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Actions of Borel Subgroups on Homogeneous Spaces of Reductive Complex Lie Groups and Integrability

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Abstract. Let G be a real reductive Lie group, K its compact subgroup. Let A be the algebra of G-invariant real-analytic functions on $T^*(G/K)$ (with respect to the Poisson bracket) and let C be the center of A. Denote by $2\varepsilon(G, K)$ the maximal number of functionally independent functions from $A \setminus C$. We prove that $\varepsilon(G, K)$ is equal to the codimension $\delta(G, K)$ of maximal dimension orbits of the Borel subgroup $B \subset G^{\mathbb{C}}$ in the complex algebraic variety $G^{\mathbb{C}}/K^{\mathbb{C}}$. Moreover, if $\delta(G, K) = 1$, then all G-invariant Hamiltonian systems on $T^*(G/K)$ are integrable in the class of the integrals generated by the symmetry group G. We also discuss related questions in the geometry of the Borel group action.

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1. Introduction

Let X be a symplectic manifold. One calls a Hamiltonian system on X completely integrable if it admits a maximal number of independent integrals in involution ($\dim X/2$ functions commuting with respect to the Poisson bracket on X). Denote by G a real connected reductive Lie group which acts on X in a Hamiltonian fashion, by K a closed connected subgroup of G. Let $P: X \to \mathfrak{g}^*$ be the moment mapping, where $\mathfrak{g}$ is the Lie algebra of G. The functions of type $h \circ P$, for $h: \mathfrak{g}^* \to \mathbb{R}$, are called *collective*. These functions are integrals for any flow on X with a G-invariant Hamiltonian H. The question arises, for which symplectic manifolds X

all G-invariant Hamiltonian systems on X are integrable in
the class of the integrals generated by its symmetry group G. $\quad (*)$

Of course, X has this property if on X there exists a completely integrable system consisting of real-analytic functions of type $h \circ P$ (so-called *collective completely integrable system* [1]). All symmetric spaces G/K admit a collective completely integrable system on the face space $T^*(G/K)$ ([2, 4–6, 10]). Moreover, the following conditions and $(*)$ (with $X = T^*(G/K)$) are equivalent [1, 7, 8]:

(1) on the phase space $T^*(G/K)$ there exists a collective completely integrable system;

(2) the codimension $\delta(G^{\mathbb{C}}, K^{\mathbb{C}})$ of maximal dimension orbits of the Borel subgroup $B \subset G^{\mathbb{C}}$ in the complex algebraic variety $G^{\mathbb{C}}/K^{\mathbb{C}}$ is 0;
(3) the subgroup $K^{\mathbb{C}}$ of $G^{\mathbb{C}}$ is spherical, i.e. the quasiregular representation of $G^{\mathbb{C}}$ on the space $\mathbb{C}[G^{\mathbb{C}}/K^{\mathbb{C}}]$ of regular functions on the affine algebraic variety $G^{\mathbb{C}}/K^{\mathbb{C}}$ has the simple spectrum;
(4) the algebra of G-invariant functions on $T^*(G/K)$ is commutative, i.e. $\varepsilon(G, K) = 0$.

The classification of spherical subgroups of *semisimple* complex (connected) Lie groups was obtained in [7, 11, 12]. Since the Poisson structure on X is nondegenerate, in the case of existence of a collective completely integrable system any G-invariant Hamiltonian H locally has the form $h \circ P$; i.e., for the integrability we don't use 'effectively' the function H. This fact was observed in [9] and used to obtain new classes of spaces with property (∗).

Let $N_{\max}(X)$ be the maximal number of independent real-analytic functions in involution on X of type $h \circ P$. If $N_{\max}(X) = (\dim X/2) - 1$ we will call the corresponding system of functions an *almost collective completely integrable system* [9] and if, in addition, $X = T^*(G/K)$ we will call such homogeneous space G/K an *almost spherical space* [9]. The classification of all almost spherical spaces G/K with compact simple (connected) Lie groups G was obtained in [9].

The purpose of this note is to prove that $N_{\max}(T^*(G/K)) = \dim T^*(G/K)/2 - \varepsilon(G, K)$. Moreover, we prove that every space $X = T^*(G/K)$ with $\varepsilon(G, K) = 1$, where the subgroup $K \subset G$ is compact, also has property (∗): for the integrability we can use either H or another G-invariant function (see [9, Prop. 1] and Prop. 12). We also show that the number $\varepsilon(G, K)$ defined above admits an equivalent definition using the equation (6). But by [16, 17] the analogous equation defines the number $\delta(G, K)$ so that $\delta(G, K) = \varepsilon(G, K)$. In particular, the almost spherical spaces G/K are affine homogeneous spaces of complexity $1 = \delta(G, K)$ classified in [17]. Here we find some new properties of pairs (G, K) of reductive Lie groups (Lemmas 15, 16) and as a consequence a new proof of the equality $\delta(G, K) = \varepsilon(G, K)$ in the reductive case (Theorem 17). We investigate properties of the function $\delta(G, K)$. The main results are the interpolation property (Theorem 8): $\delta(G, S) + \delta(S, K) \leqslant \delta(G, K)$, where $K \subset S \subset G$, and Theorem 11.

2. Integrability and Actions of Borel Subgroups on Homogeneous Spaces of Reductive Lie Groups

2.1. MOMENT MAP AND HAMILTONIAN ACTION

Let M be a homogeneous space of a real connected Lie group G, i.e. $M = G/K$, where K is a closed subgroup of G. Let $\pi: G \to M$ be the canonical projection from G onto M. The natural action τ of G on the quotient space M extends to the (left) action of G on T^*M, which we denote by τ^*. This G-action on T^*M is symplectic since it preserves the canonical 1-form λ (the form 'pdq') and thus also the symplectic 2-form $d\lambda$. For each vector ξ belonging to the Lie algebra $\mathfrak{g}$ of G the 1-parameter subgroup $\exp t\xi$ induces the Hamiltonian vector field $\hat{\xi}$ on T^*M with the Hamiltonian function $f_\xi = \lambda(\hat{\xi})$: $\mathrm{d}f_\xi = -\hat{\xi}\rfloor \mathrm{d}\lambda$. The map $\xi \mapsto f_\xi$ of $\mathfrak{g}$ into the algebra $C^\infty(T^*M)$ (with Poisson bracket) is an equivariant algebra homomorphism: $f(g^{-1}\cdot m) = f_{\mathrm{Ad}\, g(\xi)}(m), m \in T^*M$ and hence the action τ^* of G on T^*M is Poisson [3,8]. This action defines the moment map $P: T^*M \to \mathfrak{g}^*$ from T^*M to the dual space of the Lie algebra $\mathfrak{g}$ by $P(m)(\xi) = f_\xi(m)$. For arbitrary smooth functions h_1 and h_2 on $\mathfrak{g}^*$ we have $\{h_1 \circ P, h_2 \circ P\} = \{h_1, h_2\} \circ P$, where the Poisson bracket on $\mathfrak{g}^*$ is given by the formula $\{h_1, h_2\}(\beta) = \beta([\mathrm{d}h_1(\beta), \mathrm{d}h_2(\beta)]), \beta \in \mathfrak{g}^*$.

Using the map $\pi_*|_e: T_eG = \mathfrak{g} \to T_{\pi(e)}M$ we can identify the tangent space $T_{\pi(e)}M$ with the quotient space $\mathfrak{g}/\mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of the Lie group K. Thus we can identify $T^*_{\pi(e)}M$ with the subspace $\mathfrak{k}^0 \subset \mathfrak{g}^*$, where $\mathfrak{k}^0 = \{\alpha \in \mathfrak{g}^*: \alpha(\mathfrak{k}) = 0\}$, using the map $\mathfrak{k}^0 \to T^*_{\pi(e)}M, \alpha \mapsto \hat{\alpha}$ and putting $\hat{\alpha}(\pi_*(\xi)) = \alpha(\xi)$. Because of the transitive action of G on M we see that the moment map P is determined completely by the restriction $P|_{T^*_{\pi(e)}M}$. From the definitions of the canonical form λ it can be concluded that $P(\hat{\alpha}) = \alpha$ (see [6] or [4]).

