Proceedings of the Royal Society of Edinburgh, page 1 of 20
DOI:10.1017/prm.2024.71

The structure of finite groups whose elements outside a normal subgroup have prime power orders

Changguo Shao and Qinhui Jiang*
College of Science, Nanjing University of Posts and Telecommunications, 210023 Nanjing, China (cgshao@njupt.edu.cn; syjqh2001@163.com)

(Received 26 September 2023; accepted 20 May 2024)

The structure of groups in which every element has prime power order (CP-groups) is extensively studied. We first investigate the properties of group G such that each element of $G \setminus N$ has prime power order. It is proved that N is solvable or every non-solvable chief factor H/K of G satisfying $H \leqslant N$ is isomorphic to $PSL_2(3^f)$ with f a 2-power. This partially answers the question proposed by Lewis in 2023, asking whether $G \cong M_{10}$? Furthermore, we prove that if each element $x \in G\setminus N$ has prime power order and $\mathbf{C}_G(x)$ is maximal in G, then N is solvable. Relying on this, we give the structure of group G with normal subgroup N such that $\mathbf{C}_G(x)$ is maximal in G for any element $x \in G \setminus N$. Finally, we investigate the structure of a normal subgroup N when the centralizer $\mathbf{C}_G(x)$ is maximal in G for any element $x \in N \setminus \mathbf{Z}(N)$, which is a generalization of results of Zhao, Chen, and Guo in 2020, investigating a special case that $N = G$ for our main result. We also provide a new proof for Zhao, Chen, and Guo's results above.

Keywords: prime power order; centralizers; maximal subgroups; Frobenius groups; CP-groups

2020 *Mathematics Subject Classification:* 20D06; 20D45; 20D15

1. Introduction

All groups are supposed to be finite. Recall that G is called a minimal non-abelian group if G is non-abelian but every proper subgroup of G is abelian. We denote by $F(G)$ the Fitting subgroup of G. The upper nilpotent series $\{F_i(G)\}_{i\geqslant 0}$ of a group G is defined recursively by $F_0(G) = 1$ and $F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G))$ for $i \geqslant 1$. If G is a solvable group, then the smallest integer h such that $F_h(G) = G$ is called the Fitting length (or nilpotent length) of G and is denoted by $h(G)$. All unexplained notation and terminology are standard (see [19]).

A group G is called a CP-group if every element of G has prime power order. The question about CP-groups was first addressed by Higman in [16], who determined all solvable CP-groups. In [15], Heineken gave the general structure of non-solvable CP-groups and listed all simple CP-groups.

* Corresponding author.

Denote $H_{p^n}(G) := \langle x \in G | x^{p^n} \neq 1\rangle$ for some prime $p \mid |G|$. In [14], Hughes and Thompson determined the structure of group G if $H_p(G)$ is a proper subgroup of G and G is not a p-group. Some authors studied those groups G having a proper subgroup $H_{p^n}(G)$ where $n > 1$, for instance, [3, 5]. Clearly, each element of $G \setminus H_{p^n}(G)$ is a p-element. Motivated by the ideas above, we study the structure of a group G that satisfies the following:

Property ($*$): Let G be a group and N be a proper normal subgroup of G. Assume that every element of $G \setminus N$ has prime power order.

Clearly, every CP-group G trivially satisfies property ($*$). Our main theorem is:

Theorem A. *Let G be a group and N be a non-trivial normal subgroup of G. Suppose that G and N satisfy property ($*$). If N is non-solvable, then every non-solvable chief factor H/K of G satisfying $H \leqslant N$ is isomorphic to $PSL_2(3^f)$, where $f > 1$ is a 2-power. In particular, G/N is solvable.*

Remark 1. In [28, Theorem 1], it is asserted that 'Let $\Delta(q)$ be a subgroup of the automorphism group of a finite simple group $L_2(q)$ generated by its inner automorphism group and by an automorphism $\varphi\delta$, where φ and δ are the generators for the groups of field and diagonal automorphisms of $L_2(q)$, respectively. If G is a finite generalized Frobenius group with an insoluble kernel F, then $|G : F| = 2$ and $G/\mathrm{Sol}(F)$ is isomorphic to $\Delta(q)$, where $q = 3^{2^l}$ for some natural number l. Here, $\mathrm{Sol}(F)$ denotes the largest solvable normal subgroup of F'.

