


COMPARISON OF KUMMER LOGARITHMIC TOPOLOGIES WITH CLASSICAL TOPOLOGIES

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(Received 8 January 2020; revised 25 June 2021; accepted 28 June 2021; first published online 27 July 2021)

Abstract We compare the Kummer flat (resp., Kummer étale) cohomology with the flat (resp., étale) cohomology with coefficients in smooth commutative group schemes, finite flat group schemes, and Kato's logarithmic multiplicative group. We are particularly interested in the case of algebraic tori in the Kummer flat topology. We also make some computations for certain special cases of the base log scheme.

Keywords and Phrases: log schemes, Kummer flat topology, Kummer étale topology, comparison of cohomology

2020 Mathematics Subject Classification: Primary 14F20
Secondary 14A21

1. Notation and conventions

Let $X = (X, M_X)$ be an fs (fine saturated) log scheme; we denote by (fs/X) the category of fs log schemes over X and by $(\text{fs}/X)_{\text{ét}}$ (resp., $(\text{fs}/X)_{\text{fl}}$, $(\text{fs}/X)_{\text{két}}$, or $(\text{fs}/X)_{\text{kfl}}$) the classical étale site (resp., classical flat site, Kummer étale site, or Kummer flat site) on (fs/X) . In order to shorten formulas, we will mostly abbreviate $(\text{fs}/X)_{\text{ét}}$ (resp., $(\text{fs}/X)_{\text{fl}}$, $(\text{fs}/X)_{\text{két}}$, or $(\text{fs}/X)_{\text{kfl}}$) as $X_{\text{ét}}$ (resp., X_{fl} , $X_{\text{két}}$, or X_{kfl}). We refer to [7, §2.5] for the classical étale site and the Kummer étale site, and to [8, Def. 2.3] and [12, §2.1] for the Kummer flat site. The site $(\text{fs}/X)_{\text{fl}}$ is an obvious analogue of $(\text{fs}/X)_{\text{ét}}$. We have natural ‘forgetful’ maps of sites:

$$\varepsilon_{\text{ét}} : (\text{fs}/X)_{\text{két}} \rightarrow (\text{fs}/X)_{\text{ét}} \tag{1.1}$$

and

$$\varepsilon_{\text{fl}} : (\text{fs}/X)_{\text{kfl}} \rightarrow (\text{fs}/X)_{\text{fl}}. \tag{1.2}$$

We denote by (st/X) the full subcategory of (fs/X) consisting of strict fs log schemes over X . Note that (st/X) is canonically identified with the category of the schemes over the underlying scheme of X .

Kato's multiplicative group (or the log multiplicative group) $\mathbb{G}_{\text{m}, \log}$ is the sheaf on $X_{\text{ét}}$ defined by $\mathbb{G}_{\text{m}, \log}(U) = \Gamma(U, M_U^{\text{gp}})$ for any $U \in (\text{fs}/X)$, where M_U^{gp} denotes the group

envelope of the log structure M_U of U . The classical étale sheaf $\mathbb{G}_{m,\log}$ is also a sheaf on $X_{\text{két}}$ and X_{kfl} (see [12, Cor. 2.22] for a proof).

By convention, for any sheaf of abelian groups F on X_{kfl} and a subgroup sheaf G of F on X_{kfl} , we denote by $(F/G)_{X_{\text{fl}}}$ the quotient sheaf on X_{fl} , and F/G denotes the quotient sheaf on X_{kfl} . We abbreviate the quotient sheaf $\mathbb{G}_{m,\log}/\mathbb{G}_m$ on X_{kfl} as $\overline{\mathbb{G}}_{m,\log}$.

Let F be a sheaf on $X_{\text{ét}}$ (resp., X_{fl} , $X_{\text{két}}$, or X_{kfl}) and set $U \in (\text{fs}/X)$; we denote by $H_{\text{ét}}^i(U, F)$ (resp., $H_{\text{fl}}^i(U, F)$, $H_{\text{két}}^i(U, F)$, or $H_{\text{kfl}}^i(U, F)$) the i th sheaf cohomology group of the sheaf F on $X_{\text{ét}}$ (resp., X_{fl} , $X_{\text{két}}$, or X_{kfl}). For a sheaf F on $X_{\text{fl}} = (\text{fs}/X)_{\text{fl}}$, the canonical map $(\text{fs}/X)_{\text{fl}} \rightarrow (\text{st}/X)_{\text{fl}}$ of sites satisfies the conditions from [15, Tag 00XU]; hence by [15, Tag 03YU], the cohomology of the sheaf F on $(\text{fs}/X)_{\text{fl}}$ can be computed on the smaller site $(\text{st}/X)_{\text{fl}}$. Hence for $U \in (\text{st}/X)$, we will use the same notation $H_{\text{fl}}^i(U, F)$ for the cohomology of F on any of the two sites $(\text{fs}/X)_{\text{fl}}$ and $(\text{st}/X)_{\text{fl}}$.

Let G be a group scheme over X and F be a presheaf of abelian groups on (fs/X) endowed with a G -action. We denote by $H_X^i(G, F)$ the i th cohomology group of the G -module F (see [3, Chap. II, §3] for the definition of such cohomology groups). For an abstract group Γ and a Γ -module N , we denote by $H^i(\Gamma, N)$ the i th cohomology group of the Γ -module N .

2. Introduction

According to [6, §0], the notion of log structure is due to L. Illusie and J.-M. Fontaine, and the theory of log structure is essentially carried out by K. Kato. One merit of log geometry is that certain nonsmooth morphisms in classical algebraic geometry become smooth (more precisely, log smooth) in the world of log geometry. There is also the notion of log étale morphism, as well as the notion of log flat morphism, in log geometry. Using log étale morphisms (resp., Kummer log étale morphisms), K. Fujiwara, K. Kato, and C. Nakayama developed a theory of log étale topology (resp., Kummer log étale topology); see [7] for an overview. Using Kummer log flat morphisms, K. Kato developed a theory of Kummer log flat topology [8]. We will abbreviate ‘Kummer log étale’ (resp., ‘Kummer log flat’) simply as ‘Kummer étale’ (resp., ‘Kummer flat’).

In order to understand cohomology theory in the Kummer étale topology (resp., Kummer flat topology), one is led to compare the Kummer étale topology (resp., Kummer flat topology) with the classical étale topology (resp., classical flat topology). The main topic of this paper is to investigate such comparisons – that is, to investigate the functor $R^i\varepsilon_{\text{ét}*}$ (resp., $R^i\varepsilon_{\text{fl}*}$) associated to the map of sites (1.1) (resp., map (1.2)) for $i > 0$.

2.1. Main results and outline

In the first half of §3.1 we study the higher direct images $R^i\varepsilon_{\text{ét}*}G$ for G a smooth commutative group scheme with connected fibers over the base. The main result is the following theorem (see also Theorem 3.8):

Theorem 2.1. *Let X be a locally Noetherian fs log scheme, G be a smooth commutative group scheme with connected fibers over the underlying scheme of X , and i be a positive integer.*

(1) We have

$$R^i \varepsilon_{\text{ét}*} G = \varinjlim_n (R^i \varepsilon_{\text{ét}*} G)[n] = \bigoplus_{l \text{ prime}} (R^i \varepsilon_{\text{ét}*} G)[l^\infty],$$

where $(R^i \varepsilon_{\text{ét}*} G)[n]$ denotes the n -torsion subsheaf of $R^i \varepsilon_{\text{ét}*} G$ and $(R^i \varepsilon_{\text{ét}*} G)[l^\infty]$ denotes the l -primary part of $R^i \varepsilon_{\text{ét}*} G$ for a prime number l .

(2) The l -primary part $(R^i \varepsilon_{\text{ét}*} G)[l^\infty]$ is supported on the locus where l is invertible.

(3) If n is invertible on X , then

$$(R^i \varepsilon_{\text{ét}*} G)[n] = R^i \varepsilon_{\text{ét}*} G[n] = G[n](-i) \otimes_{\mathbb{Z}} \bigwedge^i (\mathbb{G}_{m, \log} / \mathbb{G}_m)_{X_{\text{ét}}}.$$

When the underlying scheme of the base X is over \mathbb{Q} , Theorem 2.1 actually gives an explicit description of $R^i \varepsilon_{\text{ét}*} G$ (see Corollary 3.9). When the underlying scheme of the base X is over a finite field, we also have an explicit description of $R^i \varepsilon_{\text{ét}*} G$ (see Corollary 3.10).

In the second half of §3.1 we study the first higher direct image $R^1 \varepsilon_{\text{fl}*} G$ for G a smooth commutative group scheme over the base. The main result is the following theorem (see also Theorem 3.14):

Theorem 2.2. *Let X be a locally Noetherian fs log scheme, and let G be either a finite flat group scheme over the underlying scheme of X or a smooth commutative group scheme over the underlying scheme of X . We endow G with the induced log structure from X . Then we have*

$$R^1 \varepsilon_{\text{fl}*} G = \varinjlim_n \mathcal{H}om_X(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} (\mathbb{G}_{m, \log} / \mathbb{G}_m)_{X_{\text{fl}}}.$$

In §3.2 we mainly study the second higher direct image $R^2 \varepsilon_{\text{fl}*} G$ for a torus G over the base. The main result is the following theorem (see also Theorem 3.23):

Theorem 2.3. *Let X be a locally Noetherian fs log scheme and let G be a torus over the underlying scheme of X .*

(1) We have

$$R^2 \varepsilon_{\text{fl}*} G = \varinjlim_n (R^2 \varepsilon_{\text{fl}*} G)[n] = \bigoplus_{l \text{ prime}} (R^2 \varepsilon_{\text{fl}*} G)[l^\infty],$$

where $(R^2 \varepsilon_{\text{fl}*} G)[n]$ denotes the n -torsion subsheaf of $R^2 \varepsilon_{\text{fl}*} G$ and $(R^2 \varepsilon_{\text{fl}*} G)[l^\infty]$ denotes the l -primary part of $R^2 \varepsilon_{\text{fl}*} G$ for a prime number l .

(2) We have $(R^2 \varepsilon_{\text{fl}*} G)[n] = R^2 \varepsilon_{\text{fl}*} G[n]$.

(3) The l -primary part $(R^2 \varepsilon_{\text{fl}*} G)[l^\infty]$ is supported on the locus where l is invertible.

(4) If n is invertible on X , then

$$(R^2 \varepsilon_{\text{fl}*} G)[n] = R^2 \varepsilon_{\text{fl}*} G[n] = G[n](-2) \otimes_{\mathbb{Z}} \bigwedge^2 (\mathbb{G}_{m, \log} / \mathbb{G}_m)_{X_{\text{fl}}}.$$

When the underlying scheme of X is a \mathbb{Q} -scheme or an \mathbb{F}_q -scheme for a finite field \mathbb{F}_q , Theorem 2.3 can be strengthened to give an explicit description of $R^2\varepsilon_{\text{fl}*}G$ (see Corollaries 3.24 and 3.25, respectively).

Another case in which we have an explicit description of $R^2\varepsilon_{\text{fl}*}G$ is related to log structure. Let $Y \in (\text{fs}/X)$ be such that the ranks of the stalks of the étale sheaf M_Y/\mathcal{O}_Y^\times are at most 1, and let (st/Y) be the full subcategory of (fs/X) consisting of strict fs log schemes over Y . Then the restriction of $R^2\varepsilon_{\text{fl}*}G$ to (st/Y) is zero (here G is as in Theorem 2.3).

Theorem 2.3 can also be slightly generalized to include a certain class of unipotent group schemes (see Theorem 3.26).

In §4 we investigate the higher direct image of Kato’s logarithmic multiplicative group $\mathbb{G}_{m,\text{log}}$. The main result is the following theorem (see also Theorem 4.1):

Theorem 2.4. *Let X be an fs log scheme with its underlying scheme locally Noetherian. Then we have the following:*

- (1) $R^r\varepsilon_{\text{fl}*}\overline{\mathbb{G}}_{m,\text{log}} = 0$ (resp., $R^r\varepsilon_{\text{ét}*}\overline{\mathbb{G}}_{m,\text{log}} = 0$) for $r \geq 1$.
- (2) The canonical map $R^r\varepsilon_{\text{fl}*}\mathbb{G}_m \rightarrow R^r\varepsilon_{\text{fl}*}\mathbb{G}_{m,\text{log}}$ (resp., $R^r\varepsilon_{\text{ét}*}\mathbb{G}_m \rightarrow R^r\varepsilon_{\text{ét}*}\mathbb{G}_{m,\text{log}}$) is an isomorphism for $r \geq 2$.

By Theorem 2.2, we have $R^1\varepsilon_{\text{fl}*}\mathbb{G}_m = \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{G}_{m,\text{log}}/\mathbb{G}_m)_{X_{\text{fl}}}$. Theorem 2.1 gives a description of $R^1\varepsilon_{\text{ét}*}\mathbb{G}_m$. In contrast to Theorem 2.4(2), the canonical maps $R^1\varepsilon_{\text{fl}*}\mathbb{G}_m \rightarrow R^1\varepsilon_{\text{fl}*}\mathbb{G}_{m,\text{log}}$ and $R^1\varepsilon_{\text{ét}*}\mathbb{G}_m \rightarrow R^1\varepsilon_{\text{ét}*}\mathbb{G}_{m,\text{log}}$ are not isomorphisms in general, by the following theorem of Kato [8, Cor. 5.2]:

Theorem 2.5 (Kato). *Let X be an fs log scheme with its underlying scheme locally Noetherian. Then we have $R^1\varepsilon_{\text{fl}*}\mathbb{G}_{m,\text{log}} = 0$ and $R^1\varepsilon_{\text{ét}*}\mathbb{G}_{m,\text{log}} = 0$.*

In §5 we apply the previous results to compute $H_{\text{kff}}^i(X, \mathbb{G}_m)$ for $i = 1, 2$ in the following two cases:

- (1) R is a discrete valuation ring with finite residue field and $X = \text{Spec } R$ endowed with the canonical log structure associated to its closed point.
- (2) K is either a number field or a function field. When K is a number field, X is the spectrum of the ring of integers of K . When K is a function field, X is the unique smooth projective curve associated to K . Let S be a finite set of closed points of X , $U := X - S$, and $j : U \hookrightarrow X$. We endow X with the log structure $j_*\mathcal{O}_U^\times \cap \mathcal{O}_X \rightarrow \mathcal{O}_X$.

