ and $\mathcal{B}$ are orthogonal matrices, their product must also be orthogonal. In order to prove that $\mathcal{C}$ is orthogonal, it suffices to show that $\mathcal{C}$ satisfies (3.16). As the first step, note that

$$\mathcal{CC}^T = \mathcal{AB}(\mathcal{AB})^T = \mathcal{ABB}^T\mathcal{A}^T\,, \tag{3.27}$$

where we have used the identity $(\mathcal{AB})^T = \mathcal{B}^T\mathcal{A}^T$. With the use of the orthogonality of $\mathcal{B}$ and $\mathcal{A}$ we can write

$$\mathcal{CC}^T = \mathcal{A}\boldsymbol{I}\mathcal{A}^T = \mathcal{AA}^T = \boldsymbol{I}\,, \tag{3.28}$$

which verifies the orthogonality of $\mathcal{C}$.

For any orthogonal matrix its unique inverse is just its transpose, and it is immediate that the transpose of an orthogonal matrix is also orthogonal because we can write $(\mathcal{A}^T)^T\mathcal{A}^T = \mathcal{AA}^T = \boldsymbol{I}$, which shows that (3.12) is satisfied with $\mathcal{A}$ replaced by $\mathcal{A}^T$.

The set of orthogonal matrices enjoys the following properties:

(O1) If $\mathcal{A}$ and $\mathcal{B}$ are elements of the set, so is the product $\mathcal{AB}$.

(O2) The product is *associative* – that is, $\mathcal{A}(\mathcal{BC}) = (\mathcal{AB})\mathcal{C}$ holds true for any elements $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of the set.

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(O3) The set contains an element $\boldsymbol{I}$, called the *identity*, such that for any element $\mathcal{A}$ of the set, $\mathcal{A}\boldsymbol{I} = \boldsymbol{I}\mathcal{A} = \mathcal{A}$.

(O4) For each element $\mathcal{A}$ of the set there exists a unique element $\mathcal{B}$ of the set such that $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A} = \boldsymbol{I}$. The element $\mathcal{B}$ is called the *inverse* of $\mathcal{A}$ and denoted by $\mathcal{A}^{-1}$.

The first and fourth properties have already been proved, the second is satisfied because matrix multiplication is associative and the third is obviously true.

A set of elements equipped with a *composition law* – also called *multiplication* or *product* – having the four properties above make up a **group** (Hamermesh, 1962; Tung, 1985). The group of 3 × 3 orthogonal matrices is denoted by $O(3)$. Group theory plays an important role in various branches of contemporary physics, most notedly in elementary particle physics.

3.2 Possible Displacements of a Rigid Body

The determinant of an orthogonal matrix can only assume certains restricted values. Indeed, taking the determinant of (3.16) there results

$$(\det \mathcal{A})(\det \mathcal{A}^T) = 1 \quad \Longrightarrow \quad (\det \mathcal{A})^2 = 1\,, \tag{3.29}$$

where we have used the following well-known properties of the determinant:

$$\det(\mathcal{A}\mathcal{B}) = (\det \mathcal{A})(\det \mathcal{B})\,, \qquad \det \mathcal{A}^T = \det \mathcal{A}\,. \tag{3.30}$$

Therefore, the determinant of an orthogonal matrix can only be +1 or −1. A simple orthogonal matrix with determinant −1 is

$$\mathcal{I} = -\boldsymbol{I} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.31}$$

The transformation effected by $\mathcal{I}$ is

$$g_1' = -g_1\,, \quad g_2' = -g_2\,, \quad g_3' = -g_3\,, \tag{3.32}$$

which corresponds to an inversion of the coordinate axes.

An inversion transforms a right-handed Cartesian system into a left-handed one. It is clear, therefore, that an inversion does not represent a possible displacement of a rigid body: once a right-handed Cartesian system of axes is attached to the body, no physical displacement of the body can change the mutual orientation of those axes unless the body is deformable, which is not the case by hypothesis. On the other hand, any orthogonal matrix with determinant −1 can be written in the form $\mathcal{A} = (-\boldsymbol{I})(-\mathcal{A}) \equiv \mathcal{I}\mathcal{B}$ where $\det \mathcal{B} \equiv \det(-\mathcal{A}) = +1$ – that is, $\mathcal{A}$ contains an inversion of the axes and does not represent a possible displacement of the rigid body. Orthogonal matrices with determinant +1 are said to be *proper*, whereas those with determinant −1 are said to be *improper*. Thus, we conclude that only proper orthogonal transformations correspond to possible displacements of a rigid body. It is easily shown that the set of proper orthogonal transformations is itself a group called the **rotation group**, denoted $SO(3)$.

Example 3.2 The scalar product and the vector product are invariant under rotations. This means that if $\mathcal{R} = (R_{ij})$ is a rotation then

$$(\mathcal{R}\mathbf{a}) \cdot (\mathcal{R}\mathbf{b}) = \mathbf{a} \cdot \mathbf{b} \tag{3.33}$$

and also

$$(\mathcal{R}\mathbf{a}) \times (\mathcal{R}\mathbf{b}) = \mathcal{R}(\mathbf{a} \times \mathbf{b}) \,. \tag{3.34}$$

Proving invariance of the scalar product is very simple with the use of the Einstein convention of sum over repeated indices (Appendix A):

$$\begin{aligned}(\mathcal{R}\mathbf{a}) \cdot (\mathcal{R}\mathbf{b}) &= (\mathcal{R}\mathbf{a})_i(\mathcal{R}\mathbf{b})_i = R_{ij}a_jR_{ik}b_k \\ &= (\mathcal{R}^T\mathcal{R})_{jk}a_jb_k = \delta_{jk}a_jb_k = a_kb_k = \mathbf{a} \cdot \mathbf{b} \,.\end{aligned} \tag{3.35}$$

As to the vector product, we have (see Appendix A)

$$\begin{aligned}[(\mathcal{R}\mathbf{a}) \times (\mathcal{R}\mathbf{b})]_i &= \epsilon_{ijk}(\mathcal{R}\mathbf{a})_j(\mathcal{R}\mathbf{b})_k \\ &= \epsilon_{ijk}R_{jm}a_mR_{kn}b_n = \epsilon_{ijk}R_{jm}R_{kn}a_mb_n \,.\end{aligned} \tag{3.36}$$

According to Appendix A,

$$\epsilon_{ijk}R_{il}R_{jm}R_{kn} = (\det \mathcal{R})\epsilon_{lmn} = \epsilon_{lmn} \,, \tag{3.37}$$

where we used $\det \mathcal{R} = 1$ because $\mathcal{R}$ is a rotation matrix. Multiplying this last equation by R_{rl}, which implies summation over the repeated index l, we find

$$\begin{aligned}R_{rl}\epsilon_{lmn} &= \epsilon_{ijk}R_{rl}R_{il}R_{jm}R_{kn} = \epsilon_{ijk}(\mathcal{R}\mathcal{R}^T)_{ri}R_{jm}R_{kn} \\ &= \epsilon_{ijk}\delta_{ri}R_{jm}R_{kn} = \epsilon_{rjk}R_{jm}R_{kn} \,.\end{aligned} \tag{3.38}$$

Finally, substituting this result into (3.36) we obtain

$$[(\mathcal{R}\mathbf{a}) \times (\mathcal{R}\mathbf{b})]_i = R_{il}\epsilon_{lmn}a_mb_n = R_{il}(\mathbf{a} \times \mathbf{b})_l = [\mathcal{R}(\mathbf{a} \times \mathbf{b})]_i \,. \tag{3.39}$$

The proof of Eq. (3.34) is complete.

A classic result, which reveals one of the fundamental features of a rigid body's motion, was established by Euler in 1776.

