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A monadicity theorem

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A monadicity theorem is established for functors which satisfy the conditions of the "first isomorphism theorem" (following Linton's terminology). An application is made to the characterisation of certain types of algebraic categories generated by linear monads.

Introduction

In this article we establish that a functor $U : B \to C$ with a left adjoint is crudely monadic if it satisfies conditions analogous to those described by Linton [6] and there called the first isomorphism theorem. For certain types of algebraic category this monadicity theorem is an important alternative to the standard monadicity criteria of Beck [7].

In Section 2 we give a characterisation theorem based on the first isomorphism theorem for certain types of algebraic category generated by linear monads.

Throughout the article we suppose, unless otherwise stated, that $V = (V, \otimes, I, \ldots)$ is a complete and cocomplete symmetric monoidal closed category and that all categorical algebra is *relative* to this V . For terminology and notation we refer to Eilenberg and Kelly [5] and Mac Lane [7].

1. The theorem

DEFINITION 1.1. The *first isomorphism theorem* is said to hold for a functor $U : B \to C$ if

FIT 0: B has coequalisers and kernel pairs,

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FIT 1: f is a coequaliser in $\mathcal{B}$ iff Uf is a coequaliser in $\mathcal{C}$,

FIT 2: (f, g) is a kernel pair in $\mathcal{B}$ if (Uf, Ug) is a kernel pair in $\mathcal{C}$. //

THEOREM 1.2. *If* $\mathcal{C}$ *has coequalisers and kernel pairs and* $F \dashv U : \mathcal{B} \to \mathcal{C}$, *then* $\mathcal{B}$ *is crudely monadic over* $\mathcal{C}$ *if the first isomorphism theorem holds for* U .

Proof. Let $\mathsf{T} = (T, \mu, \eta)$ be the monad generated by the adjunction $(\varepsilon, \eta) : F \dashv U : \mathcal{B} \to \mathcal{C}$, and let $M : \mathcal{B} \to \mathcal{C}^{\mathsf{T}}$ be the canonical comparison functor

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, shift left, "M"'] \arrow[dr, shift left, "U"] & & \mathcal{C} \arrow[ll, shift left, "\overline{M}"'] \arrow[dl, shift left, "U^{\mathsf{T}}"'] \\
& \mathcal{C} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F^{\mathsf{T}}"'] &
\end{tikzcd}$$

where MB is UB with structure $U\varepsilon_B : UFUB \to UB$. Because $\mathcal{B}$ has coequalisers, M has a left adjoint $\overline{M}$ defined by the following coequaliser:

$$FUFC \underset{\varepsilon_{FC}}{\overset{F\zeta}{\rightrightarrows}} FC \xrightarrow{e_C} \overline{M}(C, \zeta) ,$$

and the counit $\overline{\varepsilon}$ is given by:

$$\begin{tikzcd}
FUFUB \arrow[r, shift left, "FU\varepsilon_B"] \arrow[r, shift right, "\varepsilon_{FUB}"'] & FUB \arrow[r, "e_{MB}"] \arrow[dr, "\varepsilon_B"'] & \overline{M}MB \arrow[d, "\overline{\varepsilon}_B"] \\
& & B
\end{tikzcd}$$

Applying U , we have that $U\varepsilon_B = \mathrm{coequ}\left(UFU\varepsilon_B, U\varepsilon_{FUB}\right)$. But Ue_{MB} is a coequaliser in $\mathcal{C}$ by hypothesis, so Ue_{MB} is an epimorphism, so $U\overline{\varepsilon}_B$ is an isomorphism. Thus $\overline{\varepsilon}_B$ is a coequaliser (by hypothesis) so M reflects isomorphisms, so U reflects isomorphisms, so $\overline{\varepsilon}_B$ is an isomorphism. Also $\overline{\eta} : 1 \to M\overline{M}$ is given by

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$$\begin{tikzcd}
MFUFC \arrow[r, shift left, "MF\zeta"] \arrow[r, shift right, "M\varepsilon_{FC}"'] & MFC \arrow[r, "Me_C"] \arrow[dr, "\zeta_C"'] & M\bar{M}(C,\zeta) \\
& & C \arrow[u, "\bar{\eta}_C"']
\end{tikzcd} .$$

Let (ϕ_1, ϕ_2) be the kernel pair of e in C (as in C^T). The functor U preserves kernel pairs, so $Ue = \text{coequ}(U\phi_1, U\phi_2)$ in C by hypothesis. Thus $\bar{\eta}_C = \text{coequ}(\zeta . U\phi_1, \zeta . U\phi_2)$ in C. Hence it remains to show that $\bar{\eta}_C$ is a monomorphism. Consider

$$\begin{tikzcd}
MQ \arrow[rr, shift left, "M\phi_1"] \arrow[rr, shift right, "M\phi_2"'] & & MFC \arrow[r, "\zeta_C"] \arrow[dr, "Me_C"'] & C \arrow[d, "\bar{\eta}_C"] \\
& P \arrow[ul, "\theta"] \arrow[ur, shift left, "\pi_1"] \arrow[ur, shift right, "\pi_2"'] & & M\bar{M}(C,\zeta)
\end{tikzcd} .$$