2.2. History

As we have already stated, the Kummer flat topology was introduced by K. Kato in [8]. The comparison between the Kummer flat topology and the classical flat topology is initiated there. In fact, our Theorem 2.2 is a generalization of [8, Thm. 4.1] from smooth affine commutative group schemes to smooth commutative group schemes. Theorem 2.5 is also from [8, Cor. 5.2]. We would like to point out that [8] was started around 1991 and was circulated as an incomplete preprint for a long time – until 2019. The comparison results there have been reproduced by Nizioł in [12], with proofs. Besides [8], the only existing

comparison theorem between the Kummer log topologies and the classical topologies is the following one from [9]:

Theorem 2.6 (Kato and Nakayama). *Let X be an fs log scheme and F be a sheaf of abelian groups on $(\text{fs}/X)_{\acute{e}t}$ such that*

$$F = \bigcup_{n: \text{invertible on } X} \text{Ker} \left(F \xrightarrow{n} F \right).$$

Then the cup product induces an isomorphism

$$F(-q) \otimes_{\mathbb{Z}} \bigwedge^q (\mathbb{G}_{m, \log} / \mathbb{G}_m)_{X_{\acute{e}t}} \rightarrow R^q \varepsilon_{\acute{e}t*} \varepsilon_{\acute{e}t}^* F$$

for any $q \geq 0$, where $(-q)$ denotes the Tate twist.

2.3. Ideas of the proofs

As happens very often in cohomological comparison, Čech cohomology is a key tool for the proofs of the comparison results in this article.

For Theorem 2.1 (resp., Theorem 2.3), we first prove that $R^i \varepsilon_{\acute{e}t*} G$ (resp., $R^2 \varepsilon_{\text{fl}*} G$) is torsion and that $R^{i-1} \varepsilon_{\acute{e}t*} G$ (resp., $R^1 \varepsilon_{\text{fl}*} G$) is divisible. If n is invertible on the base (resp., n is arbitrary), we have a short exact sequence $0 \rightarrow G[n] \rightarrow G \xrightarrow{n} G \rightarrow 0$ on $(\text{fs}/X)_{\text{k}\acute{e}t}$ (resp., $(\text{fs}/X)_{\text{k}\text{fl}}$). This short exact sequence further gives rise to a short exact sequence

$$\begin{aligned} 0 \rightarrow R^{i-1} \varepsilon_{\acute{e}t*} G \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \rightarrow R^i \varepsilon_{\acute{e}t*} G[n] \rightarrow (R^i \varepsilon_{\acute{e}t*} G)[n] \rightarrow 0 \\ (\text{resp., } 0 \rightarrow R^1 \varepsilon_{\text{fl}*} G \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \rightarrow R^2 \varepsilon_{\text{fl}*} G[n] \rightarrow (R^2 \varepsilon_{\text{fl}*} G)[n] \rightarrow 0). \end{aligned}$$

Then we are reduced to investigating $R^i \varepsilon_{\acute{e}t*} G[n]$ (resp., $R^2 \varepsilon_{\text{fl}*} G[n]$). Clearly the sheaf $R^i \varepsilon_{\acute{e}t*} G[n]$ can be explicitly described via Theorem 2.6. And actually, the sheaf $R^1 \varepsilon_{\text{fl}*} G[n]$ can be well understood via Theorem 2.6 with some effort (see Corollary 3.21). Theorem 2.1(2) and Theorem 2.3(3) are reduced to local computations (see Corollary 3.6 and Lemma 3.20(3)).

The proof of Theorem 2.2 is a modification of that of [8, Thm. 4.1], which is the affine case of Theorem 2.2. Kato first constructs a canonical homomorphism

$$\varinjlim_n \mathcal{H}om_X(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} (\mathbb{G}_{m, \log} / \mathbb{G}_m)_{X_{\text{fl}}} \rightarrow R^1 \varepsilon_{\text{fl}*} G. \tag{2.1}$$

Then it suffices to show that this homomorphism is actually an isomorphism. One is reduced to checking this for $X = \text{Spec } A$ such that A is a Noetherian strictly Henselian local ring and X admits a chart $P \rightarrow M_X$ with P an fs monoid such that the canonical map $P \xrightarrow{\cong} M_{X,x} / \mathcal{O}_{X,x}^{\times}$ is an isomorphism, where x denotes the closed point of X . Therefore one is led to the computation of $H_{\text{k}\text{fl}}^1(X, G)$ in this case. For any positive integer m , we define

$$X_m := X \times_{\text{Spec } \mathbb{Z}[P]} \text{Spec } \mathbb{Z} \left[P^{1/m} \right] \tag{2.2}$$

endowed with the canonical log structure coming from $P^{1/m}$, where $P^{1/m}$ is a monoid endowed with a homomorphism $P \rightarrow P^{1/m}$ which can be identified with $P \xrightarrow{m} P$. We get

a Kummer flat cover $X_m \rightarrow X$, which is also a Kummer étale cover if m is coprime to the characteristic of the residue field of A . It is shown in [8, Lem. 4.6] and its proof that

$$H_{\text{kfl}}^1(X, G) = \check{H}_{\text{kfl}}^1(X, G) = \varinjlim_m \check{H}_{\text{kfl}}^1(X_m/X, G) = \varinjlim_m H^1(C_{G, m}),$$

where $C_{G, m}$ is the Čech complex for the cover $X_m \rightarrow X$ with coefficients in G . In order to show that formula (2.1) is an isomorphism, one is reduced to showing that

$$\text{Hom}_X(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} P^{\text{gp}} \rightarrow H^1(C_{G, n}) \tag{2.3}$$

is an isomorphism for each positive integer n (see [8, §4.9] for the details of the reduction). For this, Kato first deals with the case where A is complete [8, Prop. 4.10]. The proof of [8, Prop. 4.10] is first given in the Artinian case in [8, §4.11], then achieved in general by passing to the limit in [8, §4.13]. The proof of our formula (2.3) in the general case is reduced to the complete case by descent, which requires the condition that G is affine. In this article we have the following key lemma (see also Lemma 3.12), which reduces the computation of $\check{H}_{\text{kfl}}^1(X_m/X, G)$ directly to the Artinian case, so that we can remove the condition that G is affine:

Lemma 2.7 (Key lemma). *Let X be an fs log scheme whose underlying scheme is $\text{Spec } A$ with A a Noetherian strictly Henselian local ring, and let x be the closed point of X . Let $P \rightarrow M_X$ be a chart of X with P an fs monoid such that the induced map $P \xrightarrow{\cong} M_{X, x}/\mathcal{O}_{X, x}^\times$ is an isomorphism. Let X_m be as constructed in formula (2.2). We regard x as an fs log scheme with respect to the induced log structure, and $x_m := X_m \times_X x$. Let $?$ be either kfl or két; then the canonical map*

$$\check{H}_?^i(X_m/X, G) \rightarrow \check{H}_?^i(x_m/x, G)$$

is an isomorphism for all $i > 0$.

Finally, we briefly discuss the proof of Theorem 2.4. The second part of the theorem clearly follows from the first part. And the first part – that is, the vanishings of $R^r \varepsilon_{\text{fl}*} \overline{\mathbb{G}}_{m, \log}$ and $R^r \varepsilon_{\text{ét}*} \overline{\mathbb{G}}_{m, \log}$ for $r \geq 1$ – follows from the vanishings of $H_{\text{kfl}}^r(X, \overline{\mathbb{G}}_{m, \log})$ and $H_{\text{két}}^r(X, \overline{\mathbb{G}}_{m, \log})$, with X having its underlying scheme the spectrum of a Noetherian strictly Henselian local ring. The latter are given by explicit computations of Čech cohomology (see Theorem 4.1 and Lemma 4.2).

3. The higher direct images for smooth commutative group schemes

In this section we investigate the higher direct images under ε_{fl} and $\varepsilon_{\text{ét}}$ for smooth commutative group schemes. We deal with the case of $\varepsilon_{\text{ét}}$ in the first subsection, and the case of ε_{fl} in the second subsection. In the Kummer étale case, we have more tools at hand and can get results for $R^i \varepsilon_{\text{ét}*} G$ for $i \geq 2$ and G a smooth commutative group scheme with connected fibers over the base. In the Kummer flat case, we have only results for $R^2 \varepsilon_{\text{fl}*}$ for certain smooth commutative group schemes.

3.1. Kummer étale case

In this subsection we study the higher direct image $R^i\varepsilon_{\text{ét}*}G$ along the forgetful map $\varepsilon_{\text{ét}} : (\text{fs}/X)_{\text{két}} \rightarrow (\text{fs}/X)_{\text{ét}}$, where $i \geq 2$ and G is a smooth commutative group scheme with connected fibers over X regarded as a sheaf on $(\text{fs}/X)_{\text{két}}$.

In order to understand $R^i\varepsilon_{\text{ét}*}G$, we need to compute $H_{\text{két}}^i(X, G)$ first, for the case where X has its underlying scheme $\text{Spec} A$ with A a Noetherian strictly Henselian ring. We will make heavy use of Čech cohomology, for which we often refer to [10, Chap. III].

The following proposition is an analogue of [10, Chap. III, Prop. 2.9]:

Proposition 3.1. *Let ? be either fl or ét. Let X be an fs log scheme and F be a sheaf on $(\text{fs}/X)_{\text{k?}}$. Let $\underline{H}_{\text{k?}}^i(F)$ be the presheaf $U \mapsto H_{\text{k?}}^i(U, F)$ for $U \in (\text{fs}/X)$. Then we have that the 0th Čech cohomology group $\check{H}_{\text{k?}}^0(U, \underline{H}_{\text{k?}}^i(F))$ vanishes for $i > 0$ and all $U \in (\text{fs}/X)$.*

Proof. The proof of [10, Chap. III, Prop. 2.9] is purely formal, and it also works here. \square

The following corollary is an analogue of [10, Chap. III, Cor. 2.10]:

Corollary 3.2. *Let ? , X , F , and U be as in Proposition 3.1. Then the Čech cohomology to derived functor cohomology spectral sequence*

$$\check{H}_{\text{k?}}^i(U, \underline{H}_{\text{k?}}^j(F)) \Rightarrow H_{\text{k?}}^{i+j}(U, F)$$

induces isomorphisms

$$\check{H}_{\text{k?}}^i(U, F) \xrightarrow{\cong} H_{\text{k?}}^i(U, F)$$

for $i = 0, 1$, and an exact sequence

$$0 \rightarrow \check{H}_{\text{k?}}^2(U, F) \rightarrow H_{\text{k?}}^2(U, F) \rightarrow \check{H}_{\text{k?}}^1(U, \underline{H}_{\text{k?}}^1(F)) \rightarrow \check{H}_{\text{k?}}^3(U, F) \rightarrow H_{\text{k?}}^3(U, F).$$

Proof. The results follow from Proposition 3.1. \square

Corollary 3.3. *Let $X = \text{Spec} A$ be an fs log scheme with A a Noetherian strictly Henselian local ring, x the closed point of X , p the characteristic of the residue field of A , and F a sheaf on $(\text{fs}/X)_{\text{két}}$. Let $P \xrightarrow{\alpha} M_X$ be a chart of the log structure of X with P an fs monoid, such that the induced map $P \rightarrow M_{X,x}/\mathcal{O}_{X,x}^\times$ is an isomorphism. For any positive integer m , we define X_m to be the fs log scheme $X \times_{\text{Spec} \mathbb{Z}[P]} \text{Spec} \mathbb{Z}[P^{1/m}]$ endowed with the canonical log structure coming from $P^{1/m}$, where $P^{1/m}$ is a monoid endowed with a homomorphism $P \rightarrow P^{1/m}$ which can be identified with $P \xrightarrow{m} P$. We get a Kummer flat cover $f_m : X_m \rightarrow X$, which is also a Kummer étale cover if m is coprime to p .*

Set $\gamma \in H_{\text{két}}^i(X, F)$; then there exists a positive integer n with $(n, p) = 1$ such that γ maps to zero in $H_{\text{két}}^i(X_n, F)$ along $f_n : X_n \rightarrow X$.

Proof. By Proposition 3.1, there exists a Kummer étale cover

$$\left\{ Y_i \xrightarrow{g_i} X \right\}_{i \in I}$$

such that $g_i^*(\gamma) = 0$ for each $i \in I$. Let i_0 be such that $g_{i_0}(Y_{i_0})$ contains the closed point of X . By [12, Prop. 2.15], there exists a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{h} & X_n \\ g \downarrow & & \downarrow f_n \\ Y_{i_0} & \xrightarrow{g_{i_0}} & X \end{array}$$

such that the image of $g_{i_0} \circ g$ contains the closed point of X , g is Kummer étale, n is a positive integer which is invertible on X , and h is classically étale. Then we have $h^* f_n^* \gamma = g^* g_{i_0}^* \gamma = 0$. Since h is classically étale and the underlying scheme of X_n is strictly Henselian local, h has a section s . It follows that

$$f_n^* \gamma = s^* h^* (f_n^* \gamma) = s^* 0 = 0. \quad \square$$

Corollary 3.4. *Let the notation and the assumptions be as in Corollary 3.3. Then we have*

$$H_{\text{két}}^i(X, F) \cong \varinjlim_n \ker \left(H_{\text{két}}^i(X, F) \xrightarrow{f_n^*} H_{\text{két}}^i(X_n, F) \right).$$

Proof. This follows from Corollary 3.3. □

Proposition 3.5. *Let the notation and the assumptions be as in Corollary 3.3. We further define $\mathbb{N}' := \{n \in \mathbb{N} \mid (n, p) = 1\}$.*

- (1) *The family $\mathcal{X}_{\mathbb{N}} := \{X_n \rightarrow X\}_{n \geq 1}$ (resp., $\mathcal{X}_{\mathbb{N}'} := \{X_n \rightarrow X\}_{n \in \mathbb{N}'}$) of Kummer flat covers (resp., Kummer étale covers) of X satisfies condition (L3) from [1, §2], whence we have a spectral sequence*

$$\check{H}_{\text{kfl}}^i \left(\mathcal{X}_{\mathbb{N}}, \underline{H}_{\text{kfl}}^j(F) \right) \Rightarrow H_{\text{kfl}}^{i+j}(X, F) \quad (\text{resp.}, \check{H}_{\text{két}}^i \left(\mathcal{X}_{\mathbb{N}'}, \underline{H}_{\text{két}}^j(F) \right) \Rightarrow H_{\text{két}}^{i+j}(X, F)), \tag{3.1}$$

where

$$\check{H}_{\text{kfl}}^i(\mathcal{X}_{\mathbb{N}}, F) := \varinjlim_{n \in \mathbb{N}} \check{H}_{\text{kfl}}^i(X_n/X, F)$$

and

$$\check{H}_{\text{két}}^i(\mathcal{X}_{\mathbb{N}'}, F) := \varinjlim_{n \in \mathbb{N}'} \check{H}_{\text{két}}^i(X_n/X, F).$$

- (2) *We have $\check{H}_{\text{két}}^0 \left(\mathcal{X}_{\mathbb{N}'}, \underline{H}_{\text{két}}^j(F) \right) = 0$ for any $j > 0$.*

Proof. Part (1) follows from [1, Chap. II, Sec. 3, (3.3)].