Since the Poisson bracket on $\mathfrak{g}^*$ induces a symplectic structure on the orbits of the coadjoint representation Ad^* of G on $\mathfrak{g}^*$, the maximal number of functions (or linear functions $\xi \in \mathfrak{g}$) on $\mathfrak{g}^*$ independent at $\alpha = P(\hat{\alpha})$ and with pairwise Poisson bracket equal to zero (at this point) is $\dim \mathfrak{g}^\alpha + \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{g}^\alpha)$, where $\mathfrak{g}^\alpha =_{\mathrm{def}} \{\xi \in \mathfrak{g}: \mathrm{ad}^*\,\xi(\alpha) = 0\}$ is the Lie algebra of the isotropy group G^α of the point α for Ad^*-action of G. On the other hand, the isotropy group $G^{\hat{\alpha}}$ of the point $\hat{\alpha}$ is a subgroup of K. But clearly the space $T^*_{\pi(e)}M$ is invariant under the action $\tau^*|_K$ and $k \cdot \hat{\alpha} = \widehat{\mathrm{Ad}^*(k^{-1})\alpha}$. Hence $G^{\hat{\alpha}} = G^\alpha \cap K$ and $\mathfrak{k}^\alpha =_{\mathrm{def}} \mathfrak{g}^\alpha \cap \mathfrak{k}$ is the Lie algebra of $G^{\hat{\alpha}}$. That is, $\mathfrak{k}^\alpha$ is the kernel of the linear mapping $\xi \mapsto \hat{\xi}(\hat{\alpha}), \xi \in \mathfrak{g}$ and therefore the maximal number of independent functions of the form $h \circ P$ in involution at $\hat{\alpha}$ is equal to $\dim(\mathfrak{g}^\alpha/\mathfrak{k}^\alpha) + \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{g}^\alpha)$. This number does not exceed $\frac{1}{2}\dim T^*M$ (because of the nondegeneracy of the symplectic form $d\lambda$ on T^*M) so that we can define the nonnegative integer $\varepsilon = \varepsilon(\mathfrak{g}, \mathfrak{k})$ such that

$$\dim(\mathfrak{g}^\alpha/\mathfrak{k}^\alpha) + \tfrac{1}{2}\dim(\mathfrak{g}/\mathfrak{g}^\alpha) = \dim(\mathfrak{g}/\mathfrak{k}) - \varepsilon \tag{1}$$

for any α from some (Zariski) open subset of $\mathfrak{k}^0 \subset \mathfrak{g}^*$.

Now let us consider the tangent space $W(\hat{\alpha})$ to the orbit $G \cdot \hat{\alpha} \subset T^*M$. This space is generated by vectors $\hat{\xi}(x), \xi \in \mathfrak{g}$ and has the dimension $\dim(\mathfrak{g}/\mathfrak{k}^\alpha)$. Let $W(\hat{\alpha})^\perp =_{\mathrm{def}} \{X \in T_{\hat{\alpha}}T^*M: \mathrm{d}\lambda(X, W(\hat{\alpha})) = 0\}$ be the orthogonal complement to $W(\hat{\alpha})$ with respect to the symplectic structure $\mathrm{d}\lambda$. We show that the codimension of $W^\perp(\hat{\alpha}) \cap W(\hat{\alpha})$ in $W^\perp(\hat{\alpha})$ is equal to 2ε. Indeed, multiplying both sides of (1) by 2, we obtain after simple rearrangements

$$\dim(\mathfrak{g}^\alpha/\mathfrak{k}^\alpha) + \dim(\mathfrak{g}/\mathfrak{k}^\alpha) = 2\dim(\mathfrak{g}/\mathfrak{k}) - 2\varepsilon. \tag{2}$$

Because of the relations

$$\begin{aligned}\mathrm{d}\lambda(\hat{\eta}(\hat{\alpha}), \hat{\xi}(\hat{\alpha}))(\hat{\alpha}) &= \{f_\xi, f_\eta\}(\hat{\alpha}) = f_{[\xi,\eta]}(\hat{\alpha}) = P(\hat{\alpha})([\xi, \eta]) \\ &= \alpha([\xi, \eta]) = \mathrm{ad}^*\,\xi(\alpha)(\eta) \quad \text{for any} \quad \xi, \eta \in \mathfrak{g}\end{aligned}$$

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it follows that $W(\hat{\alpha}) \cap W^{\perp}(\hat{\alpha}) = \{\hat{\eta}(\hat{\alpha}), \eta \in \mathfrak{g}^{\alpha}\}$ and $\dim(W(\hat{\alpha}) \cap W^{\perp}(\hat{\alpha})) = \dim(\mathfrak{g}^{\alpha}/\mathfrak{k}^{\alpha})$. Taking into account that 2-form $\mathrm{d}\lambda$ is nondegenerate we obtain $\dim W^{\perp}(\hat{\alpha}) + \dim W(\hat{\alpha}) = \dim T^*M = 2\dim(\mathfrak{g}/\mathfrak{k})$. Thus $\dim W(\hat{\alpha})^{\perp} - \dim(W(\alpha) \cap W^{\perp}(\alpha)) = 2\varepsilon$.

Suppose that the Lie group K is compact. Then any two orbits of the linear representation $\rho = \mathrm{Ad}^*|_{\mathfrak{k}^0}$ of K in $\mathfrak{k}^0$ are separated by some invariant polynomial function on $\mathfrak{k}^0$. Since the action of G on T^*M is reduced to the action of K on $\mathfrak{k}^0 = T^*_{\pi(e)}M$ induced by ρ, we see that for some point $\hat{\alpha} \in T^*_{\pi(e)}M$ the space $W^{\perp}(\hat{\alpha})$ is spanned by values of Hamiltonian vector fields (at $\hat{\alpha}$) of G-invariant real-analytic functions on T^*M. Now applying this to the action of the group G on $M = G/K$ we obtain

PROPOSITION 1. *If the subgroup K of G is compact then on $T^*(G/K)$ there exists $2\varepsilon(\mathfrak{g}, \mathfrak{k})$ independent G-invariant real-analytic functions which are independent of functions of type $h \circ P$.*

2.2. PAIRS OF REDUCTIVE LIE ALGEBRAS

Let $\mathfrak{g}$ be a reductive real (or complex) Lie algebra. There exists a faithful representation of $\mathfrak{g}$ such that its associated bilinear form Φ is nondegenerate on $\mathfrak{g}$ (if $\mathfrak{g}$ is semi-simple we can take as Φ the Killing form associated with the adjoint representation of $\mathfrak{g}$). Let $\mathfrak{k} \subset \mathfrak{g}$ be a reductive in $\mathfrak{g}$ subalgebra, i.e. the representation $x \longmapsto \mathrm{ad}_{\mathfrak{g}} x$ of $\mathfrak{k}$ on $\mathfrak{g}$ is completely reducible. This subalgebra is necessarily reductive (in itself). For each element $x \in \mathfrak{g}$ let $\mathfrak{g}^0(x)$ (respectively $\mathfrak{g}^x$) denote the set of all $z \in \mathfrak{g}$ which satisfy $(\mathrm{ad}\, x)^n(z) = 0$ for sufficiently large n (respectively $[x, z] = 0$). Let $\mathfrak{k}^x = \mathfrak{k} \cap \mathfrak{g}^x$. By consideration of minors it is clear that the set of R-elements in $\mathfrak{m} = \mathfrak{k}^{\perp} = \{x \in \mathfrak{g} : \Phi(x, \mathfrak{k}) = 0\}$

$$R(\mathfrak{m}) = \{x \in \mathfrak{m}\colon \dim \mathfrak{g}^x \leqslant \dim \mathfrak{g}^y, \dim \mathfrak{g}^0(x) \leqslant \dim \mathfrak{g}^0(y), \\ \dim \mathfrak{k}^x \leqslant \dim \mathfrak{k}^y, \forall y \in \mathfrak{m}\} \tag{3}$$

is a non-empty Zariski open subset of $\mathfrak{m}$. Let us define the integer $\varepsilon = \varepsilon(\mathfrak{g}, \mathfrak{k})$ for the pair $(\mathfrak{g}, \mathfrak{k})$ putting

$$\dim(\mathfrak{g}^x/\mathfrak{k}^x) + \tfrac{1}{2}\dim(\mathfrak{g}/\mathfrak{g}^x) = \dim(\mathfrak{g}/\mathfrak{k}) - \varepsilon, \tag{4}$$

where $x \in R(\mathfrak{m})$. Multiplying both sides of (4) by 2 and taking into account that $\dim \mathfrak{g} = \dim \mathfrak{k} + \dim \mathfrak{m}$ we obtain after simple rearrangements

$$\dim(\mathfrak{g}^x/\mathfrak{k}^x) + \dim(\mathfrak{k}/\mathfrak{k}^x) = \dim \mathfrak{m} - 2\varepsilon. \tag{5}$$