Theorem 3.2.1 (Euler) *The most general displacement of a rigid body with one point fixed is a rotation about some axis through the fixed point.*

Proof Instead of the traditional geometric reasoning, we will resort to the previously developed algebraic techniques, which will confer the proof a rather instructive character. But, first, it is necessary to clarify the content of the theorem. Suppose the rigid body is in an initial configuration and, after moving in any way whatsoever about the fixed point, it reaches a final configuration. As already argued, the final configuration results from the

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application of a proper orthogonal transformation to the initial configuration. According to the statement of Euler's theorem, the final configuration can be attained by means of a single rotation about some axis through the fixed point – that is, a rotation about a point is always equivalent to a rotation about an axis containing the point. But a vector along the rotation axis remains unaltered, its components before and after the rotation are the same. If $\boldsymbol{\mathcal{A}}$ is the proper orthogonal matrix that takes the body from its initial to its final configuration, Euler's theorem will be proved as soon as one proves the existence of a non-zero vector $\boldsymbol{n}$ such that

$$\boldsymbol{\mathcal{A}}\boldsymbol{n} = \boldsymbol{n} \tag{3.40}$$

or, equivalently,

$$(\boldsymbol{\mathcal{A}} - \boldsymbol{I})\boldsymbol{n} = 0\,. \tag{3.41}$$

If the matrix $\boldsymbol{\mathcal{A}} - \boldsymbol{I}$ is invertible, multiplying (3.41) on the left by $(\boldsymbol{\mathcal{A}} - \boldsymbol{I})^{-1}$ one infers that $\boldsymbol{n} = 0$. Therefore, there exists a vector $\boldsymbol{n} \neq 0$ satisfying (3.41) if and only if $\boldsymbol{\mathcal{A}} - \boldsymbol{I}$ has no inverse – that is, if and only if

$$\det(\boldsymbol{\mathcal{A}} - \boldsymbol{I}) = 0\,. \tag{3.42}$$

In short, proving Euler's theorem boils down to showing that (3.42) holds. From the identity

$$\boldsymbol{\mathcal{A}} - \boldsymbol{I} = \boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}^T = \boldsymbol{\mathcal{A}}(\boldsymbol{I} - \boldsymbol{\mathcal{A}}^T) = -\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}} - \boldsymbol{I})^T = (-\boldsymbol{I})\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}} - \boldsymbol{I})^T \tag{3.43}$$

one deduces

$$\det(\boldsymbol{\mathcal{A}} - \boldsymbol{I}) = \det(-\boldsymbol{I})\det\boldsymbol{\mathcal{A}}\det(\boldsymbol{\mathcal{A}} - \boldsymbol{I}) = -\det\boldsymbol{\mathcal{A}}\det(\boldsymbol{\mathcal{A}} - \boldsymbol{I})\,. \tag{3.44}$$

Using $\det\boldsymbol{\mathcal{A}} = 1$, this last equation implies

$$\det(\boldsymbol{\mathcal{A}} - \boldsymbol{I}) = -\det(\boldsymbol{\mathcal{A}} - \boldsymbol{I})\,, \tag{3.45}$$

from which one infers (3.42), completing the proof of the theorem. □

The traditional geometric proof obscures the role played by the dimensionality of space, which crucially enters the above proof. In a space of even dimension $\det(-\boldsymbol{I}) = 1$ and the previous argument fails. For example, in a two-dimensional space no vector is left unaltered by a rotation. Indeed, for a rotation in the plane the axis of rotation is perpendicular to the plane, with the consequence that any vector along the rotation axis does not belong to the plane. In summary, Euler's theorem does not hold in even-dimensional spaces. Euler's theorem was generalised by Chasles, in 1830, in the terms below.

Theorem 3.2.2 (Chasles) *The most general displacement of a rigid body is a translation together with a rotation. The rotation axis can be so chosen that the translation is parallel to it.*

The first part of Chasles's theorem is obvious, since the removal of the constraint of having one point fixed endows the body with three translational degrees of freedom. A geometric proof of the second part of Chasles's theorem can be found in Whittaker (1944) or Pars (1965). For a modern proof, with the use of purely algebraic methods, see Corben and Stehle (1960).

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3.3 Euler Angles

In a Lagrangian formulation of rigid body dynamics, the nine direction cosines a_{ij} are not the most convenient coordinates for describing the instantaneous orientation of a rigid body because they are not mutually independent. The nine equations (3.8) impose only six conditions on the direction cosines, so they can be expressed in terms of three independent parameters. Indeed, there are three distinct conditions corresponding to the diagonal part of Eqs. (3.8), but the six conditions corresponding to the off-diagonal part ($i \neq j$) are pairwise identical. For example, Eq. (3.8) for $j = 1, k = 2$ is the same as the one for $j = 2, k = 1$.

From the practical point of view, a convenient way to parameterise the rotation matrix is by means of the Euler angles. The transformation from the Cartesian system $\Sigma(x, y, z)$ to the $\Sigma'(x', y', z')$ system is accomplished in three successive stages, each serving to define one of the Euler angles (Fig. 3.4).

(a) Rotation of axes (x, y, z) about the z-axis by angle $\phi : (x, y, z) \xrightarrow{\mathcal{D}} (\xi, \eta, \zeta)$.

The transformation equations are the same as equations (3.20) with $x_1' = \xi$ and $x_2' = \eta$ supplemented by equation $x_3' = \zeta = z$. Therefore, the rotation matrix $\mathcal{D}$ is written

$$\mathcal{D} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.46}$$

(b) Rotation of axes (ξ, η, ζ) about the ξ-axis by angle $\theta : (\xi, \eta, \zeta) \xrightarrow{\mathcal{C}} (\xi', \eta', \zeta')$.

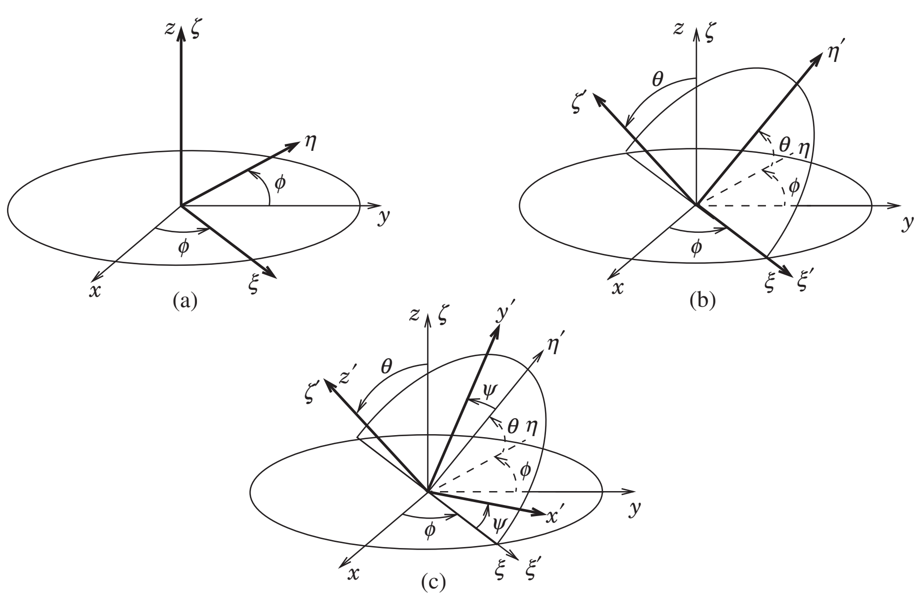

Fig. 3.4 Euler angles.

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The ξ'-axis direction is called the *line of nodes* and, in analogy with (3.20), the transformation equations take the form

$$\begin{aligned} \xi' &= \xi\,, \\ \eta' &= \eta\cos\theta + \zeta\sin\theta\,, \\ \zeta' &= -\eta\sin\theta + \zeta\cos\theta\,, \end{aligned} \tag{3.47}$$

and it follows that

$$\boldsymbol{\mathcal{C}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}. \tag{3.48}$$

(c) Rotation of axes (ξ', η', ζ') about the ζ'-axis by angle ψ : $(\xi', \eta', \zeta') \xrightarrow{\boldsymbol{\mathcal{B}}} (x', y', z')$.

The matrix $\boldsymbol{\mathcal{B}}$ represents a rotation about the third axis, thus has the same form as $\boldsymbol{\mathcal{D}}$:

$$\boldsymbol{\mathcal{B}} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.49}$$

The rotation $(x, y, z) \rightarrow (x', y', z')$ is performed by $\boldsymbol{\mathcal{A}}_E = \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}$. A direct calculation yields

$$\boldsymbol{\mathcal{A}}_E = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}. \tag{3.50}$$

The inverse transformation $(x', y', z') \rightarrow (x, y, z)$ is effected by $\boldsymbol{\mathcal{A}}_E^{-1}$ which is just the transpose of $\boldsymbol{\mathcal{A}}_E$ – that is,

$$\boldsymbol{\mathcal{A}}_E^{-1} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{pmatrix}. \tag{3.51}$$

In virtue of their definition, the Euler angles take values in the following ranges:

$$0 \le \phi \le 2\pi\,, \quad 0 \le \theta \le \pi\,, \quad 0 \le \psi \le 2\pi\,. \tag{3.52}$$

3.4 Infinitesimal Rotations and Angular Velocity

The rigid body equations of motion are differential equations, and their formulation requires the study of the body's rotation during an infinitesimal time interval.