Part (2) follows from Corollary 3.3. Indeed, we have

$$\check{H}_{\text{két}}^0 \left(\mathcal{X}_{\mathbb{N}'}, \underline{H}_{\text{két}}^j(F) \right) \hookrightarrow \varinjlim_{n \in \mathbb{N}'} H_{\text{két}}^j(X_n, F),$$

and the latter vanishes by Corollary 3.3. □

Corollary 3.6. *Let the notation and the assumptions be as in Corollary 3.3. Then the groups $H_{\text{két}}^i(X, F)$ are torsion and p -torsion-free for all $i > 0$.*

Proof. By the Kummer étale spectral sequence from formula (3.1) and Proposition 3.5(2), it suffices to show that the groups $\varinjlim_{n \in \mathbb{N}'} \check{H}_{\text{két}}^i \left(X_n/X, \underline{H}_{\text{két}}^j(F) \right)$ are torsion and p -torsion-free for all $i > 0$ and $j \geq 0$. Define $H_n := \text{Spec} \mathbb{Z} \left[(P^{1/n})^{\text{gp}} / P^{\text{gp}} \right]$, which is a group scheme over $\text{Spec} \mathbb{Z}$ such that

$$X_n \times_X X_n = X_n \times_{\text{Spec} \mathbb{Z}} H_n$$

(see [12, the second paragraph on p522] for more detailed descriptions of H_n). Regarded as a group scheme over X , H_n is constant; X_n is a Galois cover of X with Galois group H_n . By [10, Example 2.6], we have

$$\check{H}_{\text{két}}^i \left(X_n/X, \underline{H}_{\text{két}}^j(F) \right) = H^i \left(H_n, H_{\text{két}}^j(X_n, F) \right),$$

which is torsion and p -torsion-free for $i > 0$. This finishes the proof. □

Corollary 3.7. *Let X be a locally Noetherian fs log scheme and F be a sheaf on $(\text{fs}/X)_{\text{két}}$. Then the sheaves $R^i \varepsilon_{\text{ét}*} F$ are torsion for $i > 0$.*

Proof. This follows from Corollary 3.6. □

Theorem 3.8. *Let X be a locally Noetherian fs log scheme, G be a smooth commutative group scheme with connected fibers over the underlying scheme of X , and i be a positive integer.*

(1) *We have*

$$R^i \varepsilon_{\text{ét}*} G = \varinjlim_n (R^i \varepsilon_{\text{ét}*} G)[n] = \bigoplus_{l \text{ prime}} (R^i \varepsilon_{\text{ét}*} G)[l^\infty],$$

where $(R^i \varepsilon_{\text{ét}*} G)[n]$ denotes the n -torsion subsheaf of $R^i \varepsilon_{\text{ét}*} G$ and $(R^i \varepsilon_{\text{ét}*} G)[l^\infty]$ denotes the l -primary part of $R^i \varepsilon_{\text{ét}*} G$ for a prime number l .

(2) *The l -primary part $(R^i \varepsilon_{\text{ét}*} G)[l^\infty]$ is supported on the locus where l is invertible.*

(3) *If n is invertible on X , then*

$$(R^i \varepsilon_{\text{ét}*} G)[n] = R^i \varepsilon_{\text{ét}*} G[n] = G[n](-i) \otimes_{\mathbb{Z}} \bigwedge^i (\mathbb{G}_{m, \log} / \mathbb{G}_m)_{X_{\text{ét}}}.$$

Proof. Part (1) follows from Corollary 3.7.

Part (2) follows from Corollary 3.6.

We are left with part (3). Since n is invertible on X , the sequence

$$0 \rightarrow G[n] \rightarrow G \xrightarrow{n} G \rightarrow 0 \tag{3.2}$$

is a short exact sequence of sheaves of abelian groups for the classical flat topology and $G[n]$ is quasi-finite and étale. Indeed, if X is a point, then by the structure theorem of connected algebraic groups [2, Thm. 2.3, Thm. 2.4], we are reduced to checking the cases where G is a torus, a connected unipotent group, or an abelian variety, which are all clearly true. In general, it suffices to show that $G \xrightarrow{n} G$ is an epimorphism for the classical flat topology. This is clear, since it is set-theoretically surjective and flat by the fiberwise criterion of flatness, and therefore it is faithfully flat. Moreover, $G[n]$ being étale over X implies that $G \xrightarrow{n} G$ is even an epimorphism for the classical étale topology, and thus the sequence (3.2) is also exact for the classical étale topology. Since the pullback functor $\varepsilon_{\text{ét}}^*$ is exact, the sequence (3.2) remains exact on $(\text{fs}/X)_{\text{két}}$ and induces a long exact sequence

$$\rightarrow R^{i-1}\varepsilon_{\text{ét}*}G \xrightarrow{n} R^{i-1}\varepsilon_{\text{ét}*}G \rightarrow R^i\varepsilon_{\text{ét}*}G[n] \rightarrow R^i\varepsilon_{\text{ét}*}G \xrightarrow{n} R^i\varepsilon_{\text{ét}*}G.$$

This further induces a short exact sequence

$$0 \rightarrow R^{i-1}\varepsilon_{\text{ét}*}G \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \rightarrow R^i\varepsilon_{\text{ét}*}G[n] \rightarrow (R^i\varepsilon_{\text{ét}*}G)[n] \rightarrow 0$$

for each $i > 0$. We have $R^i\varepsilon_{\text{ét}*}G[n] = G[n](-i) \otimes_{\mathbb{Z}} \bigwedge^i (\mathbb{G}_{\text{m},\text{log}}/\mathbb{G}_{\text{m}})_{X_{\text{ét}}}$ by Theorem 2.6. To finish the proof, it suffices to prove that the sheaf $R^{i-1}\varepsilon_{\text{ét}*}G$ is n -divisible. We proceed by induction. For $i = 1$, this is clear, since $G \xrightarrow{n} G$ is an epimorphism of sheaves of abelian groups for the classical étale topology. Assume $R^j\varepsilon_{\text{ét}*}G$ is n -divisible; then we have $R^{j+1}\varepsilon_{\text{ét}*}G[n] \xrightarrow{\cong} (R^{j+1}\varepsilon_{\text{ét}*}G)[n]$. Therefore

$$\begin{aligned} \varinjlim_r (R^{j+1}\varepsilon_{\text{ét}*}G)[n^r] &= \varinjlim_r R^{j+1}\varepsilon_{\text{ét}*}G[n^r] \\ &= \varinjlim_r G[n^r](-j-1) \otimes_{\mathbb{Z}} \bigwedge^{j+1} (\mathbb{G}_{\text{m},\text{log}}/\mathbb{G}_{\text{m}})_{X_{\text{ét}}}, \end{aligned}$$

where the second equality follows from Theorem 2.6. Hence $\varinjlim_r (R^{j+1}\varepsilon_{\text{ét}*}G)[n^r]$ is n -divisible. It follows by part (1) that $R^{j+1}\varepsilon_{\text{ét}*}G$ is n -divisible. This finishes the induction. □

Corollary 3.9. *Let X be a locally Noetherian fs log scheme such that the underlying scheme of X is a \mathbb{Q} -scheme, and G be a smooth commutative group scheme with connected fibers over the underlying scheme of X . Then we have*

$$R^i\varepsilon_{\text{ét},*}G = \varinjlim_n G[n](-i) \otimes_{\mathbb{Z}} \bigwedge^i (\mathbb{G}_{\text{m},\text{log}}/\mathbb{G}_{\text{m}})_{X_{\text{ét}}}.$$

Proof. This follows from Theorem 3.8. □

Corollary 3.10. *Let p be a prime number. Let X be a locally Noetherian fs log scheme such that the underlying scheme of X is an \mathbb{F}_p -scheme, and G be a smooth commutative*

group scheme with connected fibers over the underlying scheme of X . Then we have

$$R^i \varepsilon_{\text{ét}*} G = \varinjlim_{(n,p)=1} G[n](-i) \otimes_{\mathbb{Z}} \bigwedge^i (\mathbb{G}_{m,\log}/\mathbb{G}_m)_{X_{\text{ét}}}.$$

Proof. This follows from Theorem 3.8. □

Now we are going to generalize Kato’s description of $R^1 \varepsilon_{\text{fl}*} G$ [8, Thm. 4.1] from smooth affine group schemes to smooth group schemes. The following lemma is analogous to [10, Proof of Thm. III.3.9, Step 2 plus Rmk. 3.11 (b)]:

Lemma 3.11. *Let X be a locally Noetherian fs log scheme endowed with a chart $P \rightarrow M_X$ with P an fs monoid satisfying $P^\times = 1$, and G be a smooth commutative group scheme over X endowed with the induced log structure from X . For a positive integer m , we define $P^{1/m}$ and $f_m : X_m \rightarrow X$ as in Corollary 3.3. Let H_m be the group scheme $\text{Spec } \mathbb{Z} \left[(P^{1/m})^{\text{gp}} / P^{\text{gp}} \right]$ over $\text{Spec } \mathbb{Z}$; then we have that the $(r + 1)$ -fold product $X_m \times_X \cdots \times_X X_m$ is isomorphic to $X_m \times_{\text{Spec } \mathbb{Z}} H_m^r$, where H_m^r denotes the r -fold product of H_m over $\text{Spec } \mathbb{Z}$.*

We define $\underline{C}^i(G)$ to be the complex of functors $\underline{C}^i(G) : (\text{st}/X) \rightarrow \text{Ab}$ such that for any $Y \in (\text{st}/X)$, $\underline{C}^i(G)(Y)$ is the Čech complex $C^i(Y_m/Y, G)$ for the Kummer flat cover $Y_m := Y \times_X X_m \rightarrow Y$. Write $\underline{Z}^i(G)$ for the functor

$$(\text{st}/X) \rightarrow \text{Ab}, Y \mapsto \ker(d^i : C^i(Y_m/Y, G) \rightarrow C^{i+1}(Y_m/Y, G)).$$

Then $d^{i-1} : \underline{C}^{i-1}(G) \rightarrow \underline{Z}^i(G)$ is representable by a smooth morphism of algebraic spaces over X for $i \geq 1$.

Proof. By definition, $\underline{C}^i(G)$ is the functor

$$(\text{st}/X) \rightarrow \text{Ab}, Y \mapsto G(Y_m \times_Y \cdots \times_Y Y_m) = G(Y \times_X X_m \times_{\text{Spec } \mathbb{Z}} H_m^i);$$

that is, it is $\hat{\pi}_* G$, where π denotes the map $X_m \times_{\text{Spec } \mathbb{Z}} H_m^i \rightarrow X$ and $\hat{\pi}$ denotes the underlying map of schemes of π . Since $\hat{\pi}$ is clearly finite and faithful flat, $\underline{C}^i(G)$ is therefore represented by the Weil restriction of scalars of $G \times_X (X_m \times_{\text{Spec } \mathbb{Z}} H_m^i)$, which is representable by a group algebraic space by [10, Chap. V, 1.4 (a)]. Hence the functor $\underline{Z}^i(G)$, as the kernel of a map $d^i : \underline{C}^i(G) \rightarrow \underline{C}^{i+1}(G)$ of group algebraic spaces over X , is representable by a group algebraic space over X .

Now we prove the smoothness of $d^{i-1} : \underline{C}^{i-1}(G) \rightarrow \underline{Z}^i(G)$. It suffices to show that for any affine X -scheme T , closed subscheme T_0 of T defined by an ideal I of square zero, and $z \in \underline{Z}^i(G)(T)$ whose image z_0 in $\underline{Z}^i(G)(T_0)$ arises from an element $c_0 \in \underline{C}^{i-1}(G)(T_0)$, there exists $c \in \underline{C}^{i-1}(G)(T)$ such that c maps to c_0 . Let N be the functor

$$(\text{Sch}/T) \rightarrow \text{Ab}, Y \mapsto \ker(G(Y) \rightarrow G(Y \times_T T_0)).$$

Let $C^\cdot(T_m/T, N)$ be the Čech complex for the Kummer flat cover $T_m := T \times_X X_m \rightarrow T$ with coefficients in N . Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C^{i-1}(T_m/T, N) & \longrightarrow & \underline{C}^{i-1}(G)(T) & \longrightarrow & \underline{C}^{i-1}(G)(T_0) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C^i(T_m/T, N) & \longrightarrow & \underline{C}^i(G)(T) & \longrightarrow & \underline{C}^i(G)(T_0) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C^{i+1}(T_m/T, N) & \longrightarrow & \underline{C}^{i+1}(G)(T) & \longrightarrow & \underline{C}^{i+1}(G)(T_0) \longrightarrow 0,
 \end{array}$$

where the exactness property at the right-hand side follows from the smoothness of G . Let $Z^i(T_m/T, N)$ be the kernel of

$$C^i(T_m/T, N) \rightarrow C^{i+1}(T_m/T, N);$$

then the diagram induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C^{i-1}(T_m/T, N) & \longrightarrow & \underline{C}^{i-1}(G)(T) & \longrightarrow & \underline{C}^{i-1}(G)(T_0) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & Z^i(T_m/T, N) & \longrightarrow & \underline{Z}^i(G)(T) & \longrightarrow & \underline{Z}^i(G)(T_0).
 \end{array}$$

By an easy diagram chasing, for the existence of c it suffices to show that the map $C^{i-1}(T_m/T, N) \rightarrow Z^i(T_m/T, N)$ is surjective – that is, that $\check{H}^i(T_m/T, N) = 0$, $i \geq 1$.

To finish the proof, we compute $\check{H}^i(T_m/T, N)$ for $i \geq 1$. Let \tilde{N} be the functor

$$(\text{Sch}/T) \rightarrow \text{Ab}, U \mapsto N(U \times_T T_m).$$

Since $T_m \times_T T_m \cong T_m \times H_m$, the group scheme H_m acts on the functor \tilde{N} , and the Čech complex $C^\cdot(T_m/T, N)$ can be identified with the standard complex $C^\cdot(H_m, \tilde{N})$ computing the cohomology of the H_m -module \tilde{N} . We claim that \tilde{N} is coherent; then the vanishing of $\check{H}^i(T_m/T, N)$ follows from H_m being diagonalizable by [4, Exposé I, Thm. 5.3.3]. For any $U \in (\text{Sch}/T)$, the smoothness of G implies that

$$\begin{aligned}
 \tilde{N}(U) &= N(U \times_T T_m) = \ker(G(U \times_T T_m) \rightarrow G(U \times_T T_m \times_T T_0)) \\
 &= \text{Lie}(G) \otimes_{\Gamma(T, \mathcal{O}_T)} \Gamma(T_m, I\mathcal{O}_{T_m}) \otimes_{\Gamma(T, \mathcal{O}_T)} \Gamma(U, \mathcal{O}_U).
 \end{aligned}$$

Therefore \tilde{N} is coherent. □

The following lemma is analogous to [10, Proof of Thm. III.3.9, Step 3 plus Rmk. 3.11 (b)]:

Lemma 3.12 (Key lemma). *Let $X, X_m, f_m, P, P^{1/m}$ be as in Lemma 3.11. We further assume that the underlying scheme of X is $\text{Spec } A$ with A a Henselian local ring, and let x be the closed point of X . We regard x as an fs log scheme with respect to the induced log structure, and $x_m := x \times_{\text{Spec } \mathbb{Z}[P]} \text{Spec } \mathbb{Z}[P^{1/m}]$ is obviously identified with $X_m \times_X x$*

canonically. Let ? be either *kfl* or *két*; then the canonical map

$$\check{H}_{\text{?}}^i(X_m/X, G) \rightarrow \check{H}_{\text{?}}^i(x_m/x, G)$$

is an isomorphism for all $i > 0$.