In fact, the chief factor in theorem A is not necessarily a simple group. Let $F := S_1 \times S_2$, where $S_1 \cong S_2 \cong PSL_2(9)$ and let further φ_i, $\delta_i \in \mathrm{Aut}(S_i)$ be field and diagonal automorphisms of S_i, for $i = 1, 2$, respectively, and $u_i = \varphi_i\delta_i$. It is easy to see that every element in S_iu_i is a 2-element by [4]. Suppose that $G = F\langle u\rangle$, where $u = u_1u_2$. Obviously, $\mathrm{Sol}(F) = 1$ and $G \setminus F = Fu = (S_1u_1) \times (S_2u_2)$. Hence, every element in $G \setminus F$ is also a 2-element. But in this case, F is not a non-abelian simple group. Hence, there must be a mistake in [28, Theorem 1].

The key ingredient for proving theorem A is theorem B, which has interest on its own.

Theorem B. *Let N be a non-abelian simple group and $Aut(N)$ be its automorphism group. If $u \in \mathrm{Aut}(N) \setminus N$ is an r-element for some prime r such that every element of Nu has prime power order, then $N \cong PSL_2(3^f)$ with integer f, $u = \delta\varphi$ is a product of a diagonal automorphism δ and a field automorphism φ of N. In particular, $o(\varphi) = f$ is a power of 2.*

Remark 2. Although this theorem is proved in [21], our method is quite different since we consider the element orders of the coset of $Soc(G)$, where G is an almost simple group, while the proof of [21] is relying on theory of classical groups. Of course, both of the proofs depend on the classification of finite simple groups.

It is well-known that maximal subgroups play an important role in researching the structure of groups. For instance, a straightforward result asserts that G is solvable when all its maximal subgroups have prime indices. On the contrary, the influence of the centralizers of elements on the structure of groups is also studied extensively.

For instance, the authors of [**6**, **25**] investigated groups in which the centralizer of any non-trivial element is nilpotent, while the authors of [**2**, **18**] studied groups with conditions on centralizers. As an application of Theorem A, we get the following result:

COROLLARY 1. *Let G be a group and $N \trianglelefteq G$. If every element $x \in G \setminus N$ has prime power order, and $\mathbf{C}_G(x)$ is maximal in G, then N is solvable.*

Furthermore, we investigate the structure of group G if the centralizer of each element in $G \setminus N$ is maximal in G, where N is a normal subgroup of G. Zhao, Chen, and Guo gave the structure of group G when $N = G$, and proved that:

THEOREM C ([**26**, Theorems A and B]). *Let G be a non-abelian group. Write $\overline{G} := G/\mathbf{Z}(G)$. If for any $x \in G \setminus \mathbf{Z}(G)$, $\mathbf{C}_G(x)$ is maximal in G, then $\overline{G}$ is either an elementary abelian p-group, or $\overline{G} = \overline{P} \rtimes \overline{Q}$ is an inner abelian group with $|\overline{P}| = p^a$ and $|\overline{Q}| = q$, where p and q are two different primes, and a is a positive integer.*

In this paper, we consider the case $\mathbf{Z}(G) < N$, and obtain that:

THEOREM D. *Let G be a group and N be a proper normal subgroup of G such that $\mathbf{Z}(G) < N$. If $\mathbf{C}_G(x)$ is maximal in G for every element $x \in G \setminus N$, then G is solvable with G/N abelian. Furthermore,*

(I) *If G is nilpotent, then G/N is a p-group for some prime p. Moreover, $G = P \times \mathbf{Z}(G)_{p'}$ and $\mathbf{C}_G(x) \trianglelefteq G$ for every $x \in P \setminus N$;*

(II) *G is non-nilpotent, then $|G : \mathbf{C}_G(x)| = r^a$ with prime r and positive integer a. Suppose that R is a Sylow r-subgroup of G and K is a Hall r'-subgroup of G, then one of the statements holds:*
(1) *If G/N is a p-group for some prime p, we have*
(1.1) *If $r = p$, write $\overline{G} := G/\mathbf{O}_p(G)\mathbf{Z}(G)$. Then $\overline{G} = \overline{K} \rtimes \overline{P}$ is a Frobenius group with abelian kernel $\overline{K}$ and complement $\overline{P}$ of order p.*

