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ORTHOMODULAR LATTICES WHICH CAN BE COVERED BY FINITELY MANY BLOCKS

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In our paper [**3**] we considered four finiteness conditions for an orthomodular lattice (abbreviated: OML) L and conjectured their equivalence. The only question left open in that paper was whether an OML L which can be covered by finitely many blocks (maximal Boolean subalgebras) has only finitely many blocks. In this paper we give an affirmative answer to this question, in fact, we prove the slightly stronger result:

THEOREM. *For every natural number $n \geqq 1$ there exists a natural number m such that every* OML *L which can be covered by n blocks contains at most m blocks.*

One of our main tools is a result proved recently by A. E. Brouwer [**1**]:

THEOREM. *If $V, U_1, U_2, \ldots, U_k$ $(k \geqq 1)$ are subspaces of a vector space X, $a_1, a_2, \ldots, a_k \in X$, $V \subseteq \bigcup_{i=1}^{k} (a_i + U_i)$, $V \subseteq \bigcup_{i \neq j} (a_i + U_j)$ $(j = 1, 2, \ldots, k)$, $W = V \wedge \bigcap_{i=1}^{k} U_i$ and if W is finite dimensional, then* $\dim V + 1 \leqq k + \dim W$.

(Brouwer states his theorem only for vector spaces over $GF(2)$, but his proof works with only minor modifications for vector spaces over an arbitrary field.) We need here the following consequence of Brouwer's result. It was conjectured in [**2**].

COROLLARY. *If a Boolean algebra B is the irredundant (set $-$) union of k subalgebras $B_1, B_2, \ldots, B_k$ and if $|\bigcap_{i=1}^{k} B_i| = 2^m$ then $|B| \leqq 2^{m+k-1}$.*

Our terminology and notation will be the same as in [**3**].

1. Some preliminary results.

LEMMA 1. *If A and B are finite blocks of an* OML *L and if there exists an isomorphism $\varphi : A \to B$ such that $x \,\mathrm{C}\, \varphi(x)$ holds for every $x \in A$ then $A = B$.*

Proof. Let $a_1, a_2, \ldots, a_n$ be the atoms of A and $b_i = \varphi(a_i)$. Suppose $A \nsubseteq B$. Then there exists an atom a_i of A which does not belong to B. Since $a_i \notin B$ there exist atoms b_j, b_k of B, $j \neq k$ such that a_i does not

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commute with b_j and b_k. Suppose

$$(a_i \vee a_j) \wedge (b_i \vee b_j) = 0.$$

Since $a_i \vee a_j \mathrm{C}\, b_i \vee b_j$ it would follow that

$$a_i \leqq a_i \vee a_j \leqq (b_i \vee b_j)' \leqq b_j',$$

contrary to the assumption that a_i and b_j do not commute. We thus have

$$0 < (a_i \vee a_j) \wedge (b_i \vee b_j).$$

But since $a_i \mathrm{C} a_j$, $a_i \mathrm{C} b_i$, $a_j \mathrm{C} b_j$ and $b_i \mathrm{C} b_j$ it follows from [**4**] that

$$(a_i \vee a_j) \wedge (b_i \vee b_j)$$
$$= (a_i \wedge b_i) \vee (a_i \wedge b_j) \vee (a_j \wedge b_i) \vee (a_j \wedge b_j).$$

Note that a_i, b_i, a_j and b_j are also atoms of L. Since $a_i \notin B$ we thus obtain $a_i \wedge b_i = a_i \wedge b_j = 0$. But $a_j \wedge b_j \neq 0$ would imply $b_j = a_j \mathrm{C}\, a_i$, contrary to the assumption $a_i \not\mathrm{C}\, b_j$. We thus obtain $a_j = b_i$. By symmetry we also obtain $a_k = b_i$ and hence $a_j = a_k$, a contradiction. We thus obtain $A \subseteq B$, from which the claim follows by symmetry.

For natural numbers k, $l \geqq 1$ define $\beta(k, l) = (2^k)^{(2^l)}$.

LEMMA 2. *Let $B_1, B_2, \ldots, B_k$ be blocks of an* OML *and let $l \geqq 1$ be a natural number. Then there exist at most $\beta(k, l)$ blocks A of L satisfying*

$$|A| = 2^l \quad \textit{and} \quad A \subseteq \bigcup_{i=1}^{k} B_i.$$

Proof. Let $A_0, A_1, \ldots, A_{\beta(k,l)}$ be blocks satisfying the assumption made for A. We have to show that there exist indices p, q with $p \neq q$ such that $A_p = A_q$. For every i, $0 \leqq i \leqq \beta(k, l)$, let $\varphi_i : A_0 \to A_i$ be an isomorphism and let α_i be a map of A_0 into the power set of $\{1, 2, \ldots, k\}$ defined by

$$\alpha_i(x) = \{j | \varphi_i(x) \in B_j\}.$$

Since there are at most $\beta(k, l)$ such maps α_i there exist indices p, q with $p \neq q$, such that $\alpha_p = \alpha_q$. Then $\varphi = \varphi_q \circ \varphi_p^{-1}$ is an isomorphism of A_p onto A_q. But for every $x \in A_p$ we have

$$\{j | x \in B_j\} = \{j | \varphi_p(\varphi_p^{-1}(x)) \in B_j\} = \alpha_p(\varphi_p^{-1}(x))$$
$$= \alpha_q(\varphi_p^{-1}(x)) = \{j | \varphi(x) \in B_j\}.$$

Since $\{j | x \in B_j\} \neq \emptyset$ for every $x \in A_p$ it follows from this that $x \mathrm{C}\, \varphi(x)$ holds for all $x \in A_p$ and hence, by Lemma 1, that $A_p = A_q$, proving Lemma 2.