Commutativity of Infinitesimal Rotations

Let $\boldsymbol{g}$ be an arbitrary vector submitted to a counterclockwise infinitesimal rotation by an angle $d\Phi$ about an axis defined by the unit vector $\hat{\boldsymbol{n}}$ (active point of view). According to (2.136) we have

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$$\boldsymbol{g}' = \boldsymbol{g} + \boldsymbol{d\Omega} \times \boldsymbol{g}, \tag{3.53}$$

where

$$\boldsymbol{d\Omega} = d\Phi\, \hat{\boldsymbol{n}}\,. \tag{3.54}$$

The symbols $d\Phi$ and $\boldsymbol{d\Omega}$ are to be understood as mere names for infinitesimal quantities, not as differentials of a scalar Φ or of a vector $\boldsymbol{\Omega}$. In fact, generally speaking, there is no vector $\boldsymbol{\Omega}$ whose differential is equal to $\boldsymbol{d\Omega}$ (Problem 3.6).

For successive rotations, with associated vectors $\boldsymbol{d\Omega}_1$ and $\boldsymbol{d\Omega}_2$, we write

$$\boldsymbol{g}' = \boldsymbol{g} + \boldsymbol{d\Omega}_1 \times \boldsymbol{g}\,, \tag{3.55}$$
$$\boldsymbol{g}'' = \boldsymbol{g}' + \boldsymbol{d\Omega}_2 \times \boldsymbol{g}'\,, \tag{3.56}$$

whence, neglecting second-order infinitesimals,

$$\boldsymbol{g}'' = \boldsymbol{g} + \boldsymbol{d\Omega}_{12} \times \boldsymbol{g} \tag{3.57}$$

with

$$\boldsymbol{d\Omega}_{12} = \boldsymbol{d\Omega}_1 + \boldsymbol{d\Omega}_2\,. \tag{3.58}$$

This last result shows that successive infinitesimal rotations commute ($\boldsymbol{d\Omega}_{12} = \boldsymbol{d\Omega}_{21}$) as a consequence of the commutativity of vector addition. Furthermore, the vector associated with successive infinitesimal rotations is the sum of the vectors associated with the individual infinitesimal rotations, a property which will be of great value for the forthcoming developments.

It is rewarding to describe infinitesimal rotations in matrix language. Adopting the active point of view, Eq. (3.53) can be written in the form

$$\boldsymbol{g}'_\Sigma = \boldsymbol{\mathcal{A}}\boldsymbol{g}_\Sigma = (\boldsymbol{I} + \boldsymbol{\varepsilon})\boldsymbol{g}_\Sigma\,, \tag{3.59}$$

where $\boldsymbol{\varepsilon}$ is an infinitesimal matrix. The commutativity of infinitesimal rotations is easily checked, since neglecting second-order infinitesimals and taking into account that matrix sum is commutative,

$$\boldsymbol{\mathcal{A}}_1\boldsymbol{\mathcal{A}}_2 = (\boldsymbol{I} + \boldsymbol{\varepsilon}_1)(\boldsymbol{I} + \boldsymbol{\varepsilon}_2) = \boldsymbol{I} + \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 = \boldsymbol{I} + \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_1 = \boldsymbol{\mathcal{A}}_2\boldsymbol{\mathcal{A}}_1\,. \tag{3.60}$$

On the other hand, since $\boldsymbol{\mathcal{A}} = \boldsymbol{I} + \boldsymbol{\varepsilon}$ represents a rotation, $\boldsymbol{\mathcal{A}}$ must be an orthogonal matrix. Then,

$$\boldsymbol{I} = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}^T = (\boldsymbol{I} + \boldsymbol{\varepsilon})(\boldsymbol{I} + \boldsymbol{\varepsilon}^T) = \boldsymbol{I} + \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^T \implies \boldsymbol{\varepsilon}^T = -\boldsymbol{\varepsilon}\,. \tag{3.61}$$

A matrix $\boldsymbol{\varepsilon}$ which obeys (3.61) is said to be anti-symmetric. In terms of the elements ϵ_{ij} of $\boldsymbol{\varepsilon}$ Eq. (3.61) means $\epsilon_{ij} = -\epsilon_{ji}$. Therefore, the most general form of the matrix $\boldsymbol{\varepsilon}$ is

$$\boldsymbol{\varepsilon} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}. \tag{3.62}$$

It follows that the the sum of the diagonal elements of ε is zero: $\operatorname{tr}\varepsilon = 0$. Requiring that the components of (3.59) be identical to those of (3.53), one easily finds

$$\boldsymbol{\varepsilon} = \begin{pmatrix} 0 & -d\Omega_3 & d\Omega_2 \\ d\Omega_3 & 0 & -d\Omega_1 \\ -d\Omega_2 & d\Omega_1 & 0 \end{pmatrix} . \tag{3.63}$$

Although, as we have just seen, it is possible to associate a vector with each infinitesimal rotation in such a way that the vector associated with two sucessive rotations is the sum of the vectors associated with the individual rotations, such a correspondence is not possible for finite rotations.[3] It should be further noted that Eq. (3.63) establishes the following bijective correspondence between anti-symmetric matrices and vectors in three dimensions:

$$\begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} . \tag{3.64}$$

Time Rate of Change of a Vector

In general, the time rate of change of a vector measured in a reference frame attached to a rotating rigid body will not coincide with that observed in an inertial reference frame external to the body. For instance, if the vector represents the position of a point fixed in the body, its time rate of change with respect to the system of axes attached to the body will be zero, but its time rate of change will not be zero with respect to the inertial system of axes relative to which the body is rotating. Consider an infinitesimal time interval dt during which an arbitrary vector $\boldsymbol{g}$ changes. Intuitively, we can write

$$(d\boldsymbol{g})_{inertial} = (d\boldsymbol{g})_{body} + (d\boldsymbol{g})_{rot} , \tag{3.65}$$

for the difference between the two systems is due exclusively to the body's rotation. But $(d\boldsymbol{g})_{rot}$ is a consequence of an infinitesimal rotation of $\boldsymbol{g}$ together with the axes attached to the body – that is,

$$(d\boldsymbol{g})_{rot} = \boldsymbol{d\Omega} \times \boldsymbol{g} , \tag{3.66}$$

as one infers from Eq. (3.53). The substitution of this last result into (3.65) leads to

$$\left(\frac{d\boldsymbol{g}}{dt}\right)_{inertial} = \left(\frac{d\boldsymbol{g}}{dt}\right)_{body} + \boldsymbol{\omega} \times \boldsymbol{g} , \tag{3.67}$$

where

$$\boldsymbol{\omega} = \frac{\boldsymbol{d\Omega}}{dt} = \frac{d\Phi}{dt}\hat{\boldsymbol{n}} \tag{3.68}$$

is the instantaneous **angular velocity** of the rigid body. Note that the magnitude of $\boldsymbol{\omega}$ is the instantaneous rotation speed whereas its direction is along the **instantaneous axis of rotation** characterised by the unit vector $\hat{\boldsymbol{n}}$. We insist that Eq. (3.68) represents

[3] A detailed study of finite rotations can be found in Konopinski (1969). See, also, Example 4.1 in the next chapter.

symbolically the limit of the ratio of two quantities, $\boldsymbol{d\Omega}$ and dt, which jointly tend to zero as $dt \to 0$, and that $\boldsymbol{\omega}$ *is not* the time derivative of a vector $\boldsymbol{\Omega}$ (Problem 3.6). For a purely analytic definition of angular velocity, the reader is referred to Problem 3.7.