The set $R(\mathfrak{m})$ consists of semi-simple elements of $\mathfrak{g}$ [7, Prop. 1.2], i.e. the centralizer $\mathfrak{g}^x$, $x \in R(\mathfrak{m})$ is a reductive (in $\mathfrak{g}$) subalgebra of $\mathfrak{g}$. Moreover, the maximal semi-simple ideal $[\mathfrak{g}^x, \mathfrak{g}^x]$ of $\mathfrak{g}^x$ is contained in the algebra $\mathfrak{k}^x$ (see [5] or [7, Prop. 1.1]) so that

$\dim(\mathfrak{g}^x/\mathfrak{k}^x) = \operatorname{rank}\mathfrak{g} - \operatorname{rank}\mathfrak{k}^x$, and so (5) can be rewritten as

$$(\operatorname{rank}\mathfrak{g} - \operatorname{rank}\mathfrak{k}^x) + \dim(\mathfrak{k}/\mathfrak{k}^x) = \dim\mathfrak{m} - 2\varepsilon. \tag{6}$$

Dividing both sides of (6) by 2 and taking into account that $\dim\mathfrak{g} = \dim\mathfrak{k} + \dim\mathfrak{m}$ we conclude that

$$\tfrac{1}{2}(\operatorname{rank}\mathfrak{k}^x + \dim\mathfrak{k}^x) = \tfrac{1}{2}(\operatorname{rank}\mathfrak{g} + \dim\mathfrak{g}) - \dim\mathfrak{m} + \varepsilon \tag{7}$$

and

$$\dim\mathfrak{g} - \varepsilon = \tfrac{1}{2}(\operatorname{rank}\mathfrak{g} + \dim\mathfrak{g}) + \dim\mathfrak{k} - \tfrac{1}{2}(\operatorname{rank}\mathfrak{k}^x + \dim\mathfrak{k}^x). \tag{8}$$

Moreover, it is evident that $\dim(\mathfrak{g}^x/\mathfrak{k}^x) \leqslant \operatorname{rank}\mathfrak{g} \leqslant \dim\mathfrak{g}^x$ and, consequently, (4) implies

$$\dim\mathfrak{k} \geqslant \tfrac{1}{2}(\dim\mathfrak{g} - \operatorname{rank}\mathfrak{g}) - \varepsilon. \tag{9}$$

Denote by $\mathbf{b}(\mathfrak{m})$ the non-negative integer $\frac{1}{2}(\operatorname{rank}\mathfrak{g} + \dim\mathfrak{g}) - \dim\mathfrak{m} + \varepsilon(\mathfrak{g},\mathfrak{k})$ (the right side of (7)). However, $\mathfrak{k}^x$ is a reductive algebra. Hence $\frac{1}{2}(\operatorname{rank}\mathfrak{k}^x + \dim\mathfrak{k}^x)$ (the left side of (7)) is the dimension of a Borel subalgebra of $\mathfrak{k}^x$ and as an immediate consequence of (7) we obtain

PROPOSITION 2. *For any $x \in R(\mathfrak{m})$ the dimension of a Borel subalgebra of $\mathfrak{k}^x$ is equal to $\mathbf{b}(\mathfrak{m})$ and, consequently, is the same for all $x \in R(\mathfrak{m})$.*

DEFINITION 3. Let $\mathfrak{g}$ be a reductive real (or complex) Lie algebra, let Φ be a nondegenerate bilinear form on $\mathfrak{g}$ associated with a faithful representation of $\mathfrak{g}$, and let $\mathfrak{k} \subset \mathfrak{g}$ be a reductive in $\mathfrak{g}$ subalgebra. Let $\mathfrak{m} = \{x \in \mathfrak{g}\colon \Phi(x,\mathfrak{k}) = 0\}$. We say that $(\mathfrak{g},\mathfrak{k})$ is an *ε-pair*, if the equivalent conditions (4)–(8) are satisfied for all points x belonging to some open subset of $\mathfrak{m}$ (or, what is the same thing, for some $x \in R(\mathfrak{m})$).

Remark 4. The reductive Lie algebra $\mathfrak{k}$ is the Lie algebra direct sum $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}_1$, where $\mathfrak{z}$ is the center of $\mathfrak{k}$ and $\mathfrak{k}_1 = [\mathfrak{k},\mathfrak{k}]$ is the maximal semi-simple ideal in $\mathfrak{k}$. Since $\mathfrak{k}$ is reductive in $\mathfrak{g}$, the center $\mathfrak{z}$ consists of semi-simple elements of $\mathfrak{g}$. Let $\mathfrak{h}_1$ be a Cartan subalgebra of the centralizer $Z(\mathfrak{k}_1)$ of $\mathfrak{k}_1$ in $\mathfrak{g}$ containing the center $\mathfrak{z}$. Then for some subspace $\mathfrak{z}_1 \subset \mathfrak{h}_1$ we have $\mathfrak{h}_1 = \mathfrak{z}_1 \oplus \mathfrak{z}$. The restrictions of Φ to the subalgebras $\mathfrak{k}_1$ and $\mathfrak{k}_2 = \mathfrak{h}_1 \oplus \mathfrak{k}_1$ are nondegenerate with $\Phi(\mathfrak{k}_1,\mathfrak{h}_1) = 0$ [13, Chapter VII, Section 1]. The algebra $\mathfrak{k}_2$ is its own normalizer in $\mathfrak{g}$ (the Cartan subalgebra $\mathfrak{h}_1$ is its own normalizer in $Z(\mathfrak{k}_1)$). Let $\mathfrak{p}_2$ be the orthogonal complement of $\mathfrak{k}_2$ in $\mathfrak{g}$ relative to Φ. Let $\mathfrak{p}$ be the $\operatorname{ad}\mathfrak{k}$-invariant complement to $\mathfrak{k}$ in $\mathfrak{g}$ (so $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$). Since, by invariance of Φ, $[\mathfrak{k},\mathfrak{m}] \subset \mathfrak{m}$ and $\mathfrak{p}_2 \subset \mathfrak{m} \subset \mathfrak{h}_1 \oplus \mathfrak{p}_2$ the representations $x \mapsto \operatorname{ad}_{\mathfrak{p}} x$, $x \mapsto \operatorname{ad}_{\mathfrak{z}_1 \oplus \mathfrak{p}_2} x$ and $x \mapsto \operatorname{ad}_{\mathfrak{m}} x$ of $\mathfrak{k}$ are isomorphic. By (6) the number $\varepsilon(\mathfrak{g},\mathfrak{k})$ is defined via the latter representation, i.e. the definition of $\varepsilon(\mathfrak{g},\mathfrak{k})$ does not depend on the choice of the form Φ.

Remark 5. It is clear that if $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{k}^{\mathbb{C}}$ are complexifications of algebras $\mathfrak{g}$ and $\mathfrak{k}$ respectively then $\varepsilon(\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}) = \varepsilon(\mathfrak{g}, \mathfrak{k})$.