(1.2) *If $r \neq p$, write $\widetilde{G} := G/\mathbf{Z}(G)_r$. Then*
(1.2.1) *If $\mathbf{O}_r(\widetilde{G}) = \widetilde{1}$, then the Fitting length $h(\widetilde{G}) = 3$ and $\widetilde{G}$ has the following normal series:*

$$\widetilde{1} \trianglelefteq \mathbf{O}_{r'}(\widetilde{G}) \trianglelefteq \mathbf{O}_{r',r}(\widetilde{G}) \trianglelefteq \widetilde{G} = \mathbf{O}_{r',r,r'}(\widetilde{G}),$$

where $\mathbf{O}_{r',r}(\widetilde{G})/\mathbf{O}_{r'}(\widetilde{G}) \cong \widetilde{R}$ is elementary abelian and $\widetilde{G}/\mathbf{O}_{r',r}(\widetilde{G})$ is a p-group;

(1.2.2) *Assume that $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. If $\mathbf{C}_G(x)_{p'} \nleq \mathbf{Z}(G)$, then the Fitting length $h(\widetilde{G}) = 2$ and $\widetilde{G}$ has the following normal series:*

$$\widetilde{1} \trianglelefteq \widetilde{R} \trianglelefteq \widetilde{N} \trianglelefteq \widetilde{G},$$

where $\widetilde{R}$ is an elementary abelian r-group and G/R is nilpotent.

(1.2.3) *Write $\widehat{G} := G/\mathbf{Z}(G)$. Assume that $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. If $\mathbf{C}_G(x)_{p'} \leqslant \mathbf{Z}(G)$, then $\widehat{G} = \widehat{R} \rtimes \widehat{P}$, where P is a Sylow p-subgroup of G*

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

and R is a Sylow r-subgroup of G with $\widehat{R}$ elementary abelian. Let $N_p = N \cap P$. If $\widehat{N_p} \trianglelefteq \widehat{G}$, then $\widehat{G}/\widehat{N_p}$ is a Frobenius group; if $\widehat{N_p} \ntrianglelefteq \widehat{G}$, then $\mathbf{N}_{\widehat{G}}(\widehat{N_p}) = \widehat{P}$.

(2) *If $|\pi(G/N)| \geqslant 2$, then one of the following statements holds:*

(2.1) *Let $\overline{G} := G/\mathbf{O}_r(G)\mathbf{Z}(G)$. Then $\overline{G} = \overline{K} \rtimes \overline{R}$ is a Frobenius group with abelian kernel $\overline{K}$ and complement $\overline{R}$ of order r;*

(2.2) *Let $\overline{G} := G/\mathbf{Z}(G)$. Then $\overline{G} = \overline{R} \rtimes \overline{K}$ is a Frobenius group with $\overline{R}$ a minimal normal subgroup of $\overline{G}$ and $\overline{K}$ cyclic. In particular, $R \leqslant N$.*

On the contrary, in a very recent paper, Zhao *et al.* [**27**] investigated the structure of a normal subgroup N of G, when $\mathbf{C}_G(x)$ is maximal for every element $x \in N \setminus \mathbf{Z}(G)$. Being inspired by the idea above, in this paper, by using less elements, we study the structure of group G if $\mathbf{C}_G(x)$ is maximal for every element $x \in N \setminus \mathbf{Z}(N)$. We obtain:

Theorem E. *Let G be a group and $N \trianglelefteq G$. Let $\overline{G} := G/\mathbf{Z}(N)$. If $\mathbf{C}_G(x)$ is maximal in G for every element $x \in N \setminus \mathbf{Z}(N)$, then one of the following statements holds:*

(1) *If $\overline{N}$ is nilpotent, then $\overline{N}$ is an elementary abelian p-group for some prime p;*

(2) *If $\overline{N}$ is non-nilpotent, then $\overline{N} = \overline{P} \rtimes \overline{Q}$ is a Frobenius group with an elementary abelian kernel $\overline{P}$ and complement $\overline{Q}$ with prime order. In particular, $\overline{P}$ is the minimal normal subgroup of $\overline{G}$.*

Remark 3. In [**20**], Lewis raised an interesting question asserting that: G is a group, N is a normal subgroup, p is a prime, P is a Sylow subgroup so that $(G, P, P \cap N)$ is a Frobenius–Wielandt triple, $G = NP$, $\mathbf{O}_p(G) = 1$, and G is non-solvable. Is this enough to imply that $G \cong M_{10}$? Theorems A and B partially answered the question above.