Let (π_1, π_2) be the kernel pair of ζ_C in C (as in C^T). Then the comparison θ is a monomorphism and $M\phi_1 \cdot \theta = \pi_i$ $(i = 1, 2)$. But $\bar{M} \dashv M$, so we have

$$\begin{tikzcd}
P \arrow[r, "\bar{\eta}_P"] \arrow[dr, "\theta"'] & M\bar{M}P \arrow[d, "M\lambda"] \\
& MQ
\end{tikzcd}$$

for a unique $\lambda : \bar{M}P \to Q$. Thus $\bar{\eta}_P$ is a monomorphism, hence is an isomorphism. Thus $(M\phi_1 . M\lambda, M\phi_2 . M\lambda)$ is a kernel pair in C^T, so, by hypothesis, $(\phi_1 . \lambda, \phi_2 . \lambda)$ is a kernel pair in B. Now, by definition of $\bar{\eta}$, λ, θ, and α_i $(i = 1, 2)$, the following diagram commutes $(i = 1, 2)$:

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$$\begin{tikzcd}
MFP \arrow[r, "\zeta_P"] \arrow[d, "Me"'] \arrow[rr, bend left, "M\alpha_i"] & P \arrow[r, "\pi_i"] \arrow[d, "\theta"] & MFC \\
MMP \arrow[ur, "\overline{\eta}_P"] \arrow[r, "M\lambda"'] & MQ \arrow[ur, "M\phi_i"'] &
\end{tikzcd}$$

But $e = \text{coequ}(\alpha_1, \alpha_2)$ in B because $\overline{M}$ is a reflection, and $\text{coequ}(\alpha_1, \alpha_2)$ in C^T is just (C, ζ) because ζ_P is an epimorphism. Thus

$$\begin{aligned} e &= \text{coequ}(\phi_1\lambda e, \phi_2\lambda e) \\ &= \text{coequ}(\phi_1\lambda, \phi_2\lambda) , \end{aligned}$$

since e is an epimorphism. Hence λ is an isomorphism because (ϕ_1, ϕ_2) and $(\phi_1\lambda, \phi_2\lambda)$ are now both kernel pairs of e. Hence θ is an isomorphism, so $\overline{\eta}_C$ is an isomorphism. //

COROLLARY 1.3. *If* C *has coequalisers and kernel pairs and* $F \dashv U : B \to C$ *is an adjunction such that* UF *preserves coequalisers of reflective pairs then* B *is crudely monadic over* C *if and only if the first isomorphism theorem holds on* U. //

2. Linear monads

Let $N : A \to C$ be a fully faithful dense functor. Following Diers [4], we say that an N-theory (T, t) is algebraic if $t^{op} : A \to T^{op}$ has a right N-adjoint R. We call an algebraic theory strictly N-hyperlinear if the mean tensor product (see [1]) $C(NA, C) * RtA$ exists in C and is N-absolute.

As usual, the category C^b of T-algebras is defined by the pullback:

$$\begin{tikzcd}
C^b \arrow[r] \arrow[d, "U"] & {[T, V]} \arrow[d, "{[t,1]}"] \\
C \arrow[r] \arrow[u, bend left, dashed, "F"] & {[A^{op}, V]} \arrow[u, bend left, dashed, "\overline{t}"]
\end{tikzcd} .$$

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As a consequence

$$[t, 1]\bar{t}\big(C(N-, C)\big) = \int^A T(tA, t-) \otimes C(NA, C)$$

$$\cong \int^A C(N-, RtA) \otimes C(NA, C) \cong C\big(N-, C(NA, C) * RtA\big) .$$

So we obtain $F \dashv U$ where UF is just the restriction of $\bar{t} \dashv [t, 1]$ to C .

$$UFC = C(NA, C) * RtA .$$

Thus $RtA \cong UFNA$ and

$$C(NB, UFC) \cong \int^A C(NA, C) \otimes C(NB, RtA) \cong \int^A C(NA, C) \otimes C(NB, UFNA) .$$

Thus the monad $T = UF$ is strictly N-hyperlinear in the sense of Day [3]. Such an algebraic category C^b will be called strongly N-algebraic.

THEOREM 2.1. *If* C *has coequalisers and kernel pairs and each* $C(NA, -)$, $A \in A$, *preserves coequalisers of reflective pairs, then a category* B *is strongly* N*-algebraic over* C *if and only if there exists* $F \dashv U : B \to C$ *such that*

(1) $\int^A C(NA, C) \otimes C(NB, UFNA) \cong C(NB, UFC)$,

(2) *the first isomorphism theorem holds for* U .

Proof. Necessity follows from the fact that $C(NA, -)$, $A \in A$, preserves coequalisers of reflective pairs, thus $UF \cong C(NA, -) * UFA$ preserves coequalisers of reflective pairs, thus U creates coequalisers of reflective pairs. For sufficiency we have that U is monadic by (2) and Theorem 1.2. By density of N we have $\int^A C(NA, C) \cdot UFNA \cong UFC$ from (1). Thus the monad generated by the theory which maps A^{op} to the full image of FN (see Day [2]) coincides with the monad UF , as required. //

This result should be compared with Lawvere's characterisation theorem (Linton [6], Corollary to Proposition 6) for the case $C = Ens$.

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