J. Austral. Math. Soc. (*Series A*) **61** (1996), 369–376

BASES FOR INDUCED CHARACTERS

GABRIEL NAVARRO

(Received 30 May 1995; revised 1 May 1996)

Communicated by R. Howlett

Abstract

If G is a finite solvable group, we show that Isaacs' theory on partial characters on Hall π-subgroups can be developed for the nilpotent injectors of G. Therefore, the irreducible characters of G are partitioned into blocks associated to some nilpotent subgroups of G.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 20C15.

1. Introduction

An interesting question is whether Isaacs' theory of partial characters on Hall subgroups can be generalized to other characteristic conjugacy classes of subgroups of finite groups. The aim of this paper is to do just that for a well-known class of subgroups of solvable groups: the nilpotent injectors. This result implies that a natural partition of the irreducible characters into 'blocks' associated to nilpotent subgroups exists in solvable groups. We hope that upcoming results on this 'theory' can give some new insights on the characters of solvable groups.

This paper grew out of a series of conversations with M. Isaacs and M. Lewis while I visited the University of California and I am very much indebted to them for many useful discussions. I also take this opportunity to thank the University of Valencia for financial support.

2. Good bases

If H is a subgroup of a finite group G and $\mathrm{cf}(H)$ is the space of complex class functions defined on H, we denote by $\mathrm{cf}(H)^G = \{\delta^G \mid \delta \in \mathrm{cf}(H)\} \subseteq \mathrm{cf}(G)$. Notice

Research partially supported by DGICYT

https://doi.org/10.1017/S1446788700000446 Published online by Cambridge University Press

that $\mathrm{cf}(H)^G$ is a subspace of $\mathrm{cf}(G)$ whose dimension is the number of conjugacy classes of G which meet H.

The aim of this section is to point out that the existence of a certain 'good' basis in $\mathrm{cf}(H)^G$ implies the existence of partial characters on $\bigcup_{g\in G} H^g$.

DEFINITION 2.1. We say that a basis $\mathscr{B}$ of $\mathrm{cf}(H)^G$ is *good* if it satisfies the following two conditions:

(I) If $\eta \in \mathscr{B}$ then there exists $\alpha \in \mathrm{Irr}(H)$ such that $\alpha^G = \eta$.
(D) If $\gamma \in \mathrm{Irr}(H)$ then $\gamma^G = \sum_{\eta\in\mathscr{B}} a_\eta \eta$, for uniquely determined non-negative integers a_η.

We show that good bases are necessarily unique.

THEOREM 2.2. *If $\mathscr{B}$ and $\mathscr{C}$ are good bases of* $\mathrm{cf}(H)^G$, *then $\mathscr{B} = \mathscr{C}$.*

PROOF. If $\eta \in \mathscr{B}$, we may write $\eta = \sum_{\gamma\in\mathscr{C}} a_{\eta\gamma}\gamma$ for some non-negative integers $a_{\eta\gamma}$. By the same argument and linear independence, it follows that the matrix $(a_{\eta\gamma})$ has an inverse with non-negative integers entries. It is easy to see that such a matrix is a permutation matrix and this proves the theorem.

Given the existence of good bases, now we derive the existence of partial characters on $G^0 = \bigcup_{g\in G} H^g$.

Let us denote by $\mathrm{cf}(G^0)$ the space of complex functions defined on G^0 which are constant on conjugacy classes. Notice that

$$\dim \mathrm{cf}(H)^G = \dim \mathrm{cf}(G^0).$$

If $\varphi \in \mathrm{cf}(G)$, we denote by φ^0 the restriction of φ to G^0. Clearly, $\varphi^0 \in \mathrm{cf}(G^0)$ and the map $\varphi \to \varphi^0$ is an epimorphism $\mathrm{cf}(G) \to \mathrm{cf}(G^0)$.