The important Eq. (3.67) can be derived in a more rigorous manner, which has the additional virtue of clearing up the meaning of each of its terms. Let $\{\hat{\mathbf{e}}_i\}_{i=1}^3$ be the unit vectors of the inertial Cartesian system and $\{\hat{\mathbf{e}}'_i\}_{i=1}^3$ the unit vectors of the Cartesian system attached to the body. We can write

$$\boldsymbol{g} = \sum_{i=1}^{3} g_i\,\hat{\mathbf{e}}_i = \sum_{i=1}^{3} g'_i\,\hat{\mathbf{e}}'_i\,. \tag{3.69}$$

To an inertial observer the $\hat{\mathbf{e}}_i$ are constant in time but the $\hat{\mathbf{e}}'_i$ vary with time because they rotate together with the body. There are, therefore, two distinct ways to calculate $(d\boldsymbol{g}/dt)_{inertial}$, namely

$$\left(\frac{d\boldsymbol{g}}{dt}\right)_{inertial} = \sum_{i=1}^{3} \frac{dg_i}{dt}\,\hat{\mathbf{e}}_i \tag{3.70}$$

and

$$\left(\frac{d\boldsymbol{g}}{dt}\right)_{inertial} = \sum_{i=1}^{3} \frac{dg'_i}{dt}\,\hat{\mathbf{e}}'_i + \sum_{i=1}^{3} g'_i\,\frac{d\hat{\mathbf{e}}'_i}{dt}\,. \tag{3.71}$$

Equation (3.70) defines $(d\boldsymbol{g}/dt)_{inertial}$ and, similarly, $(d\boldsymbol{g}/dt)_{body}$ is defined by

$$\left(\frac{d\boldsymbol{g}}{dt}\right)_{body} = \sum_{i=1}^{3} \frac{dg'_i}{dt}\,\hat{\mathbf{e}}'_i\,. \tag{3.72}$$

This last equation is perfectly natural since to an observer attached to the body the unit vectors $\hat{\mathbf{e}}'_i$ are constant. But during the time interval dt the mobile system of axes undergoes an infinitesimal rotation, the same occurring to each unit vector $\hat{\mathbf{e}}'_i$ – that is,

$$d\hat{\mathbf{e}}'_i = \boldsymbol{d\Omega} \times \hat{\mathbf{e}}'_i\,, \tag{3.73}$$

where we used (3.66) with $\boldsymbol{g} = \hat{\mathbf{e}}'_i$. Substituting (3.72) and (3.73) into (3.71) there results

$$\left(\frac{d\boldsymbol{g}}{dt}\right)_{inertial} = \left(\frac{d\boldsymbol{g}}{dt}\right)_{body} + \sum_{i=1}^{3} g'_i\,\boldsymbol{\omega} \times \hat{\mathbf{e}}'_i = \left(\frac{d\boldsymbol{g}}{dt}\right)_{body} + \boldsymbol{\omega} \times \sum_{i=1}^{3} g'_i\,\hat{\mathbf{e}}'_i\,, \tag{3.74}$$

which coincides with (3.67).

Equality (3.67) is valid no matter what the vector $\boldsymbol{g}$ is. This allows us to interpret it not only as a mere equality of vectors, but as a reflection of an equivalence of linear operators that act on vectors. This operator equality can be conveniently expressed in the symbolic form

$$\left(\frac{d}{dt}\right)_{inertial} = \left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega} \times\,. \tag{3.75}$$

Exercise 3.4.1 Show that $(d/dt)_{inertial} = (d/dt)_{body}$ when acting on a scalar. Hint: consider the scalar $s = \mathbf{g} \cdot \mathbf{h}$.

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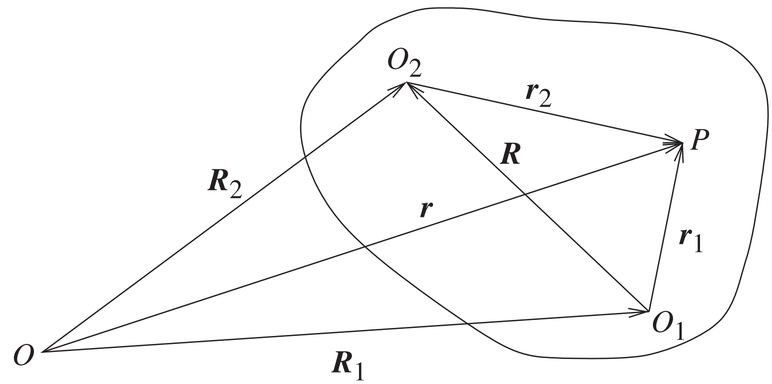

Fig. 3.5 O_1 and O_2 are the origins of two-coordinate systems fixed in the body.

Uniqueness of the Angular Velocity Vector

Intuitively, the angular velocity is expected to be a property of the rigid body as a whole, independent, therefore, of the point chosen as the origin of the coordinate system attached to the body. However, a rigorous proof of this fact is advisable (Lemos, 2000b). In Fig. 3.5, Σ is an inertial coordinate system with origin at point O, while O_1 and O_2 are the origins of two coordinate systems attached to the body, Σ_1' and Σ_2'. Let $\mathbf{r}$ be the vector from O to any point P of the rigid body and let $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ be the angular velocities associated with Σ_1' and Σ_2', respectively. Given that $\mathbf{r} = \mathbf{R}_1 + \mathbf{r}_1 = \mathbf{R}_2 + \mathbf{r}_2$, we have

$$\left(\frac{d\mathbf{r}}{dt}\right)_\Sigma = \left(\frac{d\mathbf{R}_1}{dt}\right)_\Sigma + \left(\frac{d\mathbf{r}_1}{dt}\right)_\Sigma = \left(\frac{d\mathbf{R}_1}{dt}\right)_\Sigma + \boldsymbol{\omega}_1 \times \mathbf{r}_1 \tag{3.76}$$

and, similarly,

$$\left(\frac{d\mathbf{r}}{dt}\right)_\Sigma = \left(\frac{d\mathbf{R}_2}{dt}\right)_\Sigma + \boldsymbol{\omega}_2 \times \mathbf{r}_2\,, \tag{3.77}$$

because, P being a point of the body, $\mathbf{r}_1$ and $\mathbf{r}_2$ are constant vectors with respect to the coordinate systems Σ_1' and Σ_2'. Putting $\mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1$, from (3.76) and (3.77) one deduces

$$\left(\frac{d\mathbf{R}}{dt}\right)_\Sigma = \boldsymbol{\omega}_1 \times \mathbf{r}_1 - \boldsymbol{\omega}_2 \times \mathbf{r}_2\,. \tag{3.78}$$

On the other hand,

$$\left(\frac{d\mathbf{R}}{dt}\right)_\Sigma = \left(\frac{d\mathbf{R}}{dt}\right)_{\Sigma_1'} + \boldsymbol{\omega}_1 \times \mathbf{R} = \boldsymbol{\omega}_1 \times \mathbf{R}\,, \tag{3.79}$$

because $\mathbf{R}$ is a constant vector in the coordinate system Σ_1' attached to the body. Combining (3.78) and (3.79) we get

$$\boldsymbol{\omega}_1 \times \mathbf{R} = \boldsymbol{\omega}_1 \times \mathbf{r}_1 - \boldsymbol{\omega}_2 \times \mathbf{r}_2 \tag{3.80}$$

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or, inasmuch as $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$,

$$(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) \times \mathbf{r}_2 = 0\,. \tag{3.81}$$

Since P is any point of the body, $\mathbf{r}_2$ is arbitrary and it follows that $\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1$, completing the proof.

Angular Velocity in Terms of the Euler Angles

Some important problems in rigid body dynamics require the angular velocity vector to be expressed in terms of the Euler angles. An infinitesimal rotation of the rigid body can be thought of as the result of three sucessive infinitesimal rotations whose angular velocities have, respectively, magnitudes $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$. Let $\boldsymbol{\omega}_\phi$, $\boldsymbol{\omega}_\theta$ and $\boldsymbol{\omega}_\psi$ be the corresponding angular velocity vectors. The angular velocity vector $\boldsymbol{\omega}$ is simply given by

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\phi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\psi \tag{3.82}$$

in virtue of Eq. (3.58). One can obtain the components of $\boldsymbol{\omega}$ either along inertial axes (x, y, z) or along axes (x', y', z') attached to the body. Because of its greater utility, we will consider the latter case.