Our interest now centers on what will be shown to be an important subset of $\mathfrak{m}$. Define for any $x \in \mathfrak{m}$ the subspace $\mathfrak{m}(x) \subset \mathfrak{m}$ putting

$$\mathfrak{m}(x) \stackrel{\text{def}}{=} \{z \in \mathfrak{m}: [x, z] \in \mathfrak{m}\}, \tag{10}$$

i.e. $\operatorname{ad} x(\mathfrak{m}(x)) \subset \mathfrak{m}$. By the invariance of Φ

$$\mathfrak{m}(x) = \{z \in \mathfrak{m}: \Phi(z, \operatorname{ad} x(\mathfrak{k})) = 0\}. \tag{11}$$

We continue with previous notations but throughout the remainder of this subsection it is assumed in addition that the form Φ is nondegenerate on $\mathfrak{k}$. Then $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$. Denote by $(\cdot)_{\mathfrak{m}}$ the projection into $\mathfrak{m}$ along $\mathfrak{k}$. Since $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, it is clear that $(\mathfrak{g}^x)_{\mathfrak{m}} \subset \mathfrak{m}(x)$ for any $x \in \mathfrak{m}$. An immediate consequence of (5) is the following proposition.

PROPOSITION 6. *For any $x \in R(\mathfrak{m})$ the codimension of $(\mathfrak{g}^x)_{\mathfrak{m}}$ in $\mathfrak{m}(x)$ is equal $2\varepsilon(\mathfrak{g}, \mathfrak{k})$. In particular, $\varepsilon(\mathfrak{g}, \mathfrak{k}) \geqslant 0$.*

Remark 7. If a pair $(\mathfrak{g}, \mathfrak{k})$ of real (or complex) reductive Lie algebras is a *symmetric pair*, i.e. $\mathfrak{k}$ is the algebra of fixed points of an involutory automorphism of the algebra $\mathfrak{g}$, then $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ and, consequently, for any $x \in \mathfrak{m}$: $\operatorname{ad} x(\mathfrak{m}(x)) \subset (\mathfrak{m} \cap \mathfrak{k}) = 0$. Thus $\mathfrak{m}(x) \subset \mathfrak{g}^x$ and by Proposition 6 $\varepsilon(\mathfrak{g}, \mathfrak{k}) = 0$. We say that the pair $(\mathfrak{g}, \mathfrak{k})$ of real (or complex) reductive Lie algebras is a *spherical pair* and the subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is a *spherical subalgebra* if $\varepsilon(\mathfrak{g}, \mathfrak{k}) = 0$ [7, 8]. All spherical subalgebras $\mathfrak{k}$ of compact Lie algebras $\mathfrak{g}$ are classified in [11] (for simple $\mathfrak{g}$) and [7, 12] (semi-simple case).

A theorem proved in [7] (see [7, Theorem 2.5] in the case of compact Lie algebras asserts, among other things, that if $\varepsilon(\mathfrak{g}, \mathfrak{k}) = 0$ then $\varepsilon(\mathfrak{g}, \mathfrak{s}) = 0$ and $\varepsilon(\mathfrak{s}, \mathfrak{k}) = 0$, where $\mathfrak{s}$ is any subalgebra of $\mathfrak{g}$ containing $\mathfrak{k}$. Theorem 8 below may be regarded as a generalization of this theorem to the case of arbitrary pair of reductive (not necessarily compact) Lie algebras. The proof which we have found is simple but uses limit arguments. We continue with previous notations.

THEOREM 8. *Let $\mathfrak{s}$ be a reductive subalgebra of $\mathfrak{g}$ containing $\mathfrak{k}$ such that the restriction of Φ to $\mathfrak{s}$ is non-degenerate. Then $0 \leqslant \varepsilon(\mathfrak{g}, \mathfrak{s}) + \varepsilon(\mathfrak{s}, \mathfrak{k}) \leqslant \varepsilon(\mathfrak{g}, \mathfrak{k})$.*

Proof. Let $\mathfrak{m}_1$ (respectively $\mathfrak{m}_2$) be the orthogonal complement to the subalgebra $\mathfrak{k}$ in $\mathfrak{s}$ (respectively to the subalgebra $\mathfrak{s}$ in $\mathfrak{g}$) with respect to the form Φ; i.e., $\mathfrak{s} = \mathfrak{k} \oplus \mathfrak{m}_1$ (respectively $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{m}_2$). Denote by $(\cdot)_{\mathfrak{m}_1}$ and $(\cdot)_{\mathfrak{m}_2}$ the projections into the subspaces $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively defined by the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Recall that $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Fix an element $\tilde{x}_1 + x_2 \in R(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$ such that $\tilde{x}_1 \in R(\mathfrak{m}_1)$ and $x_2 \in R(\mathfrak{m}_2)$. Consider the subspace $\mathfrak{m}_2(x_2) \subset \mathfrak{m}_2$ defined by (10). Let $V_2 = V_2(x_2)$ be some complementary subspace to $(\mathfrak{g}^{x_2})_{\mathfrak{m}_2}$ in $\mathfrak{m}_2(x_2)$. By Proposition 6 $\dim V_2 = 2\varepsilon(\mathfrak{g}, \mathfrak{s})$.

Let us show that $V_2 \cap (\mathfrak{g}^{x_1'+x_2})_{\mathfrak{m}} = 0$ for all x_1' from some nonempty Zariski open subset $Q_1(x_2) \subset R(\mathfrak{m}_1)$. Towards this end define $Q_1(x_2) \subset \mathfrak{m}_1$ as the set of all elements x_1' such that (1) $x_1' \in R(\mathfrak{m}_1)$ and $x_1' + x_2 \in R(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$; (2) the intersection $V_2'(x_1') = V_2 \cap (\mathfrak{g}^{x_1'+x_2})_{\mathfrak{m}}$ has the minimal possible dimension l. Since $V_2'(x_1') \subset (\mathfrak{g}^{x_1'+x_2})_{\mathfrak{m}}$, it follows that $\mathrm{ad}(x_1'+x_2)V_2'(x_1') \subset \mathrm{ad}(x_1'+x_2)(\mathfrak{k})$. But $V_2 \subset \mathfrak{m}_2(x_2)$ so that $[x_2, V_2'(x_1')] \subset \mathfrak{m}_2$. Taking into account the relations $[\mathfrak{k}, \mathfrak{m}_i] \subset \mathfrak{m}_i$, $i = 1, 2$ we obtain $\mathrm{ad}(x_1'+x_2)V_2'(x_1') \subset \mathrm{ad}\, x_2(\mathfrak{k})$. It is well known that the nonempty Zariski open set $Q_1(x_2)$ is dense in $\mathfrak{m}_1$. Thus we may find a sequence $x_{1,n}' \in Q_1(x_2)$, $n = 1, 2, \ldots$, converging to $0 \in \mathfrak{m}_1$. Now consider the Grassmann manifold G_l of all l-planes in $\mathfrak{m}_2$. This manifold, of course, is compact and hence we may find a subsequence $x_{1,n}$ of the sequence $x_{1,n}'$ with the property $x_{1,n} \to 0$ and the subspaces $V_2'(x_{1,n})$ converge to an l-plaine $V_2'(0)$ in the Grassmann manifold. From the above it follows immediately by taking the limit that $\mathrm{ad}\, x_2(V_2'(0)) \subset \mathrm{ad}\, x_2(\mathfrak{k}) \subset \mathrm{ad}\, x_2(\mathfrak{s})$, i.e. $V_2'(0) \subset (\mathfrak{g}^{x_2})_{\mathfrak{m}_2}$. But $V_2'(0) \subset V_2$ and $V_2 \cap (\mathfrak{g}^{x_2})_{\mathfrak{m}_2} = 0$ so $l = 0$. That is, there exists an element $x_1 \in R(\mathfrak{m}_1)$ such that $x_1 + x_2 \in R(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$ and $V_2 \cap (\mathfrak{g}^{x_1+x_2})_{\mathfrak{m}} = 0$. Let $V_1 = V_1(x_1)$ be some complementary subspace to $(\mathfrak{s}^{x_1})_{\mathfrak{m}_1}$ in $\mathfrak{m}_1(x_1)$. By Proposition 6 $\dim V_1 = 2\varepsilon(\mathfrak{s}, \mathfrak{k})$.