Proof. Since G is smooth and $X_m \times_X \cdots \times_X X_m \cong X_m \times_{\text{Spec } \mathbb{Z}} H_m^r$ is a disjoint union of spectra of Henselian local rings, the maps

$$C^i(X_m/X, G) \rightarrow C^i(x_m/x, G)$$

are surjective by [10, Chap. I, 4.13]. Thus we are reduced to showing that the complex $\ker(C^i(X_m/X, G) \rightarrow C^i(x_m/x, G))$ is exact. Let

$$z \in \ker \left(C^i(X_m/X, G) \xrightarrow{d^i} C^{i+1}(X_m/X, G) \right)$$

have image $z_0 = 0$ in $C^i(x_m/x, G)$. We seek a $c \in C^{i-1}(X_m/X, G)$ with $c_0 = 0$ such that $d^{i-1}(c) = z$. By Lemma 3.11 we know that $(d^{i-1})^{-1}(z)$ is representable by a smooth scheme over X . But $(d^{i-1})^{-1}(z)$ has a section over x , namely the zero section, and as X is Henselian, this lifts to a section of $(d^{i-1})^{-1}(z)$ over X . This finishes the proof. \square

With the help of Lemma 3.12, we can give a slightly different proof of Kato’s theorem ([8, Thm. 4.1]), which describes $R^1\varepsilon_{\text{fl}*}G$ for a smooth affine group scheme G . This alternative proof allows us to remove the affinity condition on G from [8, Thm. 4.1 (ii)].

Theorem 3.13. *Let $X = \text{Spec } A$ be an fs log scheme with A a Noetherian strictly Henselian local ring, x the closed point of X , and p the characteristic of the residue field of A , and fix a chart $P \rightarrow M_X$ satisfying $P \cong M_{X,x}/\mathcal{O}_{X,x}^\times$. Let ? be either *fl* or *ét*, let $\varepsilon_{\text{?}} : (\text{fs}/X)_{\text{k?}} \rightarrow (\text{fs}/X)_{\text{?}}$ be the canonical ‘forgetful’ map of sites, and let G be a smooth commutative group scheme over X endowed with the log structure induced from X . Then we have canonical isomorphisms*

$$H_{\text{k?}}^1(X, G) \cong \check{H}_{\text{k?}}^1(X, G) \cong \begin{cases} \varinjlim_{(n,p)=1} \text{Hom}_X(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} P^{\text{gp}} & \text{if } \text{?} = \text{ét}, \\ \varinjlim_n \text{Hom}_X(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} P^{\text{gp}} & \text{if } \text{?} = \text{fl}. \end{cases}$$

Proof. In the proof of [12, Prop. 3.13], the affineness of G is only used in the paragraph before [12, Cor. 3.17]. Its use is to extend from the complete local case [12, Lem. 3.15] to the Henselian local case. Note that the proof of [12, Lem. 3.15] deals with the Artinian local case first, then passes to the complete local case. With the help of Lemma 3.12, we can pass from the Artinian local case directly to the Henselian local case, and hence no affineness of G is needed. \square

Theorem 3.14. *Let X be a locally Noetherian fs log scheme and G be either a finite flat group scheme over the underlying scheme of X or a smooth commutative group scheme over the underlying scheme of X . We endow G with the induced log structure from X . Then we have*

$$R^1\varepsilon_{\text{fl}*}G = \varinjlim_n \text{Hom}_X(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} (\mathbb{G}_{\text{m,log}}/\mathbb{G}_{\text{m}})_{X_{\text{fl}}}.$$

Proof. The statement for the finite flat case is the same as in [8, Thm. 4.1]. We only need to deal with the other case, which follows from Theorem 3.13 in the same way as [12, Thm. 3.12] follows from [12, Prop. 3.13]. \square

The following proposition about $H_{k^?}^1(X, G)$ will be used in next subsection:

Proposition 3.15. *Let the notation and the assumptions be as in Theorem 3.13. Then we have $\varinjlim_n H_{\text{kff}}^1(X_n, G) = 0$ and $\varinjlim_{(n,p)=1} H_{\text{két}}^1(X_n, G) = 0$.*

Proof. We prove only $\varinjlim_n H_{\text{kff}}^1(X_n, G) = 0$; the other statement can be proven in the same way. By Theorem 3.13, we have

$$H_{\text{kff}}^1(X_n, G) \cong \varinjlim_r \text{Hom}_{X_n}(\mathbb{Z}/r\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} (P^{1/n})^{\text{gp}}$$

for each $n > 0$. As n varies, these isomorphisms fit into commutative diagrams of the form

$$\begin{array}{ccc} H_{\text{kff}}^1(X_n, G) & \longrightarrow & H_{\text{kff}}^1(X_{mn}, G) \\ \cong \downarrow & & \downarrow \cong \\ \varinjlim_r \text{Hom}_{X_n}(\mathbb{Z}/r\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} (P^{1/n})^{\text{gp}} & \longrightarrow & \varinjlim_r \text{Hom}_{X_{mn}}(\mathbb{Z}/r\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} (P^{1/mn})^{\text{gp}}, \end{array}$$

with the second row induced by the canonical inclusion $P^{1/n} \hookrightarrow P^{1/mn}$. The group $\text{Hom}_{X_n}(\mathbb{Z}/r\mathbb{Z}(1), G)$ is clearly torsion, and thus it follows that

$$\varinjlim_n H_{\text{kff}}^1(X_n, G) = \varinjlim_n \varinjlim_r \text{Hom}_{X_n}(\mathbb{Z}/r\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} (P^{1/n})^{\text{gp}} = 0. \quad \square$$

3.2. Kummer flat case

Throughout this subsection, X is an fs log scheme with its underlying scheme locally Noetherian and G is a smooth commutative group scheme over the underlying scheme of X . We are going to investigate the second higher direct image $R^2\varepsilon_{\text{fl}*}G$ along the forgetful map $\varepsilon_{\text{fl}} : (\text{fs}/X)_{\text{kff}} \rightarrow (\text{fs}/X)_{\text{fl}}$. We have a satisfactory result in the important case where G is a torus (as well as in a slightly more general case). The reason we can deal with $R^i\varepsilon_{\text{fl}*}G$ only for $i = 2$ and G a suitable group scheme (mainly tori) is that we are only able to do computations of higher ($i > 1$) group scheme cohomology in this case.

The following proposition is the counterpart of Corollary 3.3 in the Kummer flat topology, but only for $i = 2$ and $F = G$ for a smooth commutative group scheme G :

Proposition 3.16. *Let X, x, A, p, P, X_m , and f_m be as in Corollary 3.3. Let G be a smooth commutative group scheme over the underlying scheme of X .*

Let $\gamma \in H_{\text{kff}}^2(X, G)$; then there exists a positive integer n such that γ maps to zero in $H_{\text{kff}}^2(X_n, G)$ along $f_n : X_n \rightarrow X$.

Proof. By Proposition 3.1, we can find a Kummer flat cover $T \rightarrow X$ such that γ dies in $H_{\text{kff}}^2(T, G)$. By [12, Cor. 2.16] we may assume that for some n , we have a factorization $T \rightarrow X_n \rightarrow X$ with $T \rightarrow X_n$ a classical flat cover. It follows that the class γ on X_n is

trivialized by a classical flat cover – that is, $\gamma \in \ker(H_{\text{kfl}}^2(X_n, G) \rightarrow H_{\text{fl}}^0(X_n, R^2\varepsilon_{\text{fl}*}G))$. The seven-term exact sequence of the spectral sequence $H_{\text{fl}}^i(X_n, R^j\varepsilon_{\text{fl}*}G) \Rightarrow H_{\text{kfl}}^{i+j}(X_n, G)$ gives an exact sequence

$$\cdots \rightarrow H_{\text{fl}}^2(X_n, G) \rightarrow \ker(H_{\text{kfl}}^2(X_n, G) \rightarrow H_{\text{fl}}^0(X_n, R^2\varepsilon_{\text{fl}*}G)) \rightarrow H_{\text{fl}}^1(X_n, R^1\varepsilon_{\text{fl}*}G).$$

Hence, to show that $\gamma = 0$ in $H_{\text{kfl}}^2(X_n, G)$, it suffices to show that $H_{\text{fl}}^2(X_n, G) = H_{\text{fl}}^1(X_n, R^1\varepsilon_{\text{fl}*}G) = 0$.

Since G is smooth and A is strictly Henselian, we have

$$H_{\text{fl}}^2(X_n, G) = H_{\text{ét}}^2(X_n, G) = 0.$$

We have $R^1\varepsilon_{\text{fl}*}G = \varinjlim_m \mathcal{H}om_{X_n}(\mathbb{Z}/m\mathbb{Z}(1), G) \otimes (\mathbb{G}_{m, \log}/\mathbb{G}_m)_{X_n, \text{fl}}$ by Theorem 3.14. By Lemma A.1, the sheaf $\mathcal{H}om_{X_n}(\mathbb{Z}/m\mathbb{Z}(1), G)$ is representable by a quasi-finite étale separated group scheme over X_n . It follows that

$$\begin{aligned} H_{\text{fl}}^1(X_n, R^1\varepsilon_{\text{fl}*}G) &= H_{\text{fl}}^1\left(X_n, \varinjlim_m \mathcal{H}om_{X_n}(\mathbb{Z}/m\mathbb{Z}(1), G) \otimes (\mathbb{G}_{m, \log}/\mathbb{G}_m)_{X_n, \text{fl}}\right) \\ &= \varinjlim_m H_{\text{fl}}^1\left(X_n, \mathcal{H}om_{X_n}(\mathbb{Z}/m\mathbb{Z}(1), G) \otimes (\mathbb{G}_{m, \log}/\mathbb{G}_m)_{X_n, \text{fl}}\right) \\ &= \varinjlim_m H_{\text{fl}}^1\left(X_n, \theta^*\left(\mathcal{H}om_{X_n}(\mathbb{Z}/m\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} (\mathbb{G}_{m, \log}/\mathbb{G}_m)_{X_n, \text{ét}}\right)\right) \\ &= \varinjlim_m H_{\text{ét}}^1\left(X_n, \mathcal{H}om_{X_n}(\mathbb{Z}/m\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} (\mathbb{G}_{m, \log}/\mathbb{G}_m)_{X_n, \text{ét}}\right) \\ &= 0, \end{aligned}$$

where $\theta : (\text{fs}/X_n)_{\text{fl}} \rightarrow (\text{fs}/X_n)_{\text{ét}}$ denotes the forgetful map between these two sites. This finishes the proof. □

The following corollary is the counterpart of Corollary 3.4 and Proposition 3.5(2) for the Kummer flat topology:

Corollary 3.17. *Let the notation and the assumptions be as in Proposition 3.16. Then we have the following:*

- (1) $H_{\text{kfl}}^2(X, G) \cong \varinjlim_n \ker(H_{\text{kfl}}^2(X, G) \rightarrow H_{\text{kfl}}^2(X_n, G))$.
- (2) $\check{H}_{\text{kfl}}^0(\mathcal{X}_{\mathbb{N}}, \underline{H}_{\text{kfl}}^2(G)) = 0$.

Proof. This follows from Proposition 3.16. □

Let the notation and the assumptions be as in Proposition 3.16. Let n be a positive integer. With the help of Corollary 3.17(2), the Kummer flat Čech cohomology to derive a functor cohomology spectral sequence from formula (3.1) gives rise to an exact sequence

$$\begin{aligned} 0 &\rightarrow \varinjlim_n \check{H}_{\text{kfl}}^1(X_n/X, G) \rightarrow H_{\text{kfl}}^1(X, G) \rightarrow \varinjlim_n \check{H}_{\text{kfl}}^0(X_n/X, \underline{H}_{\text{kfl}}^1(G)) \\ &\rightarrow \varinjlim_n \check{H}_{\text{kfl}}^2(X_n/X, G) \rightarrow H_{\text{kfl}}^2(X, G) \rightarrow \varinjlim_n \check{H}_{\text{kfl}}^1(X_n/X, \underline{H}_{\text{kfl}}^1(G)) \\ &\rightarrow \varinjlim_n \check{H}_{\text{kfl}}^3(X_n/X, G). \end{aligned} \tag{3.3}$$

Lemma 3.18. *Let the notation and the assumptions be as in Proposition 3.16; then we have*

$$\varinjlim_n \check{H}_{\text{kfl}}^i(X_n/X, \underline{H}_{\text{kfl}}^1(G)) = 0$$

for $i \geq 0$.

Proof. We denote by $X_{n,i}$ the fiber product of $i + 1$ copies of X_n over X . We have $X_{n,i} = H_n^i \times_{\text{Spec } \mathbb{Z}} X_n = (H_n)_X^i \times_X X_n$, where $H_n := \text{Spec } \mathbb{Z} \left[(P^{1/n})^{\text{gp}} / P^{\text{gp}} \right]$ and $(H_n)_X := H_n \times_{\text{Spec } \mathbb{Z}} X$. Note that $(H_n)_X$ is a constant group scheme associated to the abstract group $H_n(X)$ over X if $(n, p) = 1$.