Clearly, the angular velocity $\boldsymbol{\omega}$ depends linearly on $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$. Therefore, in order to find the general form taken by $\boldsymbol{\omega}$ we can fix a pair of Euler angles at a time, determine the angular velocity associated with the variation of the third angle and then add the results (Epstein, 1982). Fixing θ and ψ, the z-axis, which is fixed in space, also becomes fixed in the body (see Fig. 3.4). So, z is the rotation axis and $\boldsymbol{\omega}_\phi$ is a vector along the z-axis with component $\dot{\phi}$, and from (3.50) we have

$$\begin{pmatrix} (\boldsymbol{\omega}_\phi)_{x'} \\ (\boldsymbol{\omega}_\phi)_{y'} \\ (\boldsymbol{\omega}_\phi)_{z'} \end{pmatrix} = \boldsymbol{\mathcal{A}}_E \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \dot{\phi}\sin\theta\, \sin\psi \\ \dot{\phi}\sin\theta\, \cos\,\psi \\ \dot{\phi}\cos\,\theta \end{pmatrix}. \tag{3.83}$$

Fixing now ϕ and ψ, the line of nodes (the ξ'-axis direction in Fig. 3.4) becomes fixed both in space and in the body, so it is the rotation axis. As a consequence, $\boldsymbol{\omega}_\theta$ is a vector with the single component $\dot{\theta}$ along the ξ'-axis and with the help of (3.49) we find

$$\begin{pmatrix} (\boldsymbol{\omega}_\theta)_{x'} \\ (\boldsymbol{\omega}_\theta)_{y'} \\ (\boldsymbol{\omega}_\theta)_{z'} \end{pmatrix} = \boldsymbol{\mathcal{B}} \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{\theta}\,\cos\,\psi \\ -\dot{\theta}\,\sin\psi \\ 0 \end{pmatrix}. \tag{3.84}$$

Finally, with θ and ϕ fixed the z'-axis, which is fixed in the body, becomes fixed in space. Therefore, z' is the rotation axis and $\boldsymbol{\omega}_\psi$ is a vector along the z'-axis with component $\dot{\psi}$, there being no need to apply any transformation matrix to $\boldsymbol{\omega}_\psi$. Collecting the corresponding components the final result is

$$\begin{aligned} \omega_{x'} &= \dot{\phi}\sin\theta\,\sin\psi + \dot{\theta}\,\cos\,\psi\,, \\ \omega_{y'} &= \dot{\phi}\sin\theta\,\cos\,\psi - \dot{\theta}\,\sin\psi\,, \\ \omega_{z'} &= \dot{\psi} + \dot{\phi}\,\cos\,\theta\,. \end{aligned} \tag{3.85}$$

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Exercise 3.4.2 Prove that the components of the angular velocity along the axes fixed in space are

$$\begin{aligned} \omega_x &= \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi\,, \\ \omega_y &= \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi\,, \\ \omega_z &= \dot{\phi} + \dot{\psi}\cos\theta\,. \end{aligned} \tag{3.86}$$

3.5 Rotation Group and Infinitesimal Generators

With the help of (3.63) we can write (3.59) in the form

$$\boldsymbol{g}'_\Sigma = \boldsymbol{\mathcal{A}}\boldsymbol{g}_\Sigma = (\boldsymbol{I} + \boldsymbol{J}_1\, d\Omega_1 + \boldsymbol{J}_2\, d\Omega_2 + \boldsymbol{J}_3\, d\Omega_3)\,\boldsymbol{g}_\Sigma\,, \tag{3.87}$$

where

$$\boldsymbol{J}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \boldsymbol{J}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{J}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.88}$$

The matrices $\boldsymbol{J}_i$ are called **infinitesimal generators**[4] of the rotation group $SO(3)$. Note that $\mathrm{tr}\,\boldsymbol{J}_i = 0$.

Exercise 3.5.1 Check that the matrices $\boldsymbol{J}_i$, $i = 1, 2, 3$, obey the algebra

$$\left[\boldsymbol{J}_i, \boldsymbol{J}_j\right] = \boldsymbol{J}_k\,, \tag{3.89}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and $[\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}] = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}} - \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{A}}$ is the commutator of matrices $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$. Show that, equivalently, one can write

$$\left[\boldsymbol{J}_i, \boldsymbol{J}_j\right] = \epsilon_{ijk}\,\boldsymbol{J}_k\,, \qquad i, j = 1, 2, 3\,, \tag{3.90}$$

with use of the convention of sum over repeated indices and of the Levi-Civita symbol ϵ_{ijk} defined in Appendix A.

The matrix associated with a finite rotation by an angle α about the direction $\hat{\mathbf{n}}$ (counterclockwise, active point of view) can be found by applying N successive infinitesimal rotations by the same angle α/N and taking the limit $N \to \infty$:

$$\boldsymbol{g}'_\Sigma = \lim_{N\to\infty}\left(\boldsymbol{I} + \frac{\alpha}{N}\hat{\mathbf{n}}\cdot\boldsymbol{J}\right)^N \boldsymbol{g}_\Sigma = e^{\alpha\hat{\mathbf{n}}\cdot\boldsymbol{J}}\,\boldsymbol{g}_\Sigma\,. \tag{3.91}$$

Thus, it turns out that a finite rotation is expressed in terms of the exponential function of the matrix $\hat{\mathbf{n}}\cdot\boldsymbol{J} = n_1\boldsymbol{J}_1 + n_2\boldsymbol{J}_2 + n_3\boldsymbol{J}_3$. There's nothing mysterious about the exponential of a matrix. By definition,

$$e^{\boldsymbol{\mathcal{A}}} = \boldsymbol{I} + \boldsymbol{\mathcal{A}} + \frac{\boldsymbol{\mathcal{A}}^2}{2!} + \frac{\boldsymbol{\mathcal{A}}^3}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{\boldsymbol{\mathcal{A}}^k}{k!}\,, \tag{3.92}$$

[4] Strictly speaking, the matrices $\boldsymbol{J}_i$ are a particular representation of the generators of $SO(3)$ called the fundamental representation.

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where $\mathcal{A}$ and its powers are $n \times n$ matrices. The series (3.92) is always convergent (Courant & Hilbert, 1953).

Matrix $\boldsymbol{J}_k$ is the generator of infinitesimal rotations about the kth Cartesian axis and all matrices in the three-dimensional rotation group $SO(3)$ can be represented in the form $\mathcal{R}(\boldsymbol{\alpha}) = e^{\boldsymbol{\alpha}\cdot\boldsymbol{J}}$ where $\boldsymbol{\alpha} \equiv \alpha\hat{\mathbf{n}}$. The matrices $\mathcal{R}(\boldsymbol{\alpha})$ form a **Lie group** whose generators obey the **Lie algebra** associated with the group, Eq. (3.90). The quantities ϵ_{ijk} that appear in Eq. (3.90) are called the **structure constants** of the Lie algebra of the three-dimensional rotation group.

Exercise 3.5.2 Making use of the identity

$$\det e^{\mathcal{A}} = e^{\operatorname{tr}\mathcal{A}}, \tag{3.93}$$

verify that all matrices in $SO(3)$ have unit determinant.

3.6 Dynamics in Non-Inertial Reference Frames

Let Σ and Σ' be reference frames *with common origin*, Σ being an inertial frame and Σ' being a rotating frame with angular velocity $\boldsymbol{\omega}$. Let $\mathbf{r}$ be the position vector of a particle of mass m whose equation of motion in Σ is

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\Sigma} = \mathbf{F}\,, \tag{3.94}$$

where $\mathbf{F}$ is the total force on the particle. Using Eq. (3.67) we have

$$\mathbf{v}_{in} \equiv \left(\frac{d\mathbf{r}}{dt}\right)_{\Sigma} = \left(\frac{d\mathbf{r}}{dt}\right)_{\Sigma'} + \boldsymbol{\omega}\times\mathbf{r} \equiv \mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}\,, \tag{3.95}$$

where $\mathbf{v}_{in}$ is the particle's velocity in the inertial frame Σ and $\mathbf{v}$ denotes the particle's velocity in the rotating frame – owing to the coincidence of origins, $\mathbf{r}$ is also the particle's position vector in Σ'. From (3.95), with further use of (3.67), one derives

$$\begin{aligned}\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\Sigma} &= \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{\Sigma} = \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{\Sigma'} + \boldsymbol{\omega}\times\mathbf{v}_{in}\\ &= \left(\frac{d\mathbf{v}}{dt}\right)_{\Sigma'} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} + 2\boldsymbol{\omega}\times\mathbf{v} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\,.\end{aligned} \tag{3.96}$$

Equation (3.67) with $\boldsymbol{g} = \boldsymbol{\omega}$ shows that the angular acceleration is the same in both frames, hence the lack of subscript in $d\boldsymbol{\omega}/dt$ in the last equation. Denoting the particle's acceleration in the rotating frame by $\mathbf{a} = (d\mathbf{v}/dt)_{\Sigma'}$ and using (3.94), the equation of motion for the particle in the rotating frame is finally obtained:

$$m\mathbf{a} = \mathbf{F} + 2m\mathbf{v}\times\boldsymbol{\omega} + m\boldsymbol{\omega}\times(\mathbf{r}\times\boldsymbol{\omega}) + m\mathbf{r}\times\frac{d\boldsymbol{\omega}}{dt}\,. \tag{3.97}$$

To an observer in the non-inertial frame Σ' it all happens as if the particle were subject to an effective force which is the actual force $\mathbf{F}$ plus: a first term known as the **Coriolis force**;

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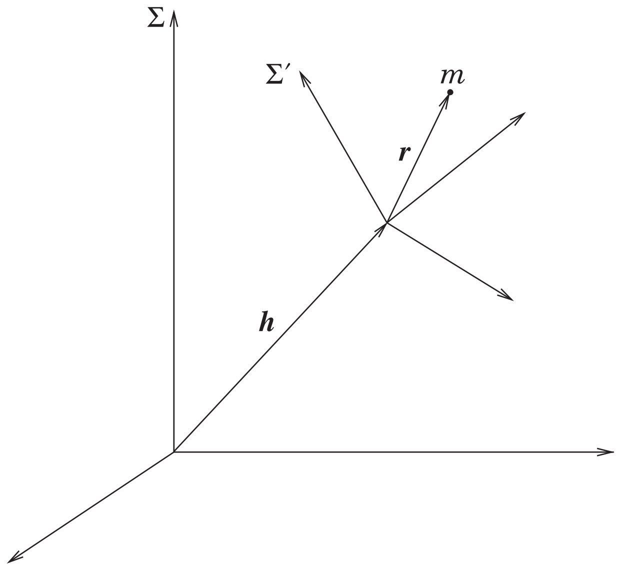

Fig. 3.6 Simultaneous rotation and translation (Exercise 3.6.1).

a second term, quadratic in $\boldsymbol{\omega}$, called the **centrifugal force**; and a third term, proportional to the angular acceleration of the non-inertial frame, sometimes called the **Euler force**.