Remark 4. Corollary 1 can also be obtained by [**20**, Theorem 1.1]. It is worth to mention that our method and result are different from Lewis' since he determined the structure of group G while we are concerning on the information of the normal subgroup N of G.

Remark 5. Both [**27**, Theorems A and B] and theorem C can be considered as corollaries of theorem 5.

2. Proof of theorem B

To show theorem A, we first prove theorem B. Here, we list some notation and lemmas, which will be used below.

We denote by $\omega(G)$ the set of the element orders of G. If $A \subseteq G$ is a subset of G, then $\omega(A)$ denotes the set of element orders of A, and $k \cdot \omega(A) = \{ka | a \in \omega(A)\}$.

If $\varepsilon \in \{+, -\}$, we may write ε instead of $\pm$ in arithmetic expressions. The notation used in this section is mainly borrowed from [**4**, **10**, **12**].

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

LEMMA 2.1 ([**22**, Lemma 2]). *Let p and q be two primes and m, n be natural numbers such that $p^m = q^n + 1$. Then one of the following statements holds:*

(1) *$n = 1$, m is a prime number, $p = 2$ and $q = 2^m - 1$ is a Mersenne prime;*

(2) *$m = 1$, n is a power of 2, $q = 2$ and $p = 2^n + 1$ is a Fermat prime;*

(3) *$p = n = 3$ and $q = m = 2$.*

LEMMA 2.2 ([**29**]). *Let a and n be integers greater than 1. Then there exists a primitive prime divisor of $a^n - 1$, that is a prime s dividing $a^n - 1$ and not dividing $a^i - 1$ for $1 \leqslant i \leqslant n - 1$, except if*

(1) *$a = 2$ and $n = 6$, or*

(2) *a is a Mersenne prime and $n = 2$.*

Proof of theorem B. Since every element of Nu has prime power order, we take $nu \in Nu$ an r_1-element for some element $n \in N$ and prime r_1. Note that $N\langle u\rangle = N\langle nu\rangle$. Then $N\langle u\rangle/N = N\langle nu\rangle/N$. Moreover, $N\langle u\rangle/N \cong \langle u\rangle/\langle u\rangle \cap N$ is an r-group and $\langle nu\rangle N/N \cong \langle nu\rangle/\langle nu\rangle \cap N$ is an r_1-group. This forces $r = r_1$. Consequently, we conclude that every element in Nu is an r-element.

If N is a sporadic simple group, we may take $N = M_{12}$ as an example. In this case, select $u \in \mathrm{Aut}(N) \setminus N$ a 2-element. By [**4**], there exists an element of order 10, against our assumption. By a similar reasoning, we rule out the case that N is a sporadic simple group.

Assume then that $N = A_n$ is an alternating group of degree n with $n \geqslant 5$ but $n \neq 6$. As $\mathrm{Out}(N) \cong C_2$, we select $u = (12)$. Take $x = (345) \in N$. Then $ux = xu \in Nu$ does not have prime power, also a contradiction.

Now, we consider $N = N(q)$ is a simple group of Lie type defined over a field of q-elements, where $q = p^f$ with p a prime. As $u \in\mathrm{Aut}(N)$, by [**9**, Theorem 2.5.1], we may write $u = \delta\varphi\tau$, where δ is a diagonal automorphism, φ is a field automorphism, and τ is a graph automorphism of N, respectively.