We first compute the Čech cohomology group $\check{H}_{\text{kfl}}^i(X_n/X, \underline{H}_{\text{kfl}}^1(G))$ for $n = p^r$. In this case, the underlying scheme of the group scheme $(H_{p^r})_X$ is strictly Henselian local. Let A_{p^r} (resp., x_{p^r}) be the underlying ring (resp., the closed point) of X_{p^r} , and let $\mathfrak{m}_{A_{p^r}}$ be the maximal ideal of A_{p^r} . Since A_{p^r} is a finite local A -algebra, it is also Henselian local by [15, Tag 04GH], and thus $(A_{p^r}, \mathfrak{m}_{A_{p^r}})$ is a Henselian pair [15, Tag 09XE]. Let B be the underlying ring of $(H_{p^r})_X^i \times_X X_{p^r}$. Clearly B is a finite local A_{p^r} -algebra. Therefore any finite B -algebra C is also a finite A_{p^r} -algebra, and we have $(\mathfrak{m}_{A_{p^r}} B) C = \mathfrak{m}_{A_{p^r}} C$. Then by an easy exercise, one can see that the equivalence [15, Tag 09XI] implies that $(B, \mathfrak{m}_{A_{p^r}} B)$ is also a Henselian pair. Since the Hom-sheaf $\mathcal{H}om_X(\mathbb{Z}/m\mathbb{Z}(1), G)$ is torsion, Gabber’s theorem [15, Tag 09ZI]) implies that the vertical maps in the canonical commutative diagram

$$\begin{CD} \text{Hom}_{X_{p^r}}(\mathbb{Z}/m\mathbb{Z}(1), G) @>>> \text{Hom}_{(H_{p^r})_X^i \times_X X_{p^r}}(\mathbb{Z}/m\mathbb{Z}(1), G) \\ @V \cong VV @VV \cong V \\ \text{Hom}_{x_{p^r}}(\mathbb{Z}/m\mathbb{Z}(1), G) @>\cong>> \text{Hom}_{(H_{p^r})_X^i \times_X x_{p^r}}(\mathbb{Z}/m\mathbb{Z}(1), G) \end{CD}$$

are isomorphisms. Since $(H_{p^r})_X^i \times_X x_{p^r}$ is an infinitesimal thickening of x_{p^r} , the lower horizontal map in the diagram is also an isomorphism by [15, Tag 03SI]. Thus the canonical map

$$\text{Hom}_{X_{p^r}}(\mathbb{Z}/m\mathbb{Z}(1), G) \xrightarrow{\cong} \text{Hom}_{(H_{p^r})_X^i \times_X X_{p^r}}(\mathbb{Z}/m\mathbb{Z}(1), G)$$

is an isomorphism. Combining with Theorem 3.13, we get

$$\begin{aligned} H_{\text{kfl}}^1(X_{p^r, i}, G) &= H_{\text{kfl}}^1\left((H_{p^r})_X^i \times_X X_{p^r}, G\right) \\ &= \varinjlim_m \text{Hom}_{(H_{p^r})_X^i \times_X X_{p^r}}(\mathbb{Z}/m\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} \left(P^{1/p^r}\right)^{\text{gp}} \\ &= \varinjlim_m \text{Hom}_{X_{p^r}}(\mathbb{Z}/m\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} \left(P^{1/p^r}\right)^{\text{gp}} \\ &= H_{\text{kfl}}^1(X_{p^r}, G). \end{aligned}$$

The Čech complex for $\underline{H}_{\text{kff}}^1(G)$ with respect to the cover X_{p^r}/X can be identified with

$$H_{\text{kff}}^1(X_{p^r}, G) \xrightarrow{0} H_{\text{kff}}^1(X_{p^r}, G) \xrightarrow{\text{Id}} H_{\text{kff}}^1(X_{p^r}, G) \xrightarrow{0} H_{\text{kff}}^1(X_{p^r}, G) \xrightarrow{\text{Id}} \dots,$$

hence

$$\check{H}_{\text{kff}}^i(X_{p^r}/X, \underline{H}_{\text{kff}}^1(G)) = \begin{cases} H_{\text{kff}}^1(X_{p^r}, G) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases} \tag{3.4}$$

In general, we write $n = p^r \cdot n'$ with $(p, n') = 1$; then we have

$$X_{n,i} = (H_{n'}^i)_X \times_X \left((H_{p^r}^i)_X \times_X X_n \right)$$

and

$$H_{\text{kff}}^1(X_{n,i}, G) = \prod_{x \in H_{n'}(X)^i} H_{\text{kff}}^1(X_n, G) = \text{Map} \left(H_{n'}(X)^i, H_{\text{kff}}^1(X_n, G) \right).$$

The Čech complex for $\underline{H}_{\text{kff}}^1(G)$ with respect to the cover X_n/X can be identified with the standard complex that computes the cohomology of $H_{\text{kff}}^1(X_n, G)$ regarded as a trivial $H_{n'}(X)$ -module. Hence we get

$$\check{H}_{\text{kff}}^i(X_n/X, \underline{H}_{\text{kff}}^1(G)) = H^i(H_{n'}(X), H_{\text{kff}}^1(X_n, G)).$$

Finally, we get

$$\begin{aligned} \varinjlim_n \check{H}_{\text{kff}}^i(X_n/X, \underline{H}_{\text{kff}}^1(G)) &= \varinjlim_{n=p^r \cdot n'} H^i(H_{n'}(X), H_{\text{kff}}^1(X_n, G)) \\ &= H^i \left(\varinjlim_{n'} H_{n'}(X), \varinjlim_{n=p^r \cdot n'} H_{\text{kff}}^1(X_n, G) \right), \end{aligned}$$

where the second identification follows from [13, §2, Prop. 8]. By Proposition 3.15 we have $\varinjlim_n H_{\text{kff}}^1(X_n, G) = 0$, and thus $\varinjlim_n \check{H}_{\text{kff}}^i(X_n/X, \underline{H}_{\text{kff}}^1(G)) = 0$. □

Theorem 3.19. *Let the notation and assumptions be as in Proposition 3.16. Then the canonical homomorphism $\varinjlim_n \check{H}_{\text{kff}}^2(X_n/X, G) \rightarrow H_{\text{kff}}^2(X, G)$ is an isomorphism.*

Proof. The result follows from Lemma 3.18 and the exact sequence (3.3). □

In order to understand the group $H_{\text{kff}}^2(X, G)$, we are reduced to computing the groups $\check{H}_{\text{kff}}^2(X_n/X, G)$.

Lemma 3.20. *Let the notation and assumptions be as in Proposition 3.16. We further assume that G is a torus. Then we have the following:*

- (1) $\check{H}_{\text{kff}}^2(X_n/X, G) = \check{H}_{\text{két}}^2(X_n/X, G)$ for $(n, p) = 1$.
- (2) $\check{H}_{\text{kff}}^2(X_{p^r}/X, G) = 0$ for $r > 0$.

(3)

$$H_{\text{kfl}}^2(X, G) = \varinjlim_{(n,p)=1} \check{H}_{\text{kfl}}^2(X_n/X, G) = \varinjlim_{(n,p)=1} \check{H}_{\text{két}}^2(X_n/X, G) = H_{\text{két}}^2(X, G);$$

in particular, $H_{\text{kfl}}^2(X, G)$ is torsion, p -torsion-free, and divisible.

Proof. (1) Since X_n is a Kummer étale cover of X whenever $(p, n) = 1$, this is clear.

(2) By Lemma 3.12, we may assume that X is a log point $\text{Spec} k$ with k separably closed. By [12, page 521-523, in particular Lem. 3.16], we have

$$\check{H}_{\text{kfl}}^i(X_{p^r}/X, G) \cong H_{X_{\text{fl}}}^i(H_{p^r}, G)$$

for $i \geq 1$, where $H_{X_{\text{fl}}}^i(H_{p^r}, G)$ denotes the i th cohomology group of the group scheme $H_{p^r} = \text{Spec} \mathbb{Z} \left[(P^{1/p^r})^{\text{gp}} / P^{\text{gp}} \right]$ acting trivially on G over the flat site X_{fl} . The group $H_{X_{\text{fl}}}^2(H_{p^r}, G)$ can be identified with the group of extension classes of H_{p^r} by G which admit a (not necessarily homomorphic) section [14, Exposé XVII, Prop. A.3.1]. By [14, Exposé XVII, Prop. 7.1.1], such extensions must be of multiplicative type, and therefore must be commutative. Since the base field k is separably closed, such extensions must be trivial. It follows that $\check{H}_{\text{kfl}}^2(X_{p^r}/X, G) \cong H_{X_{\text{fl}}}^2(H_{p^r}, G) = 0$.

(3) First we show that any class $\gamma \in H_{\text{kfl}}^2(X, G)$ vanishes in $H_{\text{kfl}}^2(X_m, G)$ for some positive integer m with $(m, p) = 1$. By Corollary 3.17, γ is annihilated by some cover $X_{m \cdot p^r}$ with $(m, p) = 1$. Let γ' be the image of γ in $H_{\text{kfl}}^2(X_m, G)$; we want to show that it is zero. The Čech-to-derived functor spectral sequence for the cover $X_{m \cdot p^r}/X_m$ gives rise to an exact sequence

$$\begin{aligned} \cdots \rightarrow \check{H}_{\text{kfl}}^2(X_{m \cdot p^r}/X_m, G) &\rightarrow \ker(H_{\text{kfl}}^2(X_m, G) \rightarrow H_{\text{kfl}}^2(X_{m \cdot p^r}, G)) \\ &\rightarrow \check{H}_{\text{kfl}}^1(X_{m \cdot p^r}/X_m, \underline{H}_{\text{kfl}}^1(G)) \rightarrow \cdots \end{aligned}$$

We have $\gamma' \in \ker(H_{\text{kfl}}^2(X_m, G) \rightarrow H_{\text{kfl}}^2(X_{m \cdot p^r}, G))$. By part (2), we have $\check{H}_{\text{kfl}}^2(X_{m \cdot p^r}/X_m, G) = 0$. By equation (3.4) we have $\check{H}_{\text{kfl}}^1(X_{m \cdot p^r}/X_m, \underline{H}_{\text{kfl}}^1(G)) = 0$. Hence we get $\gamma' = 0$. It follows that

$$\varinjlim_{(m,p)=1} \ker(H_{\text{kfl}}^2(X, G) \rightarrow H_{\text{kfl}}^2(X_m, G)) \xrightarrow{\cong} H_{\text{kfl}}^2(X, G). \tag{3.5}$$

Now consider the exact sequences

$$\begin{aligned} \rightarrow \check{H}_{\text{kfl}}^0(X_m/X, \underline{H}_{\text{kfl}}^1(G)) &\rightarrow \check{H}_{\text{kfl}}^2(X_m/X, G) \rightarrow \ker(H_{\text{kfl}}^2(X, G) \rightarrow H_{\text{kfl}}^2(X_m, G)) \\ \rightarrow \check{H}_{\text{kfl}}^1(X_m/X, \underline{H}_{\text{kfl}}^1(G)) &\rightarrow \end{aligned}$$

arising from the spectral sequence $\check{H}_{\text{kfl}}^i(X_m/X, \underline{H}_{\text{kfl}}^j(G)) \Rightarrow H_{\text{kfl}}^{i+j}(X_m, G)$ for the Kummer flat covers X_m/X with $(m, p) = 1$. Taking the direct limit, we get an exact sequence

$$\begin{aligned} \rightarrow \varinjlim_{(m,p)=1} \check{H}_{\text{kfl}}^0(X_m/X, \underline{H}_{\text{kfl}}^1(G)) &\xrightarrow{\alpha} \varinjlim_{(m,p)=1} \check{H}_{\text{kfl}}^2(X_m/X, G) \rightarrow H_{\text{kfl}}^2(X, G) \\ \rightarrow \varinjlim_{(m,p)=1} \check{H}_{\text{kfl}}^1(X_m/X, \underline{H}_{\text{kfl}}^1(G)) & \end{aligned} \tag{3.6}$$

by the identification (3.5). Similar to the general case of the proof of Lemma 3.18, we can show that $\varinjlim_{(m,p)=1} \check{H}_{\text{kfl}}^i(X_m/X, \underline{H}_{\text{kfl}}^1(G)) = 0$ for any $i \geq 0$. Then the exact sequence (3.6) tells us that

$$H_{\text{kfl}}^2(X, G) = \varinjlim_{(m,p)=1} \check{H}_{\text{kfl}}^2(X_m/X, G) = \varinjlim_{(m,p)=1} \check{H}_{\text{két}}^2(X_m/X, G) = H_{\text{két}}^2(X, G).$$

The group $H_{\text{két}}^2(X, G)$ is torsion and p -torsion-free by Corollary 3.6, and n -divisible for $(n, p) = 1$ by Theorem 3.8(3). Therefore $H_{\text{kfl}}^2(X, G)$ is torsion, p -torsion-free, and divisible. \square

Corollary 3.21. *Let the notation and assumptions be as in Lemma 3.20. Then we have $H_{\text{kfl}}^2(X, G[n]) \cong G[m](-2)(X) \otimes_{\mathbb{Z}} \bigwedge^2 P^{\text{gp}}$, where $n = m \cdot p^r$ with $(m, p) = 1$.*

Proof. We have a short exact sequence

$$0 \rightarrow H_{\text{kfl}}^1(X, G) \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \rightarrow H_{\text{kfl}}^2(X, G[n]) \rightarrow H_{\text{kfl}}^2(X, G)[n] \rightarrow 0.$$

Since G is a torus, the group $H_{\text{kfl}}^1(X, G) = \varinjlim_n \text{Hom}_{X_{\text{kfl}}}(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} P^{\text{gp}}$ is divisible, whence $H_{\text{kfl}}^1(X, G) \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$. Therefore for $n = m \cdot p^r$ with $(m, p) = 1$, we have

$$\begin{aligned} H_{\text{kfl}}^2(X, G[n]) &= H_{\text{kfl}}^2(X, G)[n] = H_{\text{két}}^2(X, G)[n] = H_{\text{két}}^2(X, G)[m] \\ &= H_{\text{két}}^2(X, G[m]) = G[m](-2)(X) \otimes_{\mathbb{Z}} \bigwedge^2 P^{\text{gp}}. \end{aligned} \quad \square$$

Corollary 3.22. *Let X be a locally Noetherian fs log scheme and G be a torus over the underlying scheme of X . Let $Y \in (\text{fs}/X)$ be such that the ranks of the stalks of the étale sheaf M_Y/\mathcal{O}_Y^\times are at most 1 and let (st/Y) be the full subcategory of (fs/X) consisting of strict fs log schemes over Y . Then we have that the restriction of $R^2\varepsilon_{\text{fl}*}G$ to (st/Y) is zero.*

Proof. This follows from Corollary 3.21. \square

Theorem 3.23. *Let X be a locally Noetherian fs log scheme and G be a torus over the underlying scheme of X .*

(1) *We have*

$$R^2\varepsilon_{\text{fl}*}G = \varinjlim_n (R^2\varepsilon_{\text{fl}*}G)[n] = \bigoplus_{l \text{ prime}} (R^2\varepsilon_{\text{fl}*}G)[l^\infty],$$

where $(R^2\varepsilon_{\text{fl}*}G)[n]$ denotes the n -torsion subsheaf of $R^2\varepsilon_{\text{fl}*}G$ and $(R^2\varepsilon_{\text{fl}*}G)[l^\infty]$ denotes the l -primary part of $R^2\varepsilon_{\text{fl}*}G$ for a prime number l .

(2) *We have $(R^2\varepsilon_{\text{fl}*}G)[n] = R^2\varepsilon_{\text{fl}*}G[n]$.*

(3) *The l -primary part $(R^2\varepsilon_{\text{fl}*}G)[l^\infty]$ is supported on the locus where l is invertible.*

(4) *If n is invertible on X , then*

$$(R^2\varepsilon_{\text{fl}*}G)[n] = R^2\varepsilon_{\text{fl}*}G[n] = G[n](-2) \otimes_{\mathbb{Z}} \bigwedge^2 (\mathbb{G}_{m, \text{log}}/\mathbb{G}_m)_{X_{\text{fl}}}.$$

Proof. By Lemma 3.20, $R^2\varepsilon_{\text{fl}*}G$ is torsion. Hence part (1) follows.