Exercise 3.6.1 If the origins of Σ and Σ' are not coincident (Fig. 3.6), show that instead of (3.97) we have

$$m\mathbf{a} = \mathbf{F} - m\mathbf{a}_{o'} + 2m\mathbf{v} \times \boldsymbol{\omega} + m\boldsymbol{\omega} \times (\mathbf{r} \times \boldsymbol{\omega}) + m\mathbf{r} \times \frac{d\boldsymbol{\omega}}{dt}\,, \tag{3.98}$$

where $\mathbf{a}_{o'} = (d^2\mathbf{h}/dt^2)_\Sigma$ is the acceleration of the origin of Σ' relative to Σ.

Dynamics on the Rotating Earth

As one of the most important applications of the previous developments, consider the motion of a particle near the Earth's surface. It is convenient to introduce a Cartesian coordinate system which rotates together with the Earth with origin at the Earth's surface, as shown in Fig. 3.7. The z-axis is vertical, that is, it is perpendicular to the Earth's surface; the x-axis is tangent to the meridian pointing south; the y-axis is tangent to the parallel pointing east. The latitude of the place on the Earth's surface taken as origin of the Cartesian system xyz is λ. Taking the origin of the inertial frame Σ at the centre of the Earth, the position vector of the origin of the rotating frame is $\mathbf{R} = R\hat{\mathbf{z}}$, where R is the radius of the Earth. Therefore, $\mathbf{R}$ is a constant vector in the rotating frame attached to the Earth's surface and the repeated use of Eq. (3.67) immediately gives

$$\mathbf{a}_{o'} = m\left(\frac{d^2\mathbf{R}}{dt^2}\right)_\Sigma = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})\,, \tag{3.99}$$

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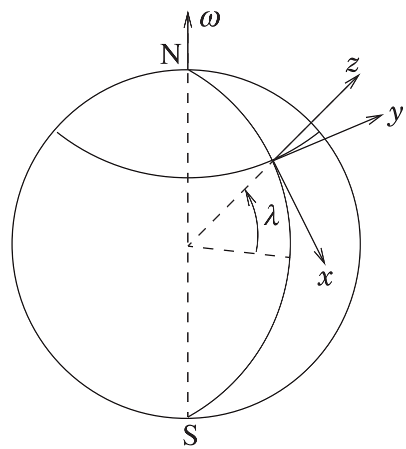

Fig. 3.7 Rotating frame *xyz* fixed on the Earth's surface.

where we have used $d\boldsymbol{\omega}/dt = 0$ because the Earth's angular velocity is practically constant. Substitution of this result into (3.98) yields

$$m\mathbf{a} = \mathbf{T} + m\mathbf{g} + m\boldsymbol{\omega} \times (\mathbf{R} \times \boldsymbol{\omega}) + 2m\mathbf{v} \times \boldsymbol{\omega} + m\boldsymbol{\omega} \times (\mathbf{r} \times \boldsymbol{\omega}), \tag{3.100}$$

where $\mathbf{F} = m\mathbf{g} + \mathbf{T}$ with $\mathbf{T}$ denoting the sum of all forces on the particle other than its weight. Equation (3.100) can be written in the form

$$m\mathbf{a} = \mathbf{T} + m\mathbf{g}_{\text{eff}} + 2m\mathbf{v} \times \boldsymbol{\omega} + m\boldsymbol{\omega} \times (\mathbf{r} \times \boldsymbol{\omega}), \tag{3.101}$$

where

$$\mathbf{g}_{\text{eff}} = \mathbf{g} + \boldsymbol{\omega} \times (\mathbf{R} \times \boldsymbol{\omega}) \tag{3.102}$$

is the effective acceleration of gravity. In order to justify this name, consider a plumb bob, that is, a particle of mass m suspended from a string under tension $\mathbf{T}$ and in equilibrium at the origin ($\mathbf{r} \equiv 0$). Equation (3.101) yields

$$\mathbf{T} + m\mathbf{g}_{\text{eff}} = 0, \tag{3.103}$$

that is, the direction of the plum line (the measured vertical direction) is determined by the effective acceleration of gravity vector. The vector $\mathbf{g}_{\text{eff}}$ does not point exactly toward the centre of the Earth. In the coordinate system of Fig. 3.7 we have $\mathbf{g} = -g\hat{\mathbf{z}}$ and

$$\boldsymbol{\omega} = -\omega \cos\lambda\, \hat{\mathbf{x}} + \omega \sin\lambda\, \hat{\mathbf{z}}, \tag{3.104}$$

whence

$$\mathbf{g}_{\text{eff}} = -(g - R\omega^2 \cos^2\lambda)\, \hat{\mathbf{z}} + R\omega^2 \sin\lambda\, \cos\lambda\, \hat{\mathbf{x}}. \tag{3.105}$$

Therefore, there is a reduction of the magnitude of the gravitational acceleration in the vertical direction and the appearence of a component in the north-south direction that

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prevents $\mathbf{g}_{\text{eff}}$ from pointing toward the centre of the Earth – the true vertical direction. The effects are small, however, because

$$\omega \approx \frac{2\pi \text{ rad}}{24 \times 3600\,\text{s}} \approx 7.3 \times 10^{-5}\,\text{rad/s} \tag{3.106}$$

and in consequence

$$R\omega^2 \approx (6400\,\text{km}) \times (7.3 \times 10^{-5}\,\text{s}^{-1})^2 \approx 3 \times 10^{-2}\,\text{m/s}^2\,, \tag{3.107}$$

which is about 300 times smaller that the acceleration of gravity. This centrifugal effect, although minute, cannot be neglected in studies of the shape of the Earth.

For objects moving near the surface of the Earth, the last term on the right-hand side of Eq. (3.100) is usually much smaller that the Coriolis term. Essentially, the centrifugal effect reduces to converting $\mathbf{g}$ into $\mathbf{g}_{\text{eff}}$, and the main modification to the motion brought about by the rotation of the Earth arises from the Coriolis force. For a supersonic airplane with speed $v \approx 2000$ km/h, the Coriolis acceleration does not exceed $2\omega v \approx 0.08$ m/s^2 or about $0.008g$. Though small, the Coriolis acceleration is important in several circumstances, as will be seen in forthcoming examples. In the northern hemisphere the Coriolis force deflects to the right an object that is moving horizontally – that is, parallelly to the Earth's surface, the deflection being to the left in the southern hemisphere (Problem 3.11). Presumably this effect explains the behaviour of cyclones, which typically rotate counterclockwise in the northern hemisphere but clockwise in the southern hemisphere (Goldstein, 1980).

Example 3.3 An object at height h falls from rest. Find the transversal deflection of the object when it hits the ground caused by the Coriolis force.