If u is a field or a graph-field automorphism of N, then by [**8**], $\mathbf{C}_N(u) = N(p^{f/t})$ is a group of the same type as N. Clearly, $\mathbf{C}_N(u) = N(p^{f/t})$ is not a prime power group by [**4**, Table 6]. Let $1 \neq u' \in \mathbf{C}_N(u)$ be an r'-element. Then $u'u = uu' \in Nu$ is not an r-element, contrary to our assumption. If u is a graph automorphism of N, according to [**4**], we see that $N \cong PSL_n^+(q)$ with $n \geqslant 3$; $P\Omega_5(q)$ with q even; $P\Omega_{2n}^+(q)$ with $n \geqslant 4$; $G_2(q)$ with $p = 3$; $F_4(q)$ with $p = 2$; or $E_6(q)$ with $3 \mid (q - 1)$. However, there will be a contradiction according to [**1**, **13**].

Consequently, $u = \delta\varphi\tau$ with $\delta \neq 1$. In the following, we will analyse it case by case.

Case 1. $u = \delta$.
As $\delta \neq 1$, according to [**4**, Table 5], we see that $N \cong PSL_n^+(q)$ with $n \geqslant 2$; $PSL_n^-(q)$ with $n \geqslant 3$; $PSp_{2n}(q)$ with $n \geqslant 2$ and q odd; $P\Omega_{2n+1}(q)$ with $n \geqslant 3$ and q odd; $P\Omega_{2n}^{\varepsilon}(q)$ with $n \geqslant 4$; $E_6^{\varepsilon}(q)$ with $3 \mid (q - \varepsilon 1)$ or $E_7(q)$ with q odd.

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

Assume first $N \cong PSL_n^+(q)$. Write $d := o(u)$. Easily, $r \mid d$. If $n = 2$, then $Nu = PGL_2(q) \setminus PSL_2(q)$ and $2 \mid (2, q-1)$. As we see that q is odd and $r = 2$. It follows by [**11**, Lemma 2.1] that both $(q + (\varepsilon 1))$ and $(q - (\varepsilon 1))$ are powers of r. Lemma 2.1 indicates that $q = p = 3$. However, in this case, $N \cong PSL_2(3)$ is not a simple group, a contradiction. Hence, $n \geqslant 3$. Then $G = N \rtimes \langle u \rangle \leqslant PGL_n^\varepsilon(q)$. Let $T := \langle x_0 \rangle$ be a Singer subgroup of $PGL_n^\varepsilon(q)$ such that $S_0 := T \cap N$ is a Singer subgroup of N. By [**17**, Theorem 2.7.3], $|T| = (q^n - (\varepsilon 1)^n)/(q - \varepsilon 1)$ and $|T \cap N| = (q^n - (\varepsilon 1)^n)/(d(q - (\varepsilon 1)))$. Note that $PGL_n^\varepsilon(q) = PSL_n^\varepsilon(q)\langle x_0 \rangle$ and $x_0^d \in PSL_n^\varepsilon(q)$, without loss of generality, we may assume that $u \in T$. Then u centralizes $T \cap N$. It follows that $(q^n - (\varepsilon 1)^n)/(d(q - (\varepsilon 1)))$ is an r-power. If $n = 3$ and $q = 2$, we have $N \cong PSL_3(2) \cong PSL_2(7)$, which is a contradiction. Hence, $(n, q) \neq (3, 2)$. If $\varepsilon = +$, according to lemma 2.1, $q^n - 1$ has a prime divisor p_1 such that $p_1 \neq r$ and $p_1 \nmid q - 1$. Hence, $p_1 \mid (q^n - 1)/(d(q - 1))$, a contradiction. If $\varepsilon = -$, we can also rule out this case by a similar argument as above.

Now, consider $N \cong P\Omega_{2n}^\varepsilon(q)$. In this case, $r \mid d = (4, q^n - \varepsilon)$, indicating $r = 2$, and thus q is odd. By [**9**, Theorem 2.5.12], we see that all diagonal automorphisms of the same order are conjugate in $\mathrm{Out}(N)$. By [**12**, Figs. 1, 2, 3], $\omega(Nu) = \omega(PCSO_{2n}^\varepsilon(q) \setminus PSO_{2n}^\varepsilon(q))$ or $\omega(Nu) = \omega(PSO_{2n}^\varepsilon(q) \setminus P\Omega_{2n}^\varepsilon(q))$. By [**12**, Lemma 2.9], each element in $\omega(PCSO_{2n}^\varepsilon(q) \setminus PSO_{2n}^\varepsilon(q))$ is not an r-number for $n \geqslant 4$, contrary to our assumption. By [**12**, Lemmas 2.6 and 2.4], there exists an element in $\omega(Nu) = \omega(PSO_{2n}^\varepsilon(q) \setminus P\Omega_{2n}^\varepsilon(q))$, whose order is not an r-number, also contradiction.