Now to prove the theorem it suffices to prove that $(V_1 \oplus V_2) \cap (\mathfrak{g}^{x_1+x_2})_{\mathfrak{m}} = 0$. Let $v_1 + v_2 \in (V_1 \oplus V_2) \cap (\mathfrak{g}^{x_1+x_2})_{\mathfrak{m}}$. Then there exists $z \in \mathfrak{k}$ such that $[x_1 + x_2, v_1 + v_2 + z] = 0$. Since $\mathrm{ad}\, x_2(\mathfrak{m}_2(x_2)) \subset \mathfrak{m}_2$, $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_2$, $[\mathfrak{k}, \mathfrak{m}_i] \subset \mathfrak{m}_i$, we see that $[x_1, v_1 + z] = 0$.Thus $v_1 \in (\mathfrak{s}^{x_1})_{\mathfrak{m}_1} \cap V_1$, i.e. $v_1 = 0$ and, consequently, $v_2 = 0$. □

The following properties of $\mathfrak{m}(x)$ are needed for the proof of Theorem 11.

PROPOSITION 9. *For any* $x \in R(\mathfrak{m})$ *we have* $[\mathfrak{m}(x), \mathfrak{k}^x] = 0$.

Proof. For the proof we will use the approach of [15]. Consider the subbundle E of the trivial bundle $R(\mathfrak{m}) \times \mathfrak{m} \times \mathfrak{k}$ consisting of points (x, y, z) with $y \in \mathfrak{m}(x)$ and $z \in \mathfrak{k}^x$. Since $\dim \mathfrak{m}(x)$ and $\dim \mathfrak{k}^x$ are constants for $x \in R(\mathfrak{m})$, this is indeed a bundle, and for any curve $x_t \subset R(\mathfrak{m})$ and vectors $y_0 \in \mathfrak{m}(x_0)$, $z_0 \in \mathfrak{k}^{x_0}$, we may construct curves y_t, z_t over x_t in E. By the definition $[x_t, \mathfrak{m}(x_t)] \subset \mathfrak{m}$ so that $\Phi(z_t, [x_t, y_t]) = 0$. Differentiating the former identity at zero and using the invariance of the form Φ we obtain

$$\Phi(\dot{z}_0, [x_0, y_0]) - \Phi(\dot{y}_0, [x_0, z_0]) + \Phi(\dot{x}_0, [y_0, z_0]) = 0. \tag{12}$$

Since $\mathfrak{k} \perp \mathfrak{m}$ and $[x, \mathfrak{k}^x] = 0$, the first two terms on the left side of (12) vanish, i.e. $\Phi(y, [y_0, z_0]) = 0$ for all $y \in \mathfrak{m}$. Using the non-singularity of Φ on $\mathfrak{m}$ and the relation $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ we prove the proposition. □

Proceeding in a manner similar to the proof of Proposition 9 with the subbundle E of the trivial bundle $R(\mathfrak{m}) \times \mathfrak{g} \times \mathfrak{g}$ consisting of points (x, y, z) with $y, z \in \mathfrak{g}^x$ (differentiating the identity $\Phi(x_t, [y_t, z_t]) = 0$) we obtain

PROPOSITION 10 ([5,7]). *For any* $x \in R(\mathfrak{m})$ *we have* $[\mathfrak{g}^x, \mathfrak{g}^x] \subset \mathfrak{k}$.

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Note. It is useful to observe that the above proof of Proposition 10 (1) depends essentially on the fact that $\mathfrak{m} = \mathfrak{k}^{\perp}$ in $\mathfrak{g}$ (2) does not use the direct sum decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$, i.e. the nonsingularity of Φ on $\mathfrak{k}$.

THEOREM 11. *Assume that* $y \in R(\mathfrak{m})$ *and* $\mathfrak{a}$ *is a reductive (in* $\mathfrak{g}$*) subalgebra of* $\mathfrak{k}^y$. *Let* $\hat{\mathfrak{g}}$ *and* $\hat{\mathfrak{k}}$ *be the centalizers of* $\mathfrak{a}$ *in* $\mathfrak{g}$ *and* $\mathfrak{k}$ *respectively. Then* $\varepsilon(\mathfrak{g}, \mathfrak{k}) = \varepsilon(\hat{\mathfrak{g}}, \hat{\mathfrak{k}})$.

Proof. Algebras $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{k}}$ are subalgebras reductive in $\mathfrak{g}$ and the restrictions of Φ to $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{k}}$ respectively are nondegenerate [13, Chapter VII, Section 1]. Let $\hat{\mathfrak{m}}$ be the orthogonal complement of $\hat{\mathfrak{k}}$ in $\hat{\mathfrak{g}}$ relative to Φ: $\hat{\mathfrak{g}} = \hat{\mathfrak{k}} \oplus \hat{\mathfrak{m}}$. Since $\mathfrak{k}$ and $\mathfrak{m}$ are stable under $\mathrm{ad}(\mathfrak{a})$ we have

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}} \cap \mathfrak{k} \oplus \hat{\mathfrak{g}} \cap \mathfrak{m} \qquad \text{and} \qquad \hat{\mathfrak{m}} = \hat{\mathfrak{g}} \cap \mathfrak{m}. \tag{13}$$

Let x be any element of the set $R(\mathfrak{m}) \cap \hat{\mathfrak{m}}$ containing y. Clearly this set is a Zariski open subset of $\hat{\mathfrak{m}}$ so that without loss we may assume $x \in R(\hat{\mathfrak{m}}) \cap R(\mathfrak{m})$. By Proposition 9 $[\mathfrak{m}(x), \mathfrak{k}^x] = 0$. But clearly $\mathfrak{a} \subset \mathfrak{k}^x$ and, consequently, $\mathfrak{m}(x) \subset \hat{\mathfrak{m}}$. Therefore by definition (10) $\hat{\mathfrak{m}}(x) = \mathfrak{m}(x)$.

The reductive Lie algebra $\mathfrak{g}^x$ is the Lie algebra direct sum $\mathfrak{g}^x = \mathfrak{z} \oplus [\mathfrak{g}^x, \mathfrak{g}^x]$, where $\mathfrak{z}$ is the center of $\mathfrak{g}^x$. The commutator $[\mathfrak{g}^x, \mathfrak{g}^x] = [\mathfrak{k}^x, \mathfrak{k}^x]$ is the maximal semi-simple ideal in $\mathfrak{g}^x$. Hence, $[\mathfrak{z}, \mathfrak{k}^x] = 0$. Recalling that $\mathfrak{a} \subset \mathfrak{k}^x$ we see $\mathfrak{z} \subset \hat{\mathfrak{g}}$. But, obviously then, $\mathfrak{z} \subset \hat{\mathfrak{g}}^x$. Since $\hat{\mathfrak{g}}^x \subset \mathfrak{g}^x$ and $(\mathfrak{g}^x)_{\mathfrak{m}} = (\mathfrak{z})_{\mathfrak{m}}$ we have $(\hat{\mathfrak{g}}^x)_{\mathfrak{m}} = (\mathfrak{g}^x)_{\mathfrak{m}}$. But $\hat{\mathfrak{m}} \subset \mathfrak{m}$ and $\hat{\mathfrak{k}} \subset \mathfrak{k}$ so that the projection $(\hat{\mathfrak{g}}^x)_{\hat{\mathfrak{m}}}$ of $\hat{\mathfrak{g}}^x \subset \hat{\mathfrak{g}}$ into $\hat{\mathfrak{m}}$ along $\hat{\mathfrak{k}}$ coincides with $(\hat{\mathfrak{g}}^x)_{\mathfrak{m}}$. Thus $(\mathfrak{g}^x)_{\mathfrak{m}} = (\hat{\mathfrak{g}}^x)_{\hat{\mathfrak{m}}}$. Now Theorem 11 follows immediately from Proposition 6. □

2.3. INTEGRABILITY

We continue with notations of the previous subsection 2.2. Let G be a real connected reductive Lie groups with the Lie algebra $\mathfrak{g}$. By K we denote its reductive subgroup with the Lie algebra $\mathfrak{k} \subset \mathfrak{g}$. Suppose also that the subgroup $K \subset G$ is closed. Since Φ is nondegenerate on $\mathfrak{g}$, we may use it to identify the spaces $\mathfrak{g}^*$ and $\mathfrak{g}$, $\mathfrak{k}^0 \subset \mathfrak{g}^*$ (the annihilator of $\mathfrak{k}$ in $\mathfrak{g}^*$) and $\mathfrak{m}$. Under this identification the coadjoint action of G on $\mathfrak{g}^*$ goes over to the adjoint action of G on $\mathfrak{g}$, and the annihilator $\mathfrak{g}^\alpha$ of $\alpha \in \mathfrak{k}^0$ coincides with the centralizer $\mathfrak{g}^x$ of the corresponding element $x(\alpha) \in \mathfrak{m}$.