Solution

If the object falls vertically, $\mathbf{v} = -v\hat{\mathbf{z}}$ with $v > 0$ and the Coriolis force is

$$\mathbf{F}_{\text{Cor}} = 2m\mathbf{v} \times \boldsymbol{\omega} = -2mv\hat{\mathbf{z}} \times (-\omega \cos\lambda\,\hat{\mathbf{x}} + \omega \sin\lambda\,\hat{\mathbf{z}}) = 2m\omega v \cos\lambda\;\hat{\mathbf{y}}\,. \tag{3.108}$$

The deviation is in the west-to-east direction in both hemispheres, since $\cos(-\lambda) = \cos\lambda$. In the vertical direction the approximate equation of motion is

$$\ddot{z} = -g \quad\Longrightarrow\quad \dot{z} = -gt \quad\Longrightarrow\quad z = h - \frac{1}{2}gt^2\,, \tag{3.109}$$

where we used the initial conditions $\dot{z}(0) = 0$, $z(0) = h$. In the west-to-east direction we have

$$\ddot{y} = 2\omega v \cos\lambda \approx 2\omega g t \cos\lambda\,. \tag{3.110}$$

Strictly speaking, only immediately after the beginning of the fall is the velocity vector vertical. As the object falls, it acquires a transversal velocity that remains much smaller that the vertical component because the Coriolis acceleration is negligible compared to the acceleration of gravity. This justifies approximating the speed v by its vertical component alone during the whole fall, as done in (3.110). With the initial conditions $y(0) = \dot{y}(0) = 0$ Eq. (3.110) implies

$$y = \frac{\omega g}{3} t^3 \cos\lambda = \frac{\omega}{3}\sqrt{\frac{8(h-z)^3}{g}} \cos\lambda\,. \tag{3.111}$$

Upon hitting the ground ($z = 0$) the transversal displacement is

$$\Delta y = \frac{\omega}{3}\sqrt{\frac{8h^3}{g}} \cos\lambda\,. \tag{3.112}$$

At the equator, where the effect is biggest, for a fall from a height $h = 100$ m the deflection is $\Delta y \approx 2.2$ cm. Although the deviation is measurable, the experiment is hard to do because one has to eliminate the influence of winds, air resistance and other disturbing factors which may mask the Coriolis deflection.

Example 3.4 Determine the effect of the rotation of the Earth on a pendulum whose suspension point can turn freely, known as the Foucault pendulum.

Solution

Disregarding the quadratic term in $\boldsymbol{\omega}$ in (3.101) and taking $\mathbf{g}_{\text{eff}} \approx \mathbf{g}$, the equation of motion for a simple pendulum becomes

$$m\mathbf{a} = \mathbf{T} + m\mathbf{g} + 2m\mathbf{v} \times \boldsymbol{\omega}\,, \tag{3.113}$$

where $\mathbf{T}$ denotes the tension in the string. It is convenient to choose the origin of the coordinate system at the pendulum's equilibrium point (Fig. 3.8). In the case of small oscillations, the pendulum's vertical displacement is much smaller that its horizontal displacement. In fact,

$$z = l(1 - \cos\theta) \approx l\frac{\theta^2}{2}\,, \tag{3.114}$$

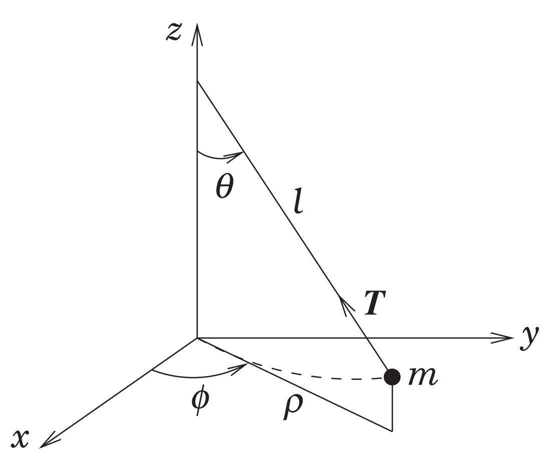

Fig. 3.8 Foucault pendulum.

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so, with $\rho = l\sin\theta \approx l\theta$,

$$\frac{z}{\rho} \approx \frac{\theta}{2} \ll 1 \tag{3.115}$$

for small θ. Therefore, in the vertical direction the pendulum remains essentially at rest with the tension in the string balancing its weight:

$$T\cos\theta \approx mg \implies T \approx mg \tag{3.116}$$

since $\cos\theta \approx 1$. The Coriolis force is transversal to the instantaneous plane of oscillation of the pendulum, which prevents it from oscillating in a fixed vertical plane. The horizontal components of (3.113) are

$$\ddot{x} = -\frac{T}{m}\sin\theta\,\cos\phi + 2\omega_z\dot{y}\,, \qquad \ddot{y} = -\frac{T}{m}\sin\theta\,\sin\phi - 2\omega_z\dot{x}\,, \tag{3.117}$$

with

$$\omega_z = \omega\sin\lambda\,. \tag{3.118}$$

Taking into account that

$$\sin\theta = \frac{\rho}{l}\,, \qquad \rho\cos\phi = x\,, \qquad \rho\sin\phi = y \tag{3.119}$$

and using (3.116), equations (3.117) take the form

$$\ddot{x} + \omega_0^2 x = 2\omega_z\dot{y}\,, \qquad \ddot{y} + \omega_0^2 y = -2\omega_z\dot{x}\,, \tag{3.120}$$

where

$$\omega_0 = \sqrt{\frac{g}{l}}\,. \tag{3.121}$$

Note that for any pendulum with $l < 1\,\text{km}$ we have $\omega_0 > 0.1\,\text{s}^{-1} \gg \omega_z$. The equations of motion for the pendulum are more easily solved with the help of the complex function $\zeta(t) = x(t) + iy(t)$ because equations (3.120) are the real and imaginary parts of

$$\ddot{\zeta} + 2i\omega_z\dot{\zeta} + \omega_0^2\zeta = 0\,. \tag{3.122}$$

This last equation admits solutions in the form $\zeta(t) = \zeta_0\,e^{-ipt}$ with

$$-p^2 + 2\omega_z p + \omega_0^2 = 0 \implies p = \omega_z \pm \sqrt{\omega_z^2 + \omega_0^2} \approx \omega_z \pm \omega_0\,. \tag{3.123}$$

The general solution for $\zeta(t)$ is

$$\zeta(t) = Ae^{-i(\omega_z-\omega_0)t} + Be^{-i(\omega_z+\omega_0)t}\,. \tag{3.124}$$

Adopting the initial conditions $x(0) = a$, $\dot{x}(0) = 0$, $y(0) = \dot{y}(0) = 0$, so that the pendulum starts oscillating in the xz-plane, we get

$$A + B = a\,, \qquad B = \frac{\omega_0 - \omega_z}{\omega_0 + \omega_z}A \approx \frac{\omega_0}{\omega_0}A = A \implies A = B = a/2\,, \tag{3.125}$$

whence

$$\zeta(t) = a\cos\omega_0 t\, e^{-i\omega_z t}\,. \tag{3.126}$$

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Therefore,

$$x(t) = \operatorname{Re}\zeta(t) = a\cos\omega_z t\,\cos\omega_0 t\,,$$
$$y(t) = \operatorname{Im}\zeta(t) = -a\sin\omega_z t\,\cos\omega_0 t\,. \tag{3.127}$$

The angle ϕ that the plane of oscillation makes with the initial plane of oscillation (the xz-plane) is such that

$$\tan\phi = \frac{y}{x} = -\tan\omega_z t \quad\Longrightarrow\quad \phi = -\omega_z t = -(\omega\sin\lambda)t\,. \tag{3.128}$$

The plane of oscillation rotates about the vertical with angular velocity $\omega\sin\lambda$, the sense, viewed from above, being clockwise in the northern hemisphere ($\lambda > 0$) and counterclockwise in the southern hemisphere ($\lambda < 0$). Each 24 hours the plane of oscilllation rotates by $360\sin\lambda$ degrees.[5] At the latitude of Rio de Janeiro ($\lambda = -22°54'$), the plane of oscillation turns 140° a day counterclockwise. In the North Pole the rate of rotation is 360° a day clockwise, which can be understood as follows: The plane of oscillation is fixed in an inertial reference frame, but the Earth rotates under the pendulum by 360° a day in the counterclockwise sense. So, to an observer on Earth the sense of rotation of the plane of oscillation is clockwise. As a final remark, note that (3.127) shows that the order of magnitude of $\dot{x}$ and $\dot{y}$ is $\omega_0 a$ since $\omega_0 \gg \omega > \omega_z$. On the other hand, the order of magnitude of $|\boldsymbol{\omega}\times(\mathbf{r}\times\boldsymbol{\omega})|$ is $\omega^2 a$. Thus, the ratio of the centrifugal and Coriolis terms in (3.101) is $\omega^2 a/2\omega_0 a\omega = \omega/2\omega_0 \ll 1$, which justifies *a posteriori* the approximation that led to (3.113).