We prove part (2). Since G is a torus, we have a short exact sequence $0 \rightarrow G[n] \rightarrow G \xrightarrow{n} G \rightarrow 0$ for any $n \geq 1$. This short exact sequence induces a short exact sequence

$$0 \rightarrow R^1\varepsilon_{\text{fl}*}G \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \rightarrow R^2\varepsilon_{\text{fl}*}G[n] \rightarrow (R^2\varepsilon_{\text{fl}*}G)[n] \rightarrow 0.$$

Since the sheaf $R^1\varepsilon_{\text{fl}*}G$ is divisible, we get $(R^2\varepsilon_{\text{fl}*}G)[n] = R^2\varepsilon_{\text{fl}*}G[n]$.

Part (3) follows from part (2) and Corollary 3.21.

We are left with part (4). By [16, §3], in particular the part between [16, Cor. 3.7] and [16, Thm. 3.8], we have a cup product for the higher direct image functors for the map of sites $\varepsilon_{\text{fl}} : (\text{fs}/X)_{\text{kfl}} \rightarrow (\text{fs}/X)_{\text{fl}}$. The cup product induces homomorphisms

$$G[n] \otimes_{\mathbb{Z}/n\mathbb{Z}} \bigwedge^2 R^1\varepsilon_{\text{fl}*}\mathbb{Z}/n\mathbb{Z} \rightarrow G[n] \otimes_{\mathbb{Z}/n\mathbb{Z}} R^2\varepsilon_{\text{fl}*}\mathbb{Z}/n\mathbb{Z} \rightarrow R^2\varepsilon_{\text{fl}*}G[n].$$

Since $G[n] \otimes_{\mathbb{Z}/n\mathbb{Z}} \bigwedge^2 R^1\varepsilon_{\text{fl}*}\mathbb{Z}/n\mathbb{Z} = G[n](-2) \otimes_{\mathbb{Z}} \bigwedge^2 (\mathbb{G}_{\text{m},\text{log}}/\mathbb{G}_{\text{m}})_{X_{\text{fl}}}$, we get a canonical homomorphism

$$G[n](-2) \otimes_{\mathbb{Z}} \bigwedge^2 (\mathbb{G}_{\text{m},\text{log}}/\mathbb{G}_{\text{m}})_{X_{\text{fl}}} \rightarrow R^2\varepsilon_{\text{fl}*}G[n].$$

By Corollary 3.21, this homomorphism is an isomorphism. This finishes the proof of part (4). □

Corollary 3.24. *Let X be a locally Noetherian fs log scheme such that the underlying scheme of X is a \mathbb{Q} -scheme, and let G be a torus over the underlying scheme of X . Then we have*

$$R^2\varepsilon_{\text{fl}*}G = \varinjlim_n G[n](-2) \otimes_{\mathbb{Z}} \bigwedge^2 (\mathbb{G}_{\text{m},\text{log}}/\mathbb{G}_{\text{m}})_{X_{\text{fl}}}.$$

Corollary 3.25. *Let p be a prime number. Let X be a locally Noetherian fs log scheme such that the underlying scheme of X is an \mathbb{F}_p -scheme, and let G be a torus over the underlying scheme of X . Then we have*

$$R^2\varepsilon_{\text{fl}*}G = \varinjlim_{(n,p)=1} G[n](-2) \otimes_{\mathbb{Z}} \bigwedge^2 (\mathbb{G}_{\text{m},\text{log}}/\mathbb{G}_{\text{m}})_{X_{\text{fl}}}.$$

Theorem 3.26. *Let X be a locally Noetherian fs log scheme and G be a smooth affine commutative group scheme over the underlying scheme of X .*

- (1) *If the fibers G_x of G over X are all unipotent and $\kappa(x)$ -solvable (see [14, Exposé XVII, Def. 5.1.0]), then we have*

$$R^1\varepsilon_{\text{fl}*}G = R^2\varepsilon_{\text{fl}*}G = 0.$$

- (2) *If G is an extension of a group U by a torus T such that the fibers U_x of U over X are all unipotent and $\kappa(x)$ -solvable, then the canonical map $R^2\varepsilon_{\text{fl}*}T \rightarrow R^2\varepsilon_{\text{fl}*}G$ is an isomorphism.*

Proof. In part (2), since G is smooth, U has to be smooth by fppf descent. Hence part (2) follows from part (1) and we are left to prove part (1).

By Theorem 3.14, we have

$$R^1 \varepsilon_{\text{fl},*} G = \varinjlim_n \mathcal{H}om_X(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} (\mathbb{G}_{\text{m,log}}/\mathbb{G}_{\text{m}})_{X_n},$$

which is zero by [14, Exposé XVII, Lem. 2.5]. To prove $R^2 \varepsilon_{\text{fl},*} G = 0$, it suffices to prove $H^2_{\text{kfl}}(X, G) = 0$ for the case where the underlying scheme of X is $\text{Spec } A$ with A a strictly Henselian ring and X admitting a chart $P \rightarrow M_X$ with $P \xrightarrow{\cong} M_{X,x}/\mathcal{O}_{X,x}^\times$. By Theorem 3.19, we have $\varinjlim_n \check{H}^2_{\text{kfl}}(X_n/X, G) \xrightarrow{\cong} H^2_{\text{kfl}}(X, G)$. By Lemma 3.12, we are further reduced to the case where X is a log point with A a separably closed field. We have $\check{H}^2_{\text{kfl}}(X_n/X, G) = H^2_{X_n}(H_n, G)$ by [12, page 521-523, in particular Lem. 3.16]. But $H^2_{X_n}(H_n, G) = 0$ for any positive integer n by [14, Exposé XVII, Thm. 5.1.1 (1) (c)] and [14, Exposé XVII, Appendice I, Prop. 3.1]. This finishes the proof. \square

4. The higher direct images of the logarithmic multiplicative group

In this section we show that $R^i \varepsilon_{\text{fl},*} \overline{\mathbb{G}}_{\text{m,log}} = 0$ (resp., $R^i \varepsilon_{\text{ét},*} \overline{\mathbb{G}}_{\text{m,log}} = 0$) for $i \geq 1$, where $\overline{\mathbb{G}}_{\text{m,log}}$ denotes the quotient of $\mathbb{G}_{\text{m,log}}$ by \mathbb{G}_{m} with respect to the Kummer flat topology (resp., Kummer étale topology). The case $i = 1$ has been treated essentially in the proof of [12, Cor. 3.21]. As a corollary, we get $R^i \varepsilon_{\text{fl},*} \mathbb{G}_{\text{m}} \cong R^i \varepsilon_{\text{fl},*} \mathbb{G}_{\text{m,log}}$ (resp., $R^i \varepsilon_{\text{ét},*} \mathbb{G}_{\text{m}} \cong R^i \varepsilon_{\text{ét},*} \mathbb{G}_{\text{m,log}}$) for $i \geq 2$. By Kato’s logarithmic Hilbert 90 [12, Cor. 3.21], we have $R^1 \varepsilon_{\text{fl},*} \mathbb{G}_{\text{m,log}} = 0$ (resp., $R^1 \varepsilon_{\text{ét},*} \mathbb{G}_{\text{m,log}} = 0$).

We start with the strictly Henselian case.

Theorem 4.1. *Let $X = \text{Spec } A$ be an fs log scheme with A a Noetherian strictly Henselian local ring, x the closed point of X , and p the characteristic of the residue field of A . We fix a chart $P \rightarrow M_X$ satisfying $P \xrightarrow{\cong} M_{X,x}/\mathcal{O}_{X,x}^\times$. Then we have the following:*

- (1) $H^r_{\text{kfl}}(X, \overline{\mathbb{G}}_{\text{m,log}}) = 0$ (resp., $H^r_{\text{két}}(X, \overline{\mathbb{G}}_{\text{m,log}}) = 0$) for $r \geq 1$.
- (2) $H^r_{\text{kfl}}(X, \mathbb{G}_{\text{m}}) \cong H^r_{\text{kfl}}(X, \mathbb{G}_{\text{m,log}})$ (resp., $H^r_{\text{két}}(X, \mathbb{G}_{\text{m}}) \cong H^r_{\text{két}}(X, \mathbb{G}_{\text{m,log}})$) for $r \geq 2$.

Before going to the proof of Theorem 4.1, we prove the following lemma:

Lemma 4.2. *Let the notation and assumptions be as in Theorem 4.1. For the Čech cohomology for the Kummer flat cover X_n/X , we have*

$$\check{H}^i_{\text{kfl}}(X_n/X, \overline{\mathbb{G}}_{\text{m,log}}) = \begin{cases} (P^{1/n})^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

If $(n, p) = 1$, for the Čech cohomology for the Kummer étale cover X_n/X we have

$$\check{H}^i_{\text{két}}(X_n/X, \overline{\mathbb{G}}_{\text{m,log}}) = \begin{cases} (P^{1/n})^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}' & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Proof. We deal only with the Kummer flat case; the Kummer étale case can be done in the same way.

It is clear that

$$\check{H}_{\text{kff}}^0(X_n/X, \overline{\mathbb{G}}_{m, \log}) = (\mathbb{G}_{m, \log}/\mathbb{G}_m)(X_n) = (P^{1/n})^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{4.1}$$

Let $n = m \cdot p^t$ with $(m, p) = 1$; then $X \times_{\text{Spec } \mathbb{Z}} H_m^r$ is a constant group scheme over X and $X \times_{\text{Spec } \mathbb{Z}} H_{p^t}^r$ is a connected group scheme over X . Therefore we have

$$\begin{aligned} \overline{\mathbb{G}}_{m, \log}(\underbrace{X_n \times_X \cdots \times_X X_n}_{r+1 \text{ times}}) &= \overline{\mathbb{G}}_{m, \log}(X_n \times_{\text{Spec } \mathbb{Z}} H_n^r) \\ &= \overline{\mathbb{G}}_{m, \log}((X_n \times_{\text{Spec } \mathbb{Z}} H_{p^t}^r) \times_X (X \times_{\text{Spec } \mathbb{Z}} H_m^r)) \\ &= \prod_{h \in H_m(X)^r} \overline{\mathbb{G}}_{m, \log}(X_n \times_{\text{Spec } \mathbb{Z}} H_{p^t}^r) \\ &= \prod_{h \in H_m(X)^r} \overline{\mathbb{G}}_{m, \log}(X_n) \\ &= \prod_{h \in H_m(X)^r} (P^{1/n})^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \\ &= \text{Map}(H_m(X)^r, (P^{1/n})^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}). \end{aligned}$$

To compute the higher Čech cohomology groups, we consider the Čech complex

$$\overline{\mathbb{G}}_{m, \log}(X_n) \xrightarrow{d_0} \overline{\mathbb{G}}_{m, \log}(X_n \times_X X_n) \xrightarrow{d_1} \overline{\mathbb{G}}_{m, \log}(X_n \times_X X_n \times_X X_n) \xrightarrow{d_2} \cdots \tag{4.2}$$

for $\overline{\mathbb{G}}_{m, \log}$ with respect to the cover X_n/X . Define $\Gamma_n := (P^{1/n})^{\text{gp}}/P^{\text{gp}}$. By [10, Chap. III, Example 2.6], the Čech nerve of the Kummer flat cover X_n/X can be identified with the sequence

$$X_n \begin{matrix} \xleftarrow{d_{1,1}} \\ \xrightarrow{d_{1,0}} \end{matrix} X_n \times H_n \begin{matrix} \xleftarrow{d_{2,2}} \\ \xrightarrow{d_{2,0}} \end{matrix} X_n \times H_n^2 \begin{matrix} \xleftarrow{d_{3,3}} \\ \xrightarrow{d_{3,0}} \end{matrix} X_n \times H_n^3 \cdots,$$

where the map $d_{r,i}$ on the ring level is given by the A -linear ring homomorphism

$$\begin{aligned} A \otimes_{\mathbb{Z}[P]} \mathbb{Z} [P^{1/n} \oplus \Gamma_n^{r-1}] &\rightarrow A \otimes_{\mathbb{Z}[P]} \mathbb{Z} [P^{1/n} \oplus \Gamma_n^r] \\ (a, \bar{a}_1, \dots, \bar{a}_{r-1}) &\mapsto \begin{cases} (a, \bar{a}, \bar{a}_1, \dots, \bar{a}_{r-1}) & \text{if } i = 0, \\ (a, \bar{a}_1, \dots, \bar{a}_i, \bar{a}_i, \dots, \bar{a}_{r-1}) & \text{if } 0 < i < r, \\ (a, \bar{a}_1, \dots, \bar{a}_{r-1}, 0) & \text{if } i = r, \end{cases} \end{aligned}$$

for any $(a, \bar{a}_1, \dots, \bar{a}_{r-1}) \in P^{1/n} \oplus \Gamma_n^{r-1}$. If $m = 1$ – that is, $n = p^t$ –, we have

$$\overline{\mathbb{G}}_{m, \log}(X_{p^t} \times_{\text{Spec } \mathbb{Z}} H_{p^t}^r) = (P^{1/p^t})^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By the description of $d_{r,i}$, the map

$$d_{r,i}^* : \overline{\mathbb{G}}_{m,\log} \left(X_{p^t} \times_{\text{Spec } \mathbb{Z}} H_{p^t}^{r-1} \right) \rightarrow \overline{\mathbb{G}}_{m,\log} \left(X_{p^t} \times_{\text{Spec } \mathbb{Z}} H_{p^t}^r \right)$$

can be identified with the identity map

$$\text{Id} : \left(P^{1/p^t} \right)^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \left(P^{1/p^t} \right)^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In general, the map

$$d_{r,i}^* : \overline{\mathbb{G}}_{m,\log} \left(X_n \times_{\text{Spec } \mathbb{Z}} H_n^{r-1} \right) \rightarrow \overline{\mathbb{G}}_{m,\log} \left(X_n \times_{\text{Spec } \mathbb{Z}} H_n^r \right)$$

can be identified with the map

$$\begin{aligned} & \text{Map} \left(H_m(X)^{r-1}, \left(P^{1/n} \right)^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \right) \xrightarrow{\partial_{r,i}} \text{Map} \left(H_m(X)^r, \left(P^{1/n} \right)^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \right) \\ f \mapsto & \left(\partial_{r,i}(f) : (h_1, \dots, h_r) \mapsto \begin{cases} f(h_2, \dots, h_r) & \text{if } i = 0, \\ f(h_1, \dots, h_i + h_{i+1}, \dots, h_r) & \text{if } 0 < i < r, \\ f(h_1, \dots, h_{r-1}) & \text{if } i = r \end{cases} \right). \end{aligned}$$

Therefore the complex (4.2) can be identified with the standard complex

$$\begin{aligned} \left(P^{1/n} \right)^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} & \rightarrow \text{Map} \left(H_m(X), \left(P^{1/n} \right)^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \right) \\ & \rightarrow \text{Map} \left(H_m(X)^2, \left(P^{1/n} \right)^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \right) \rightarrow \dots \end{aligned}$$

for computing the group cohomology of the trivial $H_m(X)$ -module $\left(P^{1/n} \right)^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$. It follows that

$$\check{H}_{\text{kfl}}^i \left(X_n/X, \overline{\mathbb{G}}_{m,\log} \right) \cong H^i \left(H_m(X), \left(P^{1/n} \right)^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \right) = \begin{cases} \left(P^{1/n} \right)^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases} \quad \square$$

Proof of Theorem 4.1. Part (2) follows from part (1) clearly. We deal only with the Kummer flat case; the Kummer étale case can be proven in the same way.