Also, if $\varphi, \theta \in \mathrm{cf}(G^0) \cup \mathrm{cf}(G)$, we write

$$[\varphi, \theta]^0 = \frac{1}{|G|} \sum_{x\in G^0} \varphi(x)\overline{\theta(x)}.$$

THEOREM 2.3. *If $\mathscr{B} = \{\eta_1, \dots, \eta_k\}$ is a basis of* $\mathrm{cf}(H)^G$ *then there is a unique basis $\mathscr{I} = \{\varphi_1, \dots, \varphi_k\}$ of* $\mathrm{cf}(G^0)$ *satisfying $[\eta_i, \varphi_j]^0 = \delta_{i,j}$.*

PROOF. Let $K_1, \dots, K_k$ be the conjugacy classes of G which meet H so that $G^0 = \bigcup_{j=1,\dots,k} K_j$ and choose $x_j \in K_j$. Hence, if $\eta, \varphi \in \mathrm{cf}(G^0) \cup \mathrm{cf}(G)$, we may write

$$[\eta, \varphi]^0 = \frac{1}{|G|} \sum_{j=1}^{k} |K_j| \eta(x_j)\overline{\varphi(x_j)}.$$

Now, for each i, we consider the set of k linear equations with unknowns $y_{i1}, \ldots, y_{ik}$

$$(i) \qquad \frac{1}{|G|}\sum_{j=1}^{k} |K_j|\eta_i(x_j)y_{rj} = \delta_{ir}$$

being $(|K_j|\eta_i(x_j)/|G|)$ the matrix of coefficients. Since we have that $\{\eta_1, \ldots, \eta_k\}$ is a basis of $\mathrm{cf}(H)^G$, we know that the matrix $(\eta_i(x_j))$ has an inverse and therefore, that the linear system (i) has a unique solution. This completely determines k functions $\{\varphi_1, \ldots, \varphi_k\}$ of $\mathrm{cf}(G^0)$.

We check that $\{\varphi_1, \ldots, \varphi_k\}$ is a basis of $\mathrm{cf}(G^0)$. It suffices to prove that these functions are linearly independent. But notice that if for some complex numbers a_j, we have that $\sum_{j=1}^k a_j\varphi_j = 0$, then $a_i = [\eta_i, \sum_{j=1}^k a_j\varphi_j] = 0$, as required.

The significance of the basis $\mathscr{I}$ is that whenever there exists a good basis for $\mathrm{cf}(H)^G$, we should view $\mathscr{I}$ as the set of 'irreducible Brauer characters' associated to H.

THEOREM 2.4. *Suppose that $\mathscr{B}$ is a good basis of* $\mathrm{cf}(H)^G$ *and let $\mathscr{I}$ be the basis of* $\mathrm{cf}(G^0)$ *associated to $\mathscr{B}$. If χ is a character of G, then $\chi^0 = \sum_{\varphi\in\mathscr{I}} d_{\chi\varphi}\varphi$ for uniquely determined non-negative integers $d_{\chi\varphi}$.*

PROOF. Since $\mathscr{I}$ is a basis of $\mathrm{cf}(G^0)$, we may write $\chi^0 = \sum_{i=1}^k d_i\varphi_i$ for uniquely determined complex numbers d_i.

By condition (I) of Definition 2.1, we know that any η_j is a character of G, and hence $[\eta_j, \chi]$ is a non-negative integer. Also, $[\eta_j, \chi] = [\eta_j, \chi^0]^0$, because, by condition (I), we also know that η_j is zero outside G^0. Now,

$$[\eta_j, \chi] = [\eta_j, \chi^0]^0 = \left[\eta_j, \textstyle\sum_{i=1}^k d_i\varphi_i\right] = d_j$$

and this proves the theorem.

3. Main results

We are now ready to state our main result.

THEOREM 3.1. *If G is a solvable group and I is a nilpotent injector of G, then there exists a good basis of* $\mathrm{cf}(I)^G$.

For the reader's convenience, let us remember that in any solvable group G, all maximal nilpotent subgroups containing $F = \mathbf{F}(G)$ are G-conjugate. This set of subgroups are the *nilpotent injectors* of G and all of them may be obtained in the following way: for any prime p dividing $|F|$, choose S_p an arbitrary Sylow p-subgroup of $\mathbf{C}_G(F_{p'})$, where $F_{p'}$ is the p-complement of F. Then $[S_p, S_q] = 1$ for different primes p, q and $\prod S_p$ is a nilpotent injector of G ([3]).

The proof of our main theorem is naturally divided into two parts. An inductive argument will lead us to distinguish whether the injector is contained in some proper normal subgroup or not. The following result is quite general and suggests, perhaps, that good bases exist for the Fischer-Gaschutz-Hartley injectors. This seems to be difficult to prove, however.