Furthermore, according to [**10**], we can rule out the cases $N \cong P\Omega_{2n+1}(q)$ and $PSp_{2n}(q)$. The remaining cases can be ruled out by [**30**].

Case 2. $u = \delta\varphi$ with $\delta \neq 1$ and $\varphi \neq 1$.
Let $S := \mathrm{Inndiag}(N)$. Then S is a simple group of Lie type, which is defined over a field of q-elements, where $q = p^f$ with prime p. Let $\delta_0 := \delta_0(q)$ be a generator of S/N (other than $N \cong P\Omega_{2n}^+(q)$ with n even) such that $\delta = \delta_0^i$ for some positive integer i, and φ_0 be a generator of the field automorphism group of N.

Since $\delta \neq 1$, by [**4**, Table 5], we see that $N \cong PSL_n^\varepsilon(q)$ with $n \geqslant 2$ and $\varepsilon \in \{+, -\}$; $P\Omega_{2n}^\varepsilon(q)$ with $n \geqslant 4$, or $P\Omega_{2n+1}(q)$ with $n \geqslant 3$ and q odd, $PSp_{2n}(q)$, with $n \geqslant 3$ and q odd, $E_6^\varepsilon(q)$ with $3 \mid (q - \varepsilon 1)$ or $E_7(q)$ with q odd.

First consider $N \cong PSL_n^\varepsilon(q)$. If $n \geqslant 3$, by [**11**, Lemma 3.3], we have $\omega(uN) = k \cdot \omega(\tau^\alpha \delta(q_0) PSL_n^\varepsilon(q_0))$, where τ is a graph automorphism of N, and $\alpha = 0$ if $\varepsilon = +$ and $\alpha = 1$ if $\varepsilon = -$. Suppose $o(\varphi) = k$. Then k is an r-power. Assume first that $\varepsilon = +$. Then $\omega(uN) = k \cdot \omega(\delta(q_0) PSL_n(q_0))$. Furthermore, $((q_0^n - 1)i)/(d(q_0 - 1)) \in \omega(\delta(q_0) PSL_n(q_0))$, forcing that $((q_0^n - 1)i)/(d(q_0 - 1))$ is an r-power. By applying a similar argument as in case 1, we get a contradiction. Assume now that $\varepsilon = -$. Then $\omega(uN) = k \cdot \omega(\tau\delta(q_0) PSL_n(q_0))$. Since $\mathrm{Out}(N) \cong \langle \delta_0 \rangle \rtimes (\langle \varphi_0 \rangle \times \langle \tau \rangle)$, we obtain that τ is conjugate to $\tau\delta$. Hence, $\omega(\tau\delta(q_0) PSL_n(q_0)) = \omega(\tau PSL_n(q_0))$. By [**11**, Lemma 4.7], we see that $\omega(\tau PSL_n(q_0)) = 2 \cdot \omega(PSp_n(q_0))$ if $p \neq 2$, against [**11**, Lemma 2.2]. Hence, $p = 2$. Since $r \mid k$ and $k \mid (n, q + 1)$, we get that r is odd. Note that $\langle \delta_0 \rangle \rtimes \langle \tau \rangle$ is a dihedral group of $2d$ with d odd. Hence, $u \in \langle \delta_0 \rangle$. That is to say, u is a diagonal automorphism of N, contrary to our assumption.

As a result, $n = 2$. That is, $N \cong PSL_2(q)$ with q odd. Easily, $r = 2$. Let $M := PGL_2(q)$. Then $\mathbf{C}_M(\varphi) = PGL_2(q_0)$, where $q_0 = p^{f/k}$. Clearly, $\delta \in \mathbf{C}