The centralizer of the semi-simple element $x \in R(\mathfrak{m})$ is the reductive Lie algebra $\mathfrak{g}^x = \mathfrak{z}(x) \oplus [\mathfrak{g}^x, \mathfrak{g}^x]$, while $\mathfrak{k}^x = \mathfrak{z}_1(x) \oplus [\mathfrak{g}^x, \mathfrak{g}^x]$ [7, Prop. 1.1 and 1.2], where $\mathfrak{z}_1(x) \subset \mathfrak{z}(x)$. On $\mathfrak{g}$ there are real polynomials $h_1, \ldots, h_p$, where $p = \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{g}^x)$, such that $h_1, \ldots, h_p$ are pairwise in involution on $\mathfrak{g}$ and the vectors $\{[x, \mathrm{grad}\, h_i(x)]\}$, $i = 1, \ldots, p$, are independent [6, Theorem 2.6]; here $\mathrm{grad}\, h_i(x)$ is the vector dual to the differential $\mathrm{d}h_i(x)$ relative to Φ. The gradients of the polynomial invariants of $\mathfrak{g}$ at the point x generate the center $\mathfrak{z}(x)$ of $\mathfrak{g}^x$ [6, Theorem 2.5]. Since $\dim(\mathfrak{g}^x/\mathfrak{k}^x) = \dim(\mathfrak{z}(x)/\mathfrak{z}_1(x))$ and we have identified the spaces $\mathfrak{m}$ and $\mathfrak{k}^0 = T^*_{\pi(e)}(G/K)$, by what was said in subsection 2.1 about $T^*(G/K)$, there exist $s = \dim(\mathfrak{g}^x/\mathfrak{k}^x) + \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{g}^x)$ functions of the form $h \circ J$ in involution and independent at $x \in \mathfrak{m} = \mathfrak{k}^0$.

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If $\varepsilon(\mathfrak{g},\mathfrak{k})=0$ then $s=\frac{1}{2}\dim T^*(G/K)$; i.e., on the manifold $T^*(G/K)$ there exists a collective completely integrable system and hence any G-invariant flow on $T^*(G/K)$ is integrable. If $\varepsilon(\mathfrak{g},\mathfrak{k})=1$ and in addition K is a compact subgroup of G then by Proposition 1 there exists a G-invariant function F on $T^*(G/K)$ which is independent of the functions $\{h_i\circ P\}, i=\overline{1,s}$. The set $\{F,h_1\circ P,\dots,h_s\circ P\}$ is the maximal involutive set of independent functions: $s+1=\frac{1}{2}\dim T^*(G/K)$. If a G-invariant function H is independent of $\{h_i\circ P\}, i=\overline{1,s}$ then the set $\{H,h_1\circ P,\dots,h_s\circ P\}$ is the maximal involutive set, if H is dependent then we have the commutative set of integrals $\{F,h_1\circ P,\dots,h_s\circ P\}$. We proved the following proposition:

PROPOSITION 12. *If K is a reductive closed subgroup of a real reductive connected Lie group G and $\varepsilon(\mathfrak{g},\mathfrak{k})=1$ then on $T^*(G/K)$ there exists an almost collective completely integrable system. If in addition the subgroup $K\subset G$ is compact then any Hamiltonian system on $T^*(G/K)$ with a G-invariant Hamiltonian function H is integrable.*

Remark 13. We say that a pair $(\mathfrak{g},\mathfrak{k})$ of real (or complex) reductive Lie algebras is an *almost spherical pair* and a subalgebra $\mathfrak{k}\subset\mathfrak{g}$ is a *almost spherical subalgebra* if $\varepsilon(\mathfrak{g},\mathfrak{k})=1$. All almost spherical subalgebras $\mathfrak{k}$ of compact simple Lie algebras $\mathfrak{g}$ are classified in [9, 17].

2.4. ACTIONS OF BOREL SUBGROUPS ON HOMOGENEOUS SPACES OF REDUCTIVE ALGEBRAIC LIE GROUPS

Let G be a connected reductive complex algebraic Lie group with reductive closed (in the Zariski topology) subgroup K, and let $\mathfrak{g}$ and $\mathfrak{k}$ be their Lie algebras. Since the subalgebra $\mathfrak{k}$ is algebraic and reductive in $\mathfrak{g}$, there exists a compact real form $\mathfrak{k}_0$ of the complex reductive Lie algebra $\mathfrak{k}$. The algebra $\mathfrak{k}_0$ is a compactly embedded subalgebra of $\mathfrak{g}$ and, consequently, $\mathfrak{k}_0$ is contained in some compact real form $\mathfrak{g}_0$ of $\mathfrak{g}$ (see [14, Chapter VI, Section 2]). Moreover, there exists a faithful representation ρ of $\mathfrak{g}$ such that its associated bilinear form Φ is nondegenerate on $\mathfrak{g}$ and negatively definite on $\mathfrak{g}_0$ (we can take as ρ the natural representation of the algebraic Lie algebra $\mathfrak{g}\subset\mathrm{End}(\mathbb{C}^n)$ or adjoint representation ad if $\mathfrak{g}$ is semi-simple). Then $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{k}$ and $\mathfrak{g}_0=\mathfrak{m}_0\oplus\mathfrak{k}_0$, where $\mathfrak{m}=\mathfrak{k}^\perp$ (in $\mathfrak{g}$) and $\mathfrak{m}_0=\mathfrak{m}\cap\mathfrak{g}_0$.

Let $x\in R(\mathfrak{m}_0)$. Then $\mathfrak{m}=\mathfrak{m}(x)\oplus\mathrm{ad}\,x(\mathfrak{k})$ and by Proposition 9 $[\mathfrak{m}(x),\mathfrak{k}^x]=0$. Moreover, using the dimensional arguments we have $\mathfrak{k}^y=\mathfrak{k}^x$ for any $y\in\mathfrak{m}(x)\cap R(\mathfrak{m})$. Therefore the set $\mathrm{Ad}(K)(\mathfrak{m}(x)\cap R(\mathfrak{m}))$ contains a Zariski open subset $R'(\mathfrak{m})\subset R(\mathfrak{m})$ of $\mathfrak{m}$ and if $z\in R'(\mathfrak{m})$ the algebra $\mathfrak{k}^z$ is conjugate to $\mathfrak{k}^x$. We proved

PROPOSITION 14. *There exists a Zariski open subset $R'(\mathfrak{m})\subset R(\mathfrak{m})$ of $\mathfrak{m}$ such that for any $y,z\in R'(\mathfrak{m})$ the algebras $\mathfrak{k}^y$ and $\mathfrak{k}^z$ are conjugate to each other.*

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Consider the action of some Borel subgroup of the Lie group G on the affine algebraic variety G/K. Let $\boldsymbol{\delta}(G, K)$ be the codimension in G/K of maximal dimension orbits of this Borel subgroup. That is, for 'almost all' Borel subalgebras $\mathfrak{b}'$ of $\mathfrak{g}$ the codimension of the subspace $\mathfrak{b}' + \mathfrak{k}$ in $\mathfrak{g}$ is equal to $\boldsymbol{\delta}(G, K)$. Below we shall prove Theorem 17 asserting that $\boldsymbol{\delta}(G, K) = \boldsymbol{\varepsilon}(\mathfrak{g}, \mathfrak{k})$. Taking into account that $\dim \mathfrak{b}' = \frac{1}{2}(\operatorname{rank} \mathfrak{g} + \dim \mathfrak{g})$, we can rewrite (8) in the form

$$\dim \mathfrak{g} - \boldsymbol{\varepsilon} = \dim \mathfrak{b}' + \dim \mathfrak{k} - \mathbf{b}(\mathfrak{m}). \tag{14}$$

That is, to prove that $\boldsymbol{\delta} = \boldsymbol{\varepsilon}$ it suffices to show that $\dim(\mathfrak{b}' \cap \mathfrak{k}) = \mathbf{b}(\mathfrak{m})$ for 'almost all' Borel subalgebras $\mathfrak{b}'$. The major point in the proof of Theorem 17 is the observation contained in the following lemma.