If H is a subgroup of G and $X \subseteq \mathrm{cf}(H)$, we denote $\{\delta^G \mid \delta \in X\}$ by X^G.

LEMMA 3.2. *Suppose that* $H \subseteq M \lhd G$ *with* $G = M\mathbf{N}_G(H)$. *If* $\mathscr{B}$ *is a good basis of* $\mathrm{cf}(H)^M$ *and* $\mathscr{C}$ *is a complete set of representatives of the orbits of* $\mathbf{N}_G(H)$ *on its action on* $\mathscr{B}$, *then* $\mathscr{C}^G$ *is a good basis of* $\mathrm{cf}(H)^G$.

PROOF. By Theorem 2.2, we know that $\mathscr{B}$ is the unique good basis of $\mathrm{cf}(H)^M$. Hence, by uniqueness, the group $\mathbf{N}_G(H)$ acts on $\mathscr{B}$ by conjugation.

Write $\mathscr{B} = \mathscr{B}_1 \cup \cdots \cup \mathscr{B}_s$, where $\mathscr{B}_i$ is the $\mathbf{N}_G(H)$-orbit of η_i and $\mathscr{C} = \{\eta_1, \ldots, \eta_s\}$. We want to prove that $\{\eta_1^G, \ldots, \eta_s^G\}$ is a good basis of $\mathrm{cf}(H)^G$.

We start by proving that they are linearly independent. Assume that $\sum_{i=1}^s a_i \eta_i^G = 0$ for some complex numbers a_i. Hence, $\sum_{i=1}^s a_i (\eta_i^G)_M = 0$. By the induction formula, we may write $(\eta_i^G)_M = b_i \sum_{\xi\in\mathscr{B}_i} \xi$ for some $b_i > 0$. Then $\sum_{i=1}^s a_i b_i (\sum_{\xi\in\mathscr{B}_i} \xi) = 0$ and thus $a_i = 0$ by linear independence of $\mathscr{B}$.

Notice that in order to complete the proof, it suffices to show that if $\delta \in \mathrm{Irr}(H)$, then $\delta^G = \sum_{\eta\in\mathscr{C}} d_\eta \eta^G$ for some non-negative integers d_η.

Since $\mathscr{B}$ is a good basis, we know that $\delta^M = \sum_{i=1}^s \sum_{\xi\in\mathscr{B}_i} a_{i,\xi}\xi$ for some non-negative integers $a_{i,\xi}$. Now, using that all elements of $\mathscr{B}_i$ are G-conjugate, we have that $\delta^G = \sum_{i=1}^s (\sum_{\xi\in\mathscr{B}_i} a_{i,\xi}) \eta_i^G$ and this finishes the proof of the lemma.

As we can see, arguing by induction, the only case left to prove Theorem 3.1 is when the nilpotent injector is not contained in any proper normal subgroup of G. Fortunately, this condition forces the group to have a very particular structure in which we will be able to use Isaacs π-theory to prove our main result.

We denote by H^G the normal closure of the subgroup H in G.

LEMMA 3.3. *Suppose that G is a solvable group and assume that I is a nilpotent injector of G with $I^G = G$. For any prime p dividing $|I|$ let $N_p = (I_p)^G\mathbf{Z}(G)$, where I_p is the Sylow p-subgroup of I. Then, I_p is a Sylow p-subgroup of N_p, $[N_p, N_q] = 1$ for $p \neq q$ and*

$$G/\mathbf{Z}(G) = \prod_p N_p/\mathbf{Z}(G)$$

is a direct product.

PROOF. Since $I^G = G$, certainly, we have that $\prod (I_p)^G = G$ and thus $\prod N_p = G$. Now we prove that $[N_p, N_q] = 1$ for different primes p and q dividing $|I|$. Notice that it suffices to show that $[(I_p)^G, (I_q)^G] = 1$. Although it is not difficult to give a direct proof of this, we use Mann's results in [3]. If $x, y \in G$, we know that $(I_p)^x$ is a Sylow p-subgroup of $\mathbf{C}_G(F_{p'})$ and $(I_q)^y$ is a Sylow q-subgroup of $\mathbf{C}_G(F_{q'})$. By Mann's construction, we have that $(I_p)^x$ and $(I_q)^y$ are different Sylow subgroups of some injector of G and hence $[(I_p)^x, (I_q)^y] = 1$, for all $x, y \in G$. This easily implies $[(I_p)^G, (I_q)^G] = 1$, as wanted.