We use induction on r to prove part (1).

First we consider the case $r = 1$. We want to prove $H_{\text{kfl}}^1 \left(X, \overline{\mathbb{G}}_{m,\log} \right) = 0$. The spectral sequence $\check{H}_{\text{kfl}}^i \left(X_n/X, \underline{H}_{\text{kfl}}^j \left(\overline{\mathbb{G}}_{m,\log} \right) \right) \Rightarrow H_{\text{kfl}}^{i+j} \left(X, \overline{\mathbb{G}}_{m,\log} \right)$ gives rise to an exact sequence

$$0 \rightarrow \check{H}_{\text{kfl}}^1 \left(X_n/X, \overline{\mathbb{G}}_{m,\log} \right) \rightarrow H_{\text{kfl}}^1 \left(X, \overline{\mathbb{G}}_{m,\log} \right) \xrightarrow{u} \check{H}_{\text{kfl}}^0 \left(X_n/X, \underline{H}_{\text{kfl}}^1 \left(\overline{\mathbb{G}}_{m,\log} \right) \right).$$

By Lemma 4.2, we have that $\check{H}_{\text{kfl}}^1 \left(X_n/X, \overline{\mathbb{G}}_{m,\log} \right)$ vanishes, and thus u is injective. We also have a canonical injection

$$v : \check{H}_{\text{kfl}}^0 \left(X_n/X, \underline{H}_{\text{kfl}}^1 \left(\overline{\mathbb{G}}_{m,\log} \right) \right) \hookrightarrow H_{\text{kfl}}^1 \left(X_n, \overline{\mathbb{G}}_{m,\log} \right).$$

The composition $v \circ u$ is nothing but the pullback map $f_n^* : H_{\text{kfl}}^1 \left(X, \overline{\mathbb{G}}_{m,\log} \right) \rightarrow H_{\text{kfl}}^1 \left(X_n, \overline{\mathbb{G}}_{m,\log} \right)$, where f_n denotes the cover map $X_n \rightarrow X$. Hence we get an injection $f_n^* : H_{\text{kfl}}^1 \left(X, \overline{\mathbb{G}}_{m,\log} \right) \hookrightarrow H_{\text{kfl}}^1 \left(X_n, \overline{\mathbb{G}}_{m,\log} \right)$. Passing to the direct limit, we get a canonical injection $H_{\text{kfl}}^1 \left(X, \overline{\mathbb{G}}_{m,\log} \right) \hookrightarrow \varinjlim_n H_{\text{kfl}}^1 \left(X_n, \overline{\mathbb{G}}_{m,\log} \right)$. Hence it suffices to show that

$\varinjlim_n H_{\text{kfl}}^1(X_n, \overline{\mathbb{G}}_{m, \log}) = 0$. Let α be an element of $H_{\text{kfl}}^1(X_n, \overline{\mathbb{G}}_{m, \log})$, and let $T \rightarrow X_n$ be a Kummer flat cover such that α dies in $H_{\text{kfl}}^1(T, \overline{\mathbb{G}}_{m, \log})$. By [12, Cor. 2.16], we may assume that for some m , we have a factorization $T \rightarrow X_{mn} \rightarrow X_n$, where $T \rightarrow X_{mn}$ is a classical flat cover. It follows that the class α on X_{mn} is trivialized by a classical flat cover – that is, α is mapped to zero along the map $H_{\text{kfl}}^1(X_{mn}, \overline{\mathbb{G}}_{m, \log}) \rightarrow H_{\text{fl}}^0(X_{mn}, R^1 \varepsilon_{\text{fl}*} \overline{\mathbb{G}}_{m, \log})$. Hence α lies in the image of the map $H_{\text{fl}}^1(X_{mn}, \varepsilon_{\text{fl}*} \overline{\mathbb{G}}_{m, \log}) \hookrightarrow H_{\text{kfl}}^1(X_{mn}, \overline{\mathbb{G}}_{m, \log})$. But

$$\begin{aligned} H_{\text{fl}}^1(X_{mn}, \varepsilon_{\text{fl}*} \overline{\mathbb{G}}_{m, \log}) &= H_{\text{fl}}^1(X_{mn}, (\mathbb{G}_{m, \log}/\mathbb{G}_m)_{X_{\text{fl}}} \otimes_{\mathbb{Z}} \mathbb{Q}) \\ &= H_{\text{ét}}^1(X_{mn}, (\mathbb{G}_{m, \log}/\mathbb{G}_m)_{X_{\text{ét}}} \otimes_{\mathbb{Z}} \mathbb{Q}) \\ &= 0. \end{aligned} \tag{4.3}$$

This finishes the proof of $H_{\text{kfl}}^1(X, \overline{\mathbb{G}}_{m, \log}) = 0$.

Now we fix a positive integer r_0 and assume that $H_{\text{kfl}}^r(Y, \overline{\mathbb{G}}_{m, \log}) = 0$ for all $0 < r \leq r_0$ and all fs log schemes Y satisfying the conditions for X in the statement of the theorem. Note that this assumption implies that $R^r \varepsilon_{\text{fl}*} \overline{\mathbb{G}}_{m, \log} = 0$ for $0 < r \leq r_0$. We are going to prove

$$H_{\text{kfl}}^{r_0+1}(X, \overline{\mathbb{G}}_{m, \log}) = 0.$$

in two steps.

In the first step, we prove that the canonical map

$$H_{\text{kfl}}^{r_0+1}(X, \overline{\mathbb{G}}_{m, \log}) \rightarrow H_{\text{kfl}}^{r_0+1}(X_n, \overline{\mathbb{G}}_{m, \log}) \tag{4.4}$$

is injective for any $n > 0$. Clearly X_n satisfies the conditions for X in the statement. For any $0 < j \leq r_0$ and any $i \geq 0$, consider the i th Čech cohomology group $\check{H}_{\text{kfl}}^i(X_n/X, \underline{H}_{\text{kfl}}^j(\overline{\mathbb{G}}_{m, \log}))$ of the Kummer flat cover X_n/X with coefficients in the presheaf $\underline{H}_{\text{kfl}}^j(\overline{\mathbb{G}}_{m, \log})$. Since

$$\underbrace{X_n \times_X \cdots \times_X X_n}_{k+1 \text{ times}} = X_n \times_{\text{Spec } \mathbb{Z}} H_n^k = (X_n \times_{\text{Spec } \mathbb{Z}} H_{p^t}^k) \times_X (X \times_{\text{Spec } \mathbb{Z}} H_{n'}^k)$$

with $n = n' \cdot p^t$ and $(n', p) = 1$, $X_n \times_{\text{Spec } \mathbb{Z}} H_{p^t}^k$ satisfies the conditions for X in the statement, and $X \times_{\text{Spec } \mathbb{Z}} H_{n'}^k$ is a constant group scheme over X , so we get

$$\Gamma\left(\underbrace{X_n \times_X \cdots \times_X X_n}_{k+1 \text{ times}}, \underline{H}_{\text{kfl}}^j(\overline{\mathbb{G}}_{m, \log})\right) = H_{\text{kfl}}^j\left(\underbrace{X_n \times_X \cdots \times_X X_n}_{k+1 \text{ times}}, \overline{\mathbb{G}}_{m, \log}\right) = 0.$$

It follows that $\check{H}_{\text{kfl}}^i(X_n/X, \underline{H}_{\text{kfl}}^j(\overline{\mathbb{G}}_{m, \log})) = 0$ for any $0 < j \leq r_0$ and any $i \geq 0$. Then the spectral sequence

$$\check{H}_{\text{kfl}}^i(X_n/X, \underline{H}_{\text{kfl}}^j(\overline{\mathbb{G}}_{m, \log})) \Rightarrow H_{\text{kfl}}^{i+j}(X, \overline{\mathbb{G}}_{m, \log})$$

implies that

$$H_{\text{kfl}}^{r_0+1}(X, \overline{\mathbb{G}}_{m, \log}) \xrightarrow{\cong} \check{H}_{\text{kfl}}^0(X_n/X, \underline{H}_{\text{kfl}}^{r_0+1}(\overline{\mathbb{G}}_{m, \log})).$$

It follows that the canonical map (4.4) is injective for any $n > 0$.

In the second step, we finish the proof of $H_{\text{kfl}}^{r_0+1}(X, \overline{\mathbb{G}}_{\text{m}, \log}) = 0$. Let β be any element of $H_{\text{kfl}}^{r_0+1}(X, \overline{\mathbb{G}}_{\text{m}, \log})$ and let $T \rightarrow X$ be a Kummer flat cover such that β dies in $H_{\text{kfl}}^{r_0+1}(T, \overline{\mathbb{G}}_{\text{m}, \log})$. By [12, Cor. 2.16], we may assume that for some m , we have a factorization $T \rightarrow X_m \rightarrow X$ such that $T \rightarrow X_m$ is a classical flat cover. It follows that the class β on X_m is trivialized by a classical flat cover – that is, it lies in the kernel of the canonical map

$$H_{\text{kfl}}^{r_0+1}(X_m, \overline{\mathbb{G}}_{\text{m}, \log}) \rightarrow H_{\text{fl}}^0(X_m, R^{r_0+1} \varepsilon_{\text{fl}*} \overline{\mathbb{G}}_{\text{m}, \log}).$$

Consider the spectral sequence

$$H_{\text{fl}}^i(X_m, R^j \varepsilon_{\text{fl}*} \overline{\mathbb{G}}_{\text{m}, \log}) \Rightarrow H_{\text{kfl}}^{i+j}(X_m, \overline{\mathbb{G}}_{\text{m}, \log}).$$

The vanishing of $R^j \varepsilon_{\text{fl}*} \overline{\mathbb{G}}_{\text{m}, \log}$ for $0 < j \leq r_0$ gives rise to an exact sequence

$$0 \rightarrow H_{\text{fl}}^{r_0+1}(X_m, \varepsilon_{\text{fl}*} \overline{\mathbb{G}}_{\text{m}, \log}) \rightarrow H_{\text{kfl}}^{r_0+1}(X_m, \overline{\mathbb{G}}_{\text{m}, \log}) \rightarrow H_{\text{fl}}^0(X_m, R^{r_0+1} \varepsilon_{\text{fl}*} \overline{\mathbb{G}}_{\text{m}, \log}).$$

Hence the class β on X_m comes from $H_{\text{fl}}^{r_0+1}(X_m, \varepsilon_{\text{fl}*} \overline{\mathbb{G}}_{\text{m}, \log})$, which is zero by the same reasoning as in equation (4.3). Hence the class β on X_m is zero. Thus $\beta = 0$ in $H_{\text{kfl}}^{r_0+1}(X, \overline{\mathbb{G}}_{\text{m}, \log})$, by the injectivity of the map (4.4). This finishes the proof of $H_{\text{kfl}}^{r_0+1}(X, \overline{\mathbb{G}}_{\text{m}, \log}) = 0$. □

Theorem 4.3. *Let X be an fs log scheme with its underlying scheme locally Noetherian. Then we have the following:*

- (1) $R^r \varepsilon_{\text{fl}*} \overline{\mathbb{G}}_{\text{m}, \log} = 0$ (resp., $R^r \varepsilon_{\text{ét}*} \overline{\mathbb{G}}_{\text{m}, \log} = 0$) for $r \geq 1$.
- (2) The canonical map $R^r \varepsilon_{\text{fl}*} \mathbb{G}_{\text{m}} \rightarrow R^r \varepsilon_{\text{fl}*} \overline{\mathbb{G}}_{\text{m}, \log}$ (resp., $R^r \varepsilon_{\text{ét}*} \mathbb{G}_{\text{m}} \rightarrow R^r \varepsilon_{\text{ét}*} \overline{\mathbb{G}}_{\text{m}, \log}$) is an isomorphism for $r \geq 2$.

Corollary 4.4. *Let X be a locally Noetherian fs log scheme such that the stalks of $M_X^{\text{gp}} / \mathcal{O}_X^\times$ for the classical étale topology have rank at most 1. Let $(\text{két}/X)_{\text{két}}$ (resp., $(\text{két}/X)_{\text{ét}}$) be the category of Kummer étale fs log schemes over X endowed with the Kummer étale topology (resp., the classical étale topology), and let $\varepsilon : (\text{két}/X)_{\text{két}} \rightarrow (\text{két}/X)_{\text{ét}}$ be the canonical forgetful map of sites. Then we have $R^2 \varepsilon_* \mathbb{G}_{\text{m}, \log} = 0$.*

5. Examples

5.1. Discrete valuation rings

Let R be a discrete valuation ring with fraction field K and residue field k . Let π be a uniformizer of R , and endow $X = \text{Spec} R$ with the log structure associated to the homomorphism $\mathbb{N} \rightarrow R, 1 \mapsto \pi$. Let x be the closed point of X and i the closed immersion $x \hookrightarrow X$, and endow x with the induced log structure from X . Let η be the generic point of X and j the open immersion $\eta \hookrightarrow X$.

Now we consider the Leray spectral sequence

$$H_{\text{fl}}^s(X, R^t \varepsilon_{\text{fl}*} \mathbb{G}_{\text{m}}) \Rightarrow H_{\text{kfl}}^{s+t}(X, \mathbb{G}_{\text{m}}). \tag{5.1}$$

We have $H_{\text{fl}}^s(X, \mathbb{G}_m) = H_{\text{ét}}^s(X, \mathbb{G}_m)$ for $s \geq 0$ by [5, Thm. 11.7]. We have

$$\begin{aligned} R^1 \varepsilon_{\text{fl}*} \mathbb{G}_m &= \varinjlim_n \mathcal{H}om_X(\mathbb{Z}/n\mathbb{Z}(1), \mathbb{G}_m) \otimes_{\mathbb{Z}} (\mathbb{G}_{m, \log}/\mathbb{G}_m)_{X_{\text{fl}}} \\ &= \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{G}_{m, \log}/\mathbb{G}_m)_{X_{\text{fl}}} \end{aligned}$$

by Theorem 3.14. Then on (st/X) , we have $(\mathbb{G}_{m, \log}/\mathbb{G}_m)_{X_{\text{fl}}} \cong i_*\mathbb{Z}$. Therefore,

$$\begin{aligned} H_{\text{fl}}^s(X, R^1 \varepsilon_{\text{fl}*} \mathbb{G}_m) &= H_{\text{fl}}^s(X, \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} i_*\mathbb{Z}) = H_{\text{fl}}^s(x, \mathbb{Q}/\mathbb{Z}) \\ &= H_{\text{ét}}^s(x, \mathbb{Q}/\mathbb{Z}) \end{aligned} \tag{5.2}$$

for $s \geq 0$. We also have

$$H_{\text{fl}}^s(X, R^2 \varepsilon_{\text{fl}*} \mathbb{G}_m) = H_{\text{fl}}^s(X, 0) = 0 \tag{5.3}$$

for $s \geq 0$, by Corollary 3.22.