Problems

3.1 In three dimensions Eq. (3.64) establishes a bijective correspondence between vectors and real antisymmetric matrices. (a) If $\mathcal{A} = (a_{kl})$ is the antisymmetric matrix associated with the vector $\mathbf{v} = (v_1, v_2, v_3)$, show that $a_{kl} = \epsilon_{kml}v_m$ with sum over repeated indices and ϵ_{kml} the Levi-Civita symbol defined in Appendix A. Show that, conversely, $v_m = \frac{1}{2}\epsilon_{kml}a_{kl}$. (b) Prove that the eigenvalues of the antisymmetric matrix associated with $\mathbf{v}$ are zero and $\pm i|\mathbf{v}|$. (c) If $\mathcal{A}$ is a real antisymmetric matrix, show that the matrices $\boldsymbol{I} \pm \mathcal{A}$ are non-singular and that the matrix $\mathcal{B} = (\boldsymbol{I} + \mathcal{A})(\boldsymbol{I} - \mathcal{A})^{-1}$ is orthogonal.

3.2 Construct the vector $\mathbf{r}'$ obtained by reflecting vector $\mathbf{r}$ in the plane whose unit normal vector is $\hat{\mathbf{n}}$. Without any calculations, using only geometric arguments, determine the eigenvalues and eigenvectors of the corresponding transformation matrix $\mathcal{A}$. If $\hat{\mathbf{n}} = (n_1, n_2, n_3)$, show that $\mathcal{A}$ has elements $a_{ij} = \delta_{ij} - 2n_i n_j$ and is an improper orthogonal matrix.

[5] Foucault intuited this result in 1851, before the mathematical treatment of the problem taking into account the Coriolis force as the cause of the rotation (Dugas, 1988).

3.3 For the matrix $\mathcal{D}$ in Eq. (3.46), prove that

$$\mathcal{D}^n = \begin{pmatrix} \cos n\phi & \sin n\phi & 0 \\ -\sin n\phi & \cos n\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for any natural number n. Was this result intuitively expected?

3.4 Any proper orthogonal matrix $\mathcal{A}$ corresponds to a rotation by some angle Φ about some direction. The rotation axis is determined by the eigenvector of $\mathcal{A}$ with eigenvalue 1. Picking coordinates such that the z-axis is along the said eigenvector the matrix $\mathcal{A}$ will have the form (3.46). Taking into account that the trace of a matrix is independent of the coordinate system, show that the rotation angle is given by $\cos\Phi = (\mathrm{tr}\,\mathcal{A} - 1)/2$.

3.5 A sphere of radius R rolls without slipping on a plane surface. If (x, y, R) are Cartesian coordinates of the centre of the sphere, show that, in terms of the Euler angles, the conditions for rolling without slipping are

$$\dot{x} - R(\dot{\theta}\,\sin\phi - \dot{\psi}\,\sin\theta\,\cos\phi) = 0\,, \qquad \dot{y} + R(\dot{\theta}\,\cos\phi + \dot{\psi}\,\sin\theta\,\sin\phi) = 0\,.$$

Prove that these constraints are not holonomic (see Appendix B).

3.6 The translational displacement of a body can be represented by a vector and the linear velocity is the time derivative of the position vector. The angular velocity vector, however, is not, in general, the time derivative of an angular displacement vector. In order to prove this, note that if there exists such a vector $\mathbf{\Omega}$, its components $\Omega_x, \Omega_y, \Omega_z$ can be expressed in terms of the Euler angles and must be such that their time derivatives equal the corresponding components of the angular velocity:

$$\omega_x = \frac{\partial\Omega_x}{\partial\theta}\dot{\theta} + \frac{\partial\Omega_x}{\partial\phi}\dot{\phi} + \frac{\partial\Omega_x}{\partial\psi}\dot{\psi}\,, \quad \omega_y = \frac{\partial\Omega_y}{\partial\theta}\dot{\theta} + \frac{\partial\Omega_y}{\partial\phi}\dot{\phi} + \frac{\partial\Omega_y}{\partial\psi}\dot{\psi}\,,$$

$$\omega_z = \frac{\partial\Omega_z}{\partial\theta}\dot{\theta} + \frac{\partial\Omega_z}{\partial\phi}\dot{\phi} + \frac{\partial\Omega_z}{\partial\psi}\dot{\psi}\,.$$

Using (3.86), prove that there can be no vector $(\Omega_x, \Omega_y, \Omega_z)$ such that these equations are satisfied.

3.7 Equation (3.67) can be derived in a fully analytic fashion without appealing to the geometric definition of the angular velocity vector. (a) By definition, the derivative of a matrix is the matrix formed by the derivatives of its elements. If $\mathcal{A}$ is a time-dependent orthogonal matrix, by differentiating the identity $\mathcal{A}^T\mathcal{A} = \boldsymbol{I}$ with respect to time prove that the matrix $\dot{\mathcal{A}}^T\mathcal{A}$ is antisymmetric – that is, $\dot{\mathcal{A}}^T\mathcal{A} = -(\dot{\mathcal{A}}^T\mathcal{A})^T$. (b) Let $\mathbf{r}_\Sigma$ and $\mathbf{r}_{\Sigma'}$ be the components of the *same* position vector $\mathbf{r}$ of a particle relative to the inertial frame Σ and the rotating frame Σ', and let $\mathcal{A}$ be the rotation matrix from Σ to Σ'. If $\mathbf{v}$ and $\mathbf{v}'$ are the velocities relative to the respective inertial and rotating frames, we have $\mathbf{v}_\Sigma = d\mathbf{r}_\Sigma/dt$ and $\mathbf{v}'_{\Sigma'} = d\mathbf{r}_{\Sigma'}/dt$. Starting from $\mathbf{r}_\Sigma = \mathcal{A}^T\mathbf{r}_{\Sigma'}$ derive

$$\mathbf{v}_\Sigma = \mathbf{v}'_\Sigma + \dot{\mathcal{A}}^T\mathcal{A}\,\mathbf{r}_\Sigma\,.$$

Taking into account part (a), the correspondence (3.64) between vectors and antisymmetric matrices and the fact that an equality of vectors which is true in one coordinate system is true in all coordinate systems, prove that there exists a vector $\boldsymbol{\omega}$ such that

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}\,,$$

which coincides with Eq. (3.67).

3.8 Let $\mathcal{P}$ be the matrix associated with a rotation through 180° about an arbitrary axis. (a) Determine $\mathcal{P}^2$ without calculations, reflecting on its meaning. (b) Letting $\mathcal{A} = (\boldsymbol{I} + \mathcal{P})/2$ and $\mathcal{B} = (\boldsymbol{I} - \mathcal{P})/2$, prove that $\mathcal{A}^2 = \mathcal{A}$ and $\mathcal{B}^2 = \mathcal{B}$. (c) Show that the matrices $\mathcal{A}$ and $\mathcal{B}$ are singular and compute their product.

3.9 A particle is fired vertically upward from the surface of the Earth with initial speed v_0, reaches its maximum height and returns to the ground. Show that the Coriolis deflection when it hits the ground has the opposite sense and is four times bigger than the deviation for a particle dropped from the same maximum height.

3.10 Show that the equation of motion (3.98) can be derived from the Lagrangian

$$L = \frac{mv^2}{2} - V - m\mathbf{r}\cdot\mathbf{a}_{O'} + \frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2 + m\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r})\,,$$

where $\mathbf{v} = \dot{\mathbf{r}}$ and V is the potential energy corresponding to the force $\mathbf{F}$. Obtain the Jacobi integral associated with this Lagrangian in the case in which $\mathbf{a}_{O'}$ and $\boldsymbol{\omega}$ are constant.

3.11 At $t = 0$ a projectile is fired horizontally near the surface of the Earth with speed v. (a) Neglecting gravity, show that, to a good approximation, the horizontal deviation of the projectile due to the Coriolis force is

$$d_H = \omega v |\sin\lambda| t^2,$$

where λ is the latitude. (b) Show that the projectile is deflected to the right in the northern hemisphere and to the left in the southern hemisphere. (c) To the same degree of approximation, express d_H in terms of the projectile's distance D from the firing point at time t. (d) How would the inclusion of the influence of gravity affect the result? (e) What changes if the projectile is fired at an angle above the horizontal? (f) It is told that during a World War I naval battle in the south Atlantic British shells missed German ships by about 90 meters to the left because the British gunsights had been corrected to a latitude 50° north (for battles in Europe) instead of 50° south where the battle took place. Conclude that the British shells would have missed the targets by twice the deflection calculated above. (g) Assuming the tale is true[6] and $v = 700$ m/s, how distant must the German ships have been?

[6] It is a pity that such a savoury story is most likely but an urban legend, as convincingly argued at http://dreadnoughtproject.org/tfs/index.php/Battle_of_the_Falkland_Islands.