Now, we prove that $N_p \cap \prod_{q \neq p} N_q = \mathbf{Z}(G)$. Since $\prod N_p = G$, it suffices to show that any $x \in N_p \cap \prod_{q \neq p} N_q$ centralizes N_r for all primes r dividing $|I|$. Such x certainly centralizes N_q for $q \neq p$, because $x \in N_p$. But also, since $x \in \prod_{q \neq p} N_q$, we also have that x centralizes N_p. This shows that $G/\mathbf{Z}(G) = \prod_p N_p/\mathbf{Z}(G)$ is a direct product.

Finally, we prove that I_p is a Sylow p-subgroup of N_p. First, notice that I_p is a Sylow p-subgroup of $(I_p)^G$, because $[(I_p)^G, (I_q)^G] = 1$ for $p \neq q$ and I is a maximal nilpotent subgroup of G. Since $\mathbf{F}(G)$ is contained in I, it follows that I_p contains the Sylow p-subgroup of $\mathbf{Z}(G)$. This implies that $N_p/(I_p)^G$ is a p'-group and the proof of the lemma is completed.

Following notation in [1], if $\prod_p I_p$ is a direct product of groups and $\alpha_p \in \mathrm{cf}(I_p)$, we will denote by $\prod_p \alpha_p$ the class function defined on $\prod_p I_p$ by $\prod_p \alpha_p(\prod_p x_p) = \prod_p(\alpha_p(x_p))$.

LEMMA 3.4. *Suppose that $G/Z = \prod_p N_p/Z$ is a direct product with $[N_p, N_q] = 1$ for $p \neq q$ and suppose that $I = \prod_p I_p$ is the direct product of its different Sylow subgroups $I_p \subseteq N_p$. Let $\alpha_p \in \mathrm{cf}(I_p)$. Then*

$$\left(\left(\prod\nolimits_p \alpha_p\right)^G\right)_I = \frac{1}{|Z|^{n-1}} \prod\nolimits_p \left(\alpha_p^{N_p}\right)_{I_p}$$

where n is the number of factors appearing in the direct product.

PROOF. Let $x = \prod_p x_p$, where $x_p \in I_p$. By the induction formula

$$\left(\prod\nolimits_p \alpha_p\right)^G (x) = \frac{1}{|I|} \sum\nolimits_{g \in G} \left(\prod\nolimits_p \alpha_p\right)^0 (gxg^{-1})$$

(where we use the notation of [1]).

Since I_p contains all the p-elements of I, we have that $gxg^{-1} \in I$ if and only if $gx_pg^{-1} \in I_p$ for all primes p. Hence,

$$\left(\prod\nolimits_p \alpha_p\right)^0 (gxg^{-1}) = \prod\nolimits_p \alpha_p^0 (gx_pg^{-1}).$$

Therefore,

$$\left(\prod_p \alpha_p\right)^G(x) = \frac{1}{|I|}\sum_{g\in G}\prod_p \alpha_p^0\left(x_p{}^{g^{-1}}\right).$$

Now, since Z is central, the group $\bar{G} = G/Z$ acts on G by conjugation, and we may write the latter sum as

$$\frac{|Z|}{|I|}\sum_{g\in\bar{G}}\prod_p \alpha_p^0\left(x_p{}^{g^{-1}}\right).$$

Now, $\bar{G} = \prod_p \bar{N}_p$ is a direct product, and then, the sum above may be written as

$$\frac{|Z|}{|I|}\sum_{\prod_q n_q\in\prod_q \bar{N}_q}\prod_p \alpha_p^0\left(x_p{}^{\prod_q n_q{}^{-1}}\right) = \frac{|Z|}{|I|}\sum_{\prod_q n_q\in\prod_q \bar{N}_q}\prod_p \alpha_p^0\left(x_p{}^{n_p^{-1}}\right)$$

since $[N_p, N_q] = 1$ for $p \neq q$.