LEMMA 3. *If L_1 and L_2 are* OML*s which are not Boolean and if $L_1 \times L_2$ can be covered by n blocks then L_1 and L_2 can be covered by fewer than n blocks.*

Proof. Let $B_1, B_2, \ldots, B_n$ be blocks covering $L_1 \times L_2$. Then $B_i = C_i \times D_i$ where the C_i are blocks of L_1 and the D_i are blocks of L_2. Clearly

$$L_1 = \bigcup_{i=1}^{n} C_i.$$

Suppose now that (say) L_1 can not be covered by fewer than n blocks. Then there would exist an element $c \in C_1 - \bigcup_{i=2}^{n} C_i$. It would follow that $\{c\} \times L_2 \subseteq B_1$, hence that $L_2 \subseteq D_1$, contrary to our assumption that L_2 is not Boolean, proving Lemma 3.

2. Proof of the theorem. Let α be a map of the set of all natural numbers $n \geqq 1$ into itself satisfying $\alpha(1) = 1$ and (K running over all non-empty subsets of $\{1, 2, \ldots, n\}$)

$$(*) \quad \sum_K \left((\alpha(n-1))^2 + \sum_{l=1}^{|K|} \beta(|K|, l)\right) \leqq \alpha(n) \quad (n \geqq 2).$$

It is easy to see that such a function α exists. We now prove the theorem with $m = \alpha(n)$ by induction on n. If $n = 1$ the OML L is Boolean and the theorem is trivially true. Assume now that $n \geqq 2$ and that $\{B_1, B_2, \ldots, B_n\}$ is a covering of an OML L by n blocks. We say that a set X is *irredundantly* covered by a family $(X_i)_{i \in I}$ of sets if and only if

$$X \subseteq \bigcup_{i \in I} X_i \quad \text{and} \quad X \not\subseteq \bigcup_{j \in I - \{i\}} X_j \quad (\text{all } i \in I).$$

For every non-empty subset K of $\{1, 2, \ldots, n\}$ let $\mathfrak{A}_K$ be the set of all blocks A of L which are irredundantly covered by $(B_i)_{i \in K}$. Clearly $\bigcup_K \mathfrak{A}_K = \mathfrak{A}(L)$ (= the set of all blocks of L).

Let K be an arbitrary non-empty subset of $\{1, 2, \ldots, n\}$. Define

$$M = \bigcap_{i \in K} B_i \quad \text{and} \quad L_1 = C(M).$$

Then the blocks of L_1 are exactly those blocks of L which contain M as a subset, in particular, the blocks B_i with $i \in K$ are also blocks of L_1. Since $A \in \mathfrak{A}_K$ implies $A \subseteq \bigcup_{i \in K} B_i \subseteq C(M)$ and hence $A \in \mathfrak{A}(L_1)$, it follows that $\mathfrak{A}_K$ is also the set of all blocks of L_1 which are irredundantly covered by $(B_i)_{i \in K}$. Since

$$M = \bigcap_{i \in K} B_i \supseteq C(L_1) = \bigcap \mathfrak{A}(L_1) \supseteq M,$$

the center $C(L_1)$ of L_1 equals M. By Proposition 3.2 of [**3**], L_1 is isomorphic with a direct product $B \times L_2$, where B is a Boolean algebra and L_2 has no non-trivial Boolean factor. Under this isomorphism the blocks of L_1 are in a one to one correspondence with the products $B \times C$, $C \in \mathfrak{A}(L_2)$. In particular, the blocks B_i $(i \in K)$ correspond to blocks $B \times C_i$ with $C_i \in \mathfrak{A}(L_2)$ and a block C of L_2 is irredundantly covered by the family $(C_i)_{i\in K}$ if and only if the block of L_1 corresponding to $B \times C$ is irredundantly covered by $(B_i)_{i\in K}$. Thus the number of blocks in $\mathfrak{A}_K$ is the same as the number of blocks of L_2 which are irredundantly covered by $(C_i)_{i\in K}$. Furthermore, the center $C(L_2)$ of L_2 is $\bigcap_{i\in K} C_i$.

If L_2 is irreducible this implies that $\bigcap_{i\in K} C_i = \{0, 1\}$. Thus, if C is a block of L_2 which is irredundantly covered by $(C_i)_{i\in K}$ we have

$$C = \bigcup_{i\in K} (C \cap C_i) \quad \text{(irredundantly) and} \quad \bigcap_{i\in K} (C \cap C_i) = \{0, 1\}.$$

It follows from the corollary of Brouwer's theorem that C has at most $2^{|K|}$ elements. Using Lemma 2 we thus obtain

$$|\mathfrak{A}_K| \leqq \sum_{l=1}^{|K|} \beta(|K|, l).$$

But if L_2 is reducible, $L_2 \simeq L_3 \times L_4$, where L_3 and L_4 are not Boolean. By Lemma 3 they can be covered by fewer than n blocks and, by inductive hypothesis, have at most $\alpha(n - 1)$ blocks. It follows that $\mathfrak{A}_K$ has at most $(\alpha(n - 1))^2$ blocks. In any case, $\mathfrak{A}_K$ has at most $(\alpha(n - 1))^2 + \sum_{l=1}^{|K|} \beta(|K|, l)$ blocks, from which the claim follows by (*).

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