Theorem 5.1. *Assume that k is a finite field. Then we have*

$$H_{\text{kfl}}^1(X, \mathbb{G}_m) \cong H_{\text{ét}}^0(x, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$$

and

$$H_{\text{kfl}}^2(X, \mathbb{G}_m) \cong H_{\text{ét}}^1(x, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}.$$

Proof. We have

$$H_{\text{fl}}^s(X, \mathbb{G}_m) = H_{\text{ét}}^s(X, \mathbb{G}_m) = 0$$

for $s > 0$ by [11, Chap. II, Prop. 1.5 (a)]. Since k is a finite field, it has absolute Galois group $\hat{\mathbb{Z}}$. Therefore we can identify the group $H_{\text{ét}}^1(x, \mathbb{Q}/\mathbb{Z})$ with the Galois cohomology

$$H^1(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}.$$

Then the results follow from the spectral sequence (5.1) with the help of equations (5.2) and (5.3). □

5.2. Global Dedekind domains

Through this subsection, let K be a global field. When K is a number field, X denotes the spectrum of the ring of integers in K , and when K is a function field, k denotes the field of constants of K and X denotes the unique connected smooth projective curve over k having K as its function field. Let S be a finite set of closed points of X , $U := X - S$, $j : U \hookrightarrow X$, and $i_x : x \hookrightarrow X$ for each closed point $x \in X$. We endow X with log structure $j_*\mathcal{O}_U^\times \cap \mathcal{O}_X \rightarrow \mathcal{O}_X$. In the case of a number field, define $S_\infty := S \cup \{\text{infinite places of } K\}$, and in the case of function field, just set $S_\infty := S$.

On (st/X) , we have $R^1 \varepsilon_{\text{fl}*} \mathbb{G}_m = \bigoplus_{x \in S} i_{x,*} \mathbb{Q}/\mathbb{Z}$ and $R^2 \varepsilon_{\text{fl}*} \mathbb{G}_m = 0$. The Leray spectral sequence

$$H_{\text{fl}}^s(X, R^t \varepsilon_{\text{fl}*} \mathbb{G}_m) \Rightarrow H_{\text{kfl}}^{s+t}(X, \mathbb{G}_m)$$

gives rise to a long exact sequence

$$\begin{aligned}
 0 \rightarrow H_{\text{fl}}^1(X, \mathbb{G}_m) &\rightarrow H_{\text{kfl}}^1(X, \mathbb{G}_m) \xrightarrow{\alpha} H_{\text{fl}}^0(X, R^1\varepsilon_{\text{fl}*}\mathbb{G}_m) \\
 &\rightarrow H_{\text{fl}}^2(X, \mathbb{G}_m) \rightarrow H_{\text{kfl}}^2(X, \mathbb{G}_m) \rightarrow H_{\text{fl}}^1(X, R^1\varepsilon_{\text{fl}*}\mathbb{G}_m) \\
 &\rightarrow H_{\text{fl}}^3(X, \mathbb{G}_m) \rightarrow H_{\text{kfl}}^3(X, \mathbb{G}_m).
 \end{aligned}
 \tag{5.4}$$

The Leray spectral sequences for \mathbb{G}_m and $\mathbb{G}_{m,\log}$ together give rise to the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 & & H_{\text{fl}}^1(X, \mathbb{G}_m) & \longrightarrow & H_{\text{fl}}^1(X, \mathbb{G}_{m,\log}) \\
 & & \downarrow & & \downarrow \\
 & & H_{\text{kfl}}^1(X, \mathbb{G}_m) & \longrightarrow & H_{\text{kfl}}^1(X, \mathbb{G}_{m,\log}) \\
 & & \downarrow \alpha & & \downarrow \\
 & & H_{\text{fl}}^0(X, R^1\varepsilon_{\text{fl}*}\mathbb{G}_m) & \longrightarrow & H_{\text{fl}}^0(X, R^1\varepsilon_{\text{fl}*}\mathbb{G}_{m,\log}) \\
 & & \downarrow & & \downarrow \\
 H_{\text{fl}}^1(X, (\mathbb{G}_{m,\log}/\mathbb{G}_m)_{X_{\text{fl}}}) & \longrightarrow & H_{\text{fl}}^2(X, \mathbb{G}_m) & \longrightarrow & H_{\text{fl}}^2(X, \mathbb{G}_{m,\log}).
 \end{array}$$

We have $R^1\varepsilon_{\text{fl}*}\mathbb{G}_{m,\log} = 0$ by Kato’s logarithmic Hilbert 90 [12, Cor. 3.21], and

$$H_{\text{fl}}^1\left(X, (\mathbb{G}_{m,\log}/\mathbb{G}_m)_{X_{\text{fl}}}\right) = H_{\text{fl}}^1\left(X, \bigoplus_{x \in S} i_{x,*}\mathbb{Z}\right) = \bigoplus_{x \in S} H_{\text{fl}}^1(x, \mathbb{Z}) = 0.$$

By diagram chasing, we find that the map α is surjective. We have

$$H_{\text{fl}}^0(X, R^1\varepsilon_{\text{fl}*}\mathbb{G}_m) = \bigoplus_{x \in S} H_{\text{fl}}^0(X, i_{x,*}\mathbb{Q}/\mathbb{Z}) = \bigoplus_{x \in S} H_{\text{fl}}^0(x, \mathbb{Q}/\mathbb{Z}) = \bigoplus_{x \in S} \mathbb{Q}/\mathbb{Z}$$

and

$$H_{\text{fl}}^1(X, R^1\varepsilon_{\text{fl}*}\mathbb{G}_m) = \bigoplus_{x \in S} H_{\text{fl}}^1(X, i_{x,*}\mathbb{Q}/\mathbb{Z}) = \bigoplus_{x \in S} H_{\text{fl}}^1(x, \mathbb{Q}/\mathbb{Z}).$$

We also have

$$H_{\text{fl}}^2(X, \mathbb{G}_m) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{r-1} & \text{if } K \text{ has } r > 0 \text{ real places,} \\ 0 & \text{otherwise,} \end{cases}$$

and $H_{\text{fl}}^3(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$ by [11, Chap. II, Prop. 2.1]. Therefore the exact sequence (5.4) splits into two exact sequences

$$0 \rightarrow H_{\text{fl}}^1(X, \mathbb{G}_m) \rightarrow H_{\text{kfl}}^1(X, \mathbb{G}_m) \xrightarrow{\alpha} H_{\text{fl}}^0(X, R^1\varepsilon_{\text{fl}*}\mathbb{G}_m) \rightarrow 0
 \tag{5.5}$$

and

$$0 \rightarrow H_{\mathfrak{H}}^2(X, \mathbb{G}_m) \rightarrow H_{\text{kff}}^2(X, \mathbb{G}_m) \rightarrow H_{\mathfrak{H}}^1(X, R^1 \varepsilon_{\text{fl}*} \mathbb{G}_m) \rightarrow H_{\mathfrak{H}}^3(X, \mathbb{G}_m), \tag{5.6}$$

which can be identified with

$$0 \rightarrow H_{\mathfrak{H}}^1(X, \mathbb{G}_m) \rightarrow H_{\text{kff}}^1(X, \mathbb{G}_m) \xrightarrow{\alpha} \bigoplus_{x \in S} \mathbb{Q}/\mathbb{Z} \rightarrow 0 \tag{5.7}$$

and

$$0 \rightarrow H_{\mathfrak{H}}^2(X, \mathbb{G}_m) \rightarrow H_{\text{kff}}^2(X, \mathbb{G}_m) \rightarrow \bigoplus_{x \in S} H_{\mathfrak{H}}^1(x, \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}, \tag{5.8}$$

with $H_{\mathfrak{H}}^2(X, \mathbb{G}_m) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{r-1} & \text{if } K \text{ has } r > 0 \text{ real places,} \\ 0 & \text{otherwise.} \end{cases}$

Let $\text{Pic}(X)$ (resp., $\text{Pic}(X^{\text{log}})$) denote the group $H_{\mathfrak{H}}^1(X, \mathbb{G}_m)$ (resp., $H_{\text{kff}}^1(X, \mathbb{G}_m)$). We have a canonical degree map $\text{deg} : \text{Pic}(X) \rightarrow \mathbb{Z}$ in both the number-field case and the function-field case. By the short exact sequence (5.7), the degree map $\text{deg} : \text{Pic}(X) \rightarrow \mathbb{Z}$ extends uniquely to a map

$$\text{deg} : \text{Pic}(X^{\text{log}}) \rightarrow \mathbb{Q}, \tag{5.9}$$

and the two degree maps fit into the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Pic}(X) & \longrightarrow & \text{Pic}(X^{\text{log}}) & \longrightarrow & \bigoplus_{x \in S} \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \\ & & \downarrow \text{deg} & & \downarrow \text{deg} & & \downarrow \text{sum} \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} \longrightarrow 0. \end{array} \tag{5.10}$$

To summarize, we get the following proposition:

Proposition 5.2. *Let the notation and assumptions be as in the beginning of this subsection. Then we have the following:*

- (1) *The group $\text{Pic}(X^{\text{log}}) := H_{\text{kff}}^1(X, \mathbb{G}_m)$ admits a canonical degree map into \mathbb{Q} which extends the canonical degree map on $\text{Pic}(X)$, and the two degree maps fit into the following commutative diagram with exact rows:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Pic}(X) & \longrightarrow & \text{Pic}(X^{\text{log}}) & \longrightarrow & \bigoplus_{x \in S} \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \\ & & \downarrow \text{deg} & & \downarrow \text{deg} & & \downarrow \text{sum} \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} \longrightarrow 0. \end{array}$$

- (2) *The group $H_{\text{kff}}^2(X, \mathbb{G}_m)$ fits into an exact sequence*

$$0 \rightarrow H_{\mathfrak{H}}^2(X, \mathbb{G}_m) \rightarrow H_{\text{kff}}^2(X, \mathbb{G}_m) \rightarrow \bigoplus_{x \in S} H_{\mathfrak{H}}^1(x, \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

If K is a function field with its field of constants algebraically closed, then we have $H_{\text{kff}}^2(X, \mathbb{G}_m) \cong H_{\mathfrak{H}}^2(X, \mathbb{G}_m) = 0$. If the residue fields at the points of S are finite, then

the exact sequence becomes

$$0 \rightarrow H_{\text{fl}}^2(X, \mathbb{G}_m) \rightarrow H_{\text{kfl}}^2(X, \mathbb{G}_m) \rightarrow \bigoplus_{x \in S} \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z},$$

$$\text{with } H_{\text{fl}}^2(X, \mathbb{G}_m) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{r-1} & \text{if } K \text{ has } r > 0 \text{ real places,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We are left with checking $H_{\text{fl}}^2(X, \mathbb{G}_m) = 0$ for X a smooth projective curve over an algebraically closed field, and $H_{\text{fl}}^1(x, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ for x a point with finite residue field. The first follows from [15, Tag 03RM], and the second follows from

$$H_{\text{fl}}^1(x, \mathbb{Q}/\mathbb{Z}) \cong H_{\text{ét}}^1(x, \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}. \quad \square$$

Remark 5.1. A homomorphism $(\mathbb{Q}/\mathbb{Z})^n \rightarrow \mathbb{Q}/\mathbb{Z}$ is either zero or surjective. It follows that in the case where the residue fields at the points of S are finite, we have either a short exact sequence

$$0 \rightarrow H_{\text{fl}}^2(X, \mathbb{G}_m) \rightarrow H_{\text{kfl}}^2(X, \mathbb{G}_m) \rightarrow \bigoplus_{x \in S} \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

or an exact sequence

$$0 \rightarrow H_{\text{fl}}^2(X, \mathbb{G}_m) \rightarrow H_{\text{kfl}}^2(X, \mathbb{G}_m) \rightarrow \bigoplus_{x \in S} \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$

Acknowledgments The author has benefited a lot from regular discussions on this project with Professor Ulrich Görtz in the past one and a half years. This article would never exist without his help. The author thanks Professor Chikara Nakayama for very helpful correspondences, as well as for telling him about reference [16] for cup products on an arbitrary site. The author thanks the anonymous referee for corrections and very helpful suggestions. This work was partially supported by SFB/TR 45 ‘Periods, moduli spaces and arithmetic of algebraic varieties’.

Appendix A. A representability lemma about Hom-Sheaves

Lemma A.1. *Let S be a scheme, F be a finite flat commutative group scheme of multiplicative type over S which is killed by some positive integer n , and G be a commutative group scheme over S satisfying one of the following two conditions:*

- (1) G is smooth and affine over S or
- (2) $G[n] := \text{Ker}(G \xrightarrow{n} G)$ is finite flat over S .

Then the fppf sheaf $H := \mathcal{H}om_S(F, G)$ is representable by an étale quasi-finite separated group scheme over S .

Proof. We first deal with case (1). By [14, Exposé XI, Cor. 4.2], the sheaf $\mathcal{H}om_S(F, G)$ is representable by a smooth separated group scheme over S . The fibers of the group scheme $\mathcal{H}om_S(F, G)$ over S are finite by the structure theorem [14, Exposé XVII, Thm. 7.2.1] of commutative group schemes and [14, Exposé XVII, Prop. 2.4]. It follows that $\mathcal{H}om_S(F, G)$ is a quasi-finite étale separated group scheme over S .

Now we deal with case (2). Clearly we have $\mathcal{H}om_S(F, G) = \mathcal{H}om_S(F, G[n])$. Let $0 \rightarrow G[n] \rightarrow G_1 \rightarrow G_2 \rightarrow 0$ be the canonical smooth resolution of G with G_1, G_2 affine smooth commutative group schemes over S (see [11, Thm. A.5]). Then we have an exact sequence

$$0 \rightarrow \mathcal{H}om_S(F, G) \rightarrow \mathcal{H}om_S(F, G_1) \xrightarrow{\alpha} \mathcal{H}om_S(F, G_2)$$

of fppf sheaves of abelian groups over S . By case (1), the sheaves $\mathcal{H}om_S(F, G_i)$ for $i = 1, 2$ are representable by étale quasi-finite separated group schemes over S . Hence $H = \mathcal{H}om_S(F, G)$ as the kernel of α is representable. Furthermore, by [15, Tag 02GW], α is étale. It is also separated. It follows that H is étale, separated, and quasi-finite. \square

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