But now, we may interchange sums and products to write the sum as

$$\begin{aligned}\frac{|Z|}{|I|}\prod_p\left(\sum_{n_p\in\bar{N}_p}\alpha_p^0\left(x_p{}^{n_p^{-1}}\right)\right) &= \frac{|Z|}{|I|}\frac{1}{|Z|^n}\prod_p\left(\sum_{n_p\in N_p}\alpha_p^0\left(x_p{}^{n_p^{-1}}\right)\right)\\ &= \frac{1}{|Z|^{n-1}}\prod_p\left(\frac{1}{|I_p|}\sum_{n_p\in N_p}\alpha_p^0\left(x_p{}^{n_p^{-1}}\right)\right)\\ &= \frac{1}{|Z|^{n-1}}\prod_p\left(\alpha_p{}^{N_p}(x_p)\right) = \frac{1}{|Z|^{n-1}}\left(\prod_p\left(\alpha_p^{N_p}\right)_{I_p}\right)(x),\end{aligned}$$

as required.

When the Fitting subgroup of a solvable group is only divisible by a single prime p, then the nilpotent injectors of G are the Sylow p-subgroups of G. As we will see, this case is covered by using Isaacs π-theory. We refer the reader to [2] for notation and properties of partial π-characters.

THEOREM 3.5. *Suppose that G is π-separable and let H be a Hall π-subgroup of G. For each $\varphi \in \mathrm{I}_\pi(G)$ let $\alpha_\varphi \in \mathrm{Irr}(H)$ be a Fong character associated to φ. Then $\{\alpha_\varphi^G \mid \varphi \in \mathrm{I}_\pi(G)\}$ is the good basis of* $\mathrm{cf}(H)^G$.

PROOF. Write $\Phi_\varphi = (\alpha_\varphi)^G$. For $\varphi \in \mathrm{I}_\pi(G)$, let $\chi_\varphi \in \mathrm{Irr}(G)$ such that $\chi_\varphi^0 = \varphi$ (Theorem 2.3 of [2]). Now, if $\theta, \varphi \in \mathrm{I}_\pi(G)$, we have that

$$[\Phi_\varphi, \chi_\theta] = [(\alpha_\varphi)^G, \chi_\theta] = [\alpha_\varphi, (\chi_\theta)_H] = [\alpha_\varphi, \theta_H] = \delta_{\varphi,\theta}$$

by the results of Section 2 in [2]. Observe that this is enough to establish that the characters Φ_φ are linearly independent. Since $|\mathrm{I}_\pi(G)|$ is the dimension of $\mathrm{cf}(H)^G$, we have proved that these characters form a basis which satisfy the property (I).

Now we prove property (D). In fact, we show more: if $\chi \in \mathrm{Char}(G) \cap \mathrm{cf}(H)^G$, we may write $\chi = \sum_{\varphi \in \mathrm{I}_\pi(G)} a_\varphi \Phi_\varphi$ for some complex numbers a_φ. Since $a_\varphi = [\chi, \chi_\varphi]$, this shows that a_φ is a non-negative integer. This proves property (D), as required.

The proof of our main theorem will be completed once we prove the following.

THEOREM 3.6. *With the hypotheses of* 3.3, *for each prime* p *and* $\varphi \in \mathrm{I}_p(N_p)$ *choose a Fong character for* φ *in* I_p *and let* $\mathscr{S}_p$ *be the set of these characters. If* $\mathscr{T} = \{\prod_p \alpha_p \mid \alpha_p \in \mathscr{S}_p\}$ *then* $\mathscr{T}^G$ *is a good basis for* $\mathrm{cf}(I)^G$.

PROOF. Let $\gamma \in \mathrm{Irr}(I)$ and write $\gamma = \prod_p \gamma_p$, where $\gamma_p \in \mathrm{Irr}(I_p)$. Since by Lemma 3.3, I_p is a Sylow p-subgroup of N_p, by applying Theorem 3.5, we may write ${\gamma_p}^{N_p} = \sum_{\alpha_p \in \mathscr{S}_p} a_{\alpha_p} {\alpha_p}^{N_p}$ for some non-negative integers a_{α_p}.