LEMMA 15. *For any $x \in \mathfrak{m} \cap R(\mathfrak{m}_0)$ there exists a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing x such that (1) $\mathfrak{b} \cap \mathfrak{k} = \mathfrak{b} \cap \mathfrak{k}^x$; (2) the algebra $\mathfrak{b} \cap \mathfrak{k}^x$ is a Borel subalgebra of the reductive Lie algebra $\mathfrak{k}^x$ (of dimension $\mathbf{b}(\mathfrak{m})$); (3) the codimension of the subspace $\mathfrak{b} + \mathfrak{k}$ in $\mathfrak{g}$ is equal to $\boldsymbol{\varepsilon}(\mathfrak{g}, \mathfrak{k})$; (4) $\mathfrak{b} + \mathfrak{k} + \mathfrak{m}(x) = \mathfrak{g}$.*

Proof. Let τ denote conjugation (an involution) of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$: $\mathfrak{g}_0 = (1 + \tau)\mathfrak{g}$, $\mathfrak{m} = \mathfrak{m}_0 + i\mathfrak{m}_0$. Fix an element $x \in R(\mathfrak{m}_0) \subset R(\mathfrak{m})$ and a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ containing x and invariant with respect to τ. Let R be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Choose eigenvectors $E_\alpha \in \mathfrak{g}$ corresponding to the roots $\alpha \in R$ such that $\tau(E_\alpha) = E_{-\alpha}$ and $\Phi(E_\alpha, E_{-\alpha}) = -1$. Let $R_0 = \{\alpha \in R : \alpha(x) = 0\}$. Since $[\mathfrak{g}^x, \mathfrak{g}^x] \subset \mathfrak{k}^x$, we see that R_0 is the root system of the semi-simple ideal $[\mathfrak{g}^x, \mathfrak{g}^x]$ of the centralizers $\mathfrak{k}^x$ and $\mathfrak{g}^x$, i.e. $E_\alpha \in \mathfrak{k}^x (\in \mathfrak{g}^x)$ iff $\alpha \in R_0$ and $\mathfrak{g}^x = \mathfrak{h} \oplus \sum_{\alpha \in R_0} \mathbb{C}E_\alpha$. Moreover, the centers of algebras $\mathfrak{k}^x$ and $\mathfrak{g}^x$ are subalgebras of the Cartan subalgebra $\mathfrak{h}$. Since $\tau(x) = x$, there exists the basis (the system of simple roots) B for the root system R such that $\beta(ix) \geqslant 0$ for all $\beta \in B$. Let $\mathfrak{b}$ be the Borel subalgebra defined by the set of positive roots R^+ in R relative to B.

Assume that $z \in \mathfrak{b} \cap \mathfrak{k}$. Since $\mathfrak{k} \perp \mathfrak{m}$ and $\tau(\mathfrak{k}) = \mathfrak{k}$, by invariance of Φ we have $\Phi(ix, [z, \tau(z)]) = \Phi([ix, z], \tau(z)) = 0$. If $z = \sum_{\alpha \in R^+} c_\alpha E_\alpha + h$ with $h \in \mathfrak{h}$, then from the definition of the vectors $\{E_\alpha\}$ we obtain $\sum_{\alpha \in R^+} c_\alpha \bar{c}_\alpha \alpha(ix) = 0$. All terms in this sum are non-negative, and so $c_\alpha \neq 0$ only in the case when $\alpha(ix) = 0$, i.e. $z \in \mathfrak{k}^x$. Hence $\mathfrak{b} \cap \mathfrak{k} = \mathfrak{b} \cap \mathfrak{k}^x$. By the definition of the basis B the set $B \cap R_0$ is the basis of the root system R_0 so that $R^+ \cap R_0$ is the set of positive roots of R_0 and $\mathfrak{b} \cap \mathfrak{k}^x$ is a Borel subalgebra of $\mathfrak{k}^x$. Moreover, since $\operatorname{ad} x(E_\alpha) = \alpha(x)E_\alpha$, we have $\mathfrak{b} \cap \operatorname{ad} x(\mathfrak{k}) = 0$.

By Proposition 2 the dimension of a Borel subalgebra of $\mathfrak{k}^x$ is equal to $\mathbf{b}(\mathfrak{m})$, i.e. $\dim(\mathfrak{b} \cap \mathfrak{k}) = \mathbf{b}(\mathfrak{m})$ and assertions (1)–(3) are proved.

Now our goal is to show that $\mathfrak{g} = \mathfrak{b} + \mathfrak{k} + \mathfrak{m}(x)$. Let V be the orthogonal complement to $\mathfrak{b} + \mathfrak{k} + \mathfrak{m}(x)$ in $\mathfrak{g}$ (relative to Φ). It is well known that $\mathfrak{b}^\perp$ is the maximal nilpotent subalgebra $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] = \sum_{\alpha \in R^+} \mathbb{C}E_\alpha$ of $\mathfrak{b}$. On the other hand, by (11) $(\mathfrak{k} + \mathfrak{m}(x))^\perp = \operatorname{ad} x(\mathfrak{k})$. Thus $V \subset \operatorname{ad} x(\mathfrak{k}) \cap \mathfrak{n}$. But then recalling that $\mathfrak{b} \cap \operatorname{ad} x(\mathfrak{k}) = 0$ and $\mathfrak{n} \subset \mathfrak{b}$ we obtain $V = 0$. □

We note the following corollary of the proof of Lemma 15

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LEMMA 16. *Let* $x \in R(\mathfrak{m})$ *and let* $\mathfrak{b}'$ *be any Borel subalgebra of* $\mathfrak{g}$ *containing* x. *Then* $\mathfrak{b}' \cap \mathfrak{k}^x$ *is Borel subalgebra of the reductive Lie algebra* $\mathfrak{k}^x$ *and* $\dim(\mathfrak{b}' \cap \mathfrak{k}) \geqslant \mathbf{b}(\mathfrak{m})$.

Proof. To prove this lemma we will need some results about a Cartan subalgebras of arbitrary Lie algebras. Our reference for all definitions will be [13] where for us the fixed algebraically closed field is of course $\mathbb{C}$. Let $\mathfrak{b}'^x$ be the centralizer of x in $\mathfrak{b}'$ and let $\mathfrak{h}$ be some Cartan subalgebra of the algebra $\mathfrak{b}'^x$. Since $\mathfrak{h}$ is its normalizer in $\mathfrak{b}'^x$, we have $x \in \mathfrak{h}$. But x is a semi-simple element of $\mathfrak{g}$ so that the endomorphisms $\mathrm{ad}_{\mathfrak{g}}\, x$ and $\mathrm{ad}_{\mathfrak{b}'}\, x$ are semi-simple. Then by Proposition 10 in [13, Chapter VII, Section 2] $\mathfrak{h}$ is a Cartan subalgebra of the Borel subalgebra $\mathfrak{b}'$. Since $\mathfrak{h}$ is a Cartan subalgebra of the reductive Lie algebra $\mathfrak{g}$ [13, Chapter VIII, Section 3, Corol. to Prop. 9], we have the root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathbb{C}E_\alpha$, where R is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Moreover, because of the inclusion $\mathfrak{h} \subset \mathfrak{b}'$ there is the system $R^+ \subset R$ of positive roots of R such that $\mathfrak{b}' = \mathfrak{h} \oplus \sum_{\alpha \in R^+} \mathbb{C}E_\alpha$ [13, Chapter VIII, Section 3, Prop. 11]. Hence, $R_0 = \{\alpha \in R\colon \alpha(x) = 0\}$ is the root system of the semi-simple ideal $[\mathfrak{g}^x, \mathfrak{g}^x] = [\mathfrak{k}^x, \mathfrak{k}^x]$. Since R^+ is determined by the lexicographic ordering induced by some basis of $\mathfrak{h}$, it follows that $R_0^+ = R_0 \cap R^+$ is the system of positive roots of the root system R_0. Thus $\mathfrak{b}' \cap \mathfrak{g}^x$ and $\mathfrak{b}' \cap \mathfrak{k}^x$ are the Borel subalgebras of the reductive Lie algebras $\mathfrak{g}^x$ and $\mathfrak{k}^x$ respectively. By Proposition 2 $\dim(\mathfrak{b}' \cap \mathfrak{k}^x) = \mathbf{b}(\mathfrak{m})$. □

THEOREM 17. *All orbits of maximal dimension of some Borel subgroup on* G/K *have the codimension* $\varepsilon(\mathfrak{g}, \mathfrak{k})$ *in* G/K.

Proof. Since G is a connected algebraic reductive group, it follows that G has the structure of an affine variety and the adjoint representation of G is a polynomial mapping (morphism). For any $x \in \mathfrak{m}$ define the Zariski closed subset $N(x) = \{g \in G\colon \mathrm{Ad}\, g(x) \in \mathfrak{m}\}$ of G. Its irreducible component containing the identity element e of G is denoted by $N_e(x)$. It is evident that $K \subset N(x)$. Let $N_e'(x)$ be the Zariski open (non-empty) subset of $N_e(x)$ of simple points; i.e., $N_e'(x)$ is a smooth complex manifold. Let $Q \subset R(\mathfrak{m})$ be the subset of all $y \in R(\mathfrak{m})$ such that e is a simple point of the variety $N_e(y)$. We wish to prove now that Q is a Zariski open dense subset of $R(\mathfrak{m})$ and, consequently, if $Q \neq \emptyset$, the intersection of Q with the real form $\mathfrak{m}_0$ is not empty. Indeed, because of the evident relation $N(\mathrm{Ad}\, a(x)) = N(x)a^{-1}$, where $a \in N(x)$, the identity element $e \in G$ is a simple point of all varieties $N_e(z), z = \mathrm{Ad}\, g(x), g \in N_e'(x)$. This, density $N_e'(x)$ in $N_e(x)$ and openness of $R(\mathfrak{m}) \subset \mathfrak{m}$ imply $Q \neq \emptyset$. But the tangent space $T_e(N_e(z))$ coincides with $\mathfrak{m}(z) \oplus \mathfrak{k}$ and if $z \in R(\mathfrak{m})$ has minimal possible constant dimension N. Hence the complement to Q in $R(\mathfrak{m})$ is the set of common zeros of some polynomials in $\mathfrak{m}$ (all $n \times n$ minors of some Jacobi matrix at point e depending polynomially on a parameter $x \in R(\mathfrak{m})$, where $n = \dim G - N$), i.e. Q is a Zariski open subset of $\mathfrak{m}$.

Now let $x \in R(\mathfrak{m}_0) \cap Q$. It is evident that $\tilde{N}_e(x) = \{g \in N_e(x)\colon \mathrm{Ad}\, g(x) \in R(\mathfrak{m})\}$ is the Zariski open subset of $N_e(x)$ containing e. Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$ as in Lemma 15. By B denote the Borel subgroup of G corresponding to $\mathfrak{b} \subset \mathfrak{g}$ and consider the morphism $\Theta\colon \tilde{N}_e(x) \times B \to G, (n, b) \mapsto nb$. Since $\mathrm{Im}\colon \mathrm{d}\Theta(e, e) =$

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$\mathfrak{m}(x)+\mathfrak{k}+\mathfrak{b}=\mathfrak{g}$, the set $\tilde{N}_e(x)B \subset G$ contains a Zariski open subset O_G of G. Moreover, for any $g = nb \in O_G$ we have $\mathrm{Ad}\, n(x) \in R(\mathfrak{m}) \cap \mathrm{Ad}\, g(\mathfrak{b})$. By Lemma 16 $\dim(\mathfrak{k} \cap \mathrm{Ad}\, g(\mathfrak{b})) \geqslant \mathbf{b}(\mathfrak{m})$. But $\dim(\mathfrak{k} \cap \mathfrak{b}) = \mathbf{b}(\mathfrak{m})$ so that for 'almost all' Borel subalgebras $\mathfrak{b}'$ of $\mathfrak{g}$ the dimension of the subspace $\mathfrak{b}' \cap \mathfrak{k}$ is equal to $\mathbf{b}(\mathfrak{m})$, i.e. $\boldsymbol{\delta}(G, K) = \boldsymbol{\varepsilon}(\mathfrak{g}, \mathfrak{k})$. □

COROLLARY 17.1. *Let* $\mathfrak{b}$ *be a Borel subalgebra of* $\mathfrak{g}$. *There is a Zariski open subset* $O \subset G$ *such that for any* $g \in O$ *we have: (1)* $\mathrm{Ad}\, g(\mathfrak{b}) \cap R(\mathfrak{m}) \neq \emptyset$; *(2)* $\mathrm{Ad}\, g(\mathfrak{b}) \cap \mathfrak{k} = \mathrm{Ad}\, g(\mathfrak{b}) \cap \mathfrak{k}^x$ *for every* $x \in \mathfrak{b} \cap R(\mathfrak{m})$ *and* $\mathrm{Ad}\, g(\mathfrak{b}) \cap \mathfrak{k}^x$ *is a Borel subalgebra of* $\mathfrak{k}^x$.

Assume that S is a connected reductive closed (in the Zariski topology) subgroup of G containing K. Denote by $\mathfrak{s}$ its Lie algebra and by $\mathfrak{m}_2$ the orthogonal complement to $\mathfrak{s}$ in $\mathfrak{g}$ with respect to the form Φ; i.e., $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{m}_2$. Let B be some Borel subgroup of G with a Lie algebra $\mathfrak{b} \subset \mathfrak{g}$ such that $B\{K\}$ and $B\{S\}$ are its maximal dimension orbits in G/K and G/S respectively. It is evident that $\boldsymbol{\delta}(G, S) \leqslant \boldsymbol{\delta}(G, K)$. Assume $\boldsymbol{\delta}(G, S) = \boldsymbol{\delta}(G, K)$. That is the codimensions of $\mathfrak{b}+\mathfrak{k}$ and $\mathfrak{b}+\mathfrak{s}$ in $\mathfrak{g}$ are the same. But $\mathfrak{k} \subset \mathfrak{s}$ so that $\mathfrak{b}+\mathfrak{s} = \mathfrak{b}+\mathfrak{k}$ and, consequently, $\mathfrak{s} = (\mathfrak{b} \cap \mathfrak{s}) + \mathfrak{k}$. By virtue of Corollary 17.1 we may assume that $\mathfrak{b} \cap \mathfrak{s} = \mathfrak{b} \cap \mathfrak{s}^{x_2}$ for some $x_2 \in R(\mathfrak{m}_2)$. Then $\mathfrak{s} = \mathfrak{s}^{x_2} + \mathfrak{k}$.

COROLLARY 17.2. *Let* $\mathfrak{k} \subset \mathfrak{s}$ *be two reductive algebraic Lie subalgebras of a reductive complex Lie algebra* $\mathfrak{g}$. *Assume* $\boldsymbol{\varepsilon}(\mathfrak{g}, \mathfrak{k}) = \boldsymbol{\varepsilon}(\mathfrak{g}, \mathfrak{s})$. *There exists a Zariski open subset* $Q_2 \subset R(\mathfrak{m}_2)$ *such that for any* $x_2 \in Q_2$ *we have* $\mathfrak{s} = \mathfrak{s}^{x_2} + \mathfrak{k}$, *where* $\mathfrak{s}^{x_2} = \mathfrak{g}^{x_2} \cap \mathfrak{s}$.

Note. Corollary above may be regarded as a generalization of one assertion of Theorem 2.5 in [7], where $\boldsymbol{\varepsilon}(\mathfrak{g}, \mathfrak{k}) = \boldsymbol{\varepsilon}(\mathfrak{g}, \mathfrak{s}) = 0$.

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