By applying Lemmas 3.3 and 3.4, if $Z = \mathbf{Z}(G)$, we have that

$$
\begin{aligned}
(\gamma^G)_I &= \left(\left(\textstyle\prod_p \gamma_p\right)^G\right)_I = \frac{1}{|Z|^{n-1}} \textstyle\prod_p ({\gamma_p}^{N_p})_{I_p} \\
&= \frac{1}{|Z|^{n-1}} \prod_p \left(\sum_{\alpha_p \in \mathscr{S}_p} a_{\alpha_p} ({\alpha_p}^{N_p})_{I_p} \right) = \frac{1}{|Z|^{n-1}} \sum_{\prod_p \alpha_p \in \mathscr{T}} \left(\textstyle\prod_p a_{\alpha_p}\right) \left(\textstyle\prod_p \left({\alpha_p}^{N_p}\right)_{I_p}\right) \\
&= \sum_{\prod_p \alpha_p \in \mathscr{T}} \left(\textstyle\prod_p a_{\alpha_p}\right) \left(\left(\textstyle\prod_p \alpha_p\right)^G\right)_I .
\end{aligned}
$$

Hence, we conclude that

$$
\gamma^G = \sum_{\prod_p \alpha_p \in \mathscr{T}} \left(\textstyle\prod_p a_{\alpha_p}\right) \left(\textstyle\prod_p \alpha_p\right)^G
$$

since both characters are zero outside G-conjugates of I.

To complete the proof of the theorem, it suffices to show that the dimension of $\mathrm{cf}(I)^G$ equals $|\mathscr{T}^G|$. Since, we have already shown that $\mathscr{T}^G$ spans $\mathrm{cf}(I)^G$ and certainly, $|\mathscr{T}^G| \le |\mathscr{T}|$, it suffices to show that $\dim \mathrm{cf}(I)^G = |\mathscr{T}|$.

First of all, notice that if $x_p, y_p \in N_p$ are p-elements, then $\prod_p x_p$ and $\prod_p y_p$ are G-conjugate if and only if x_p and y_p are conjugate in N_p for each p. This follows from the facts that $G = \prod_p N_p$ is a central product together with the uniqueness of the decomposition of an element in commuting p-p' parts.

Now, let $\mathscr{X}_p$ be a complete set of representatives of the N_p-conjugacy classes of p-elements in N_p so that $|\mathscr{X}_p| = |\mathscr{S}_p|$. Since I_p is a Sylow p-subgroup of N_p we may take $\mathscr{X}_p \subseteq I_p$. Hence $\{\mathrm{cl}_G(\prod_p x_p) \mid x_p \in \mathscr{X}_p\}$ is a set of $|\mathscr{T}|$ conjugacy classes of G meeting I.

On the other hand, assume that the class of $g \in G$ meets I. Hence, some G-conjugate of g has the form $\prod_p u_p$ for some $u_p \in I_p$. Now, $u_p = x_p^{n_p}$ for some $x_p \in \mathscr{X}_p$ and $n_p \in N_p$. Hence, $(\prod_p x_p)^{\prod_p n_p} = \prod_p u_p$ which is G-conjugate to g. Therefore, the class of g is the class of $\prod_p x_p$ and this shows that $\{\mathrm{cl}_G(\prod_p x_p) \mid x_p \in \mathscr{X}_p\}$ is exactly the set of conjugacy classes of G meeting I.

Since we know that $\dim(\mathrm{cf}(I)^G)$ is the number of conjugacy classes of G meeting I, we have that $\dim \mathrm{cf}(I)^G = |\mathscr{T}|$ and this completes the proof of the theorem.

PROOF OF THEOREM 3.1. We argue by induction on $|G|$. If I is contained in some proper normal subgroup of G, then apply Lemma 3.2. If $I^G = G$, then the theorem follows from Lemma 3.3 and Theorem 3.6.

References

[1] I. M. Isaacs, *Character theory of finite groups* (Dover, New York, 1994).
[2] ———, 'Fong characters in π-separable groups', *J. Algebra* **99** (1986), 89–107.
[3] A. Mann, 'Injectors and normal subgroups of finite groups', *Israel J. Math.* **9** (1971), 554–558.

Departament d'Algebra
Facultat de Matematiques
Universitat de Valencia
46100 Burjassot, Valencia
Spain
e-mail: gabriel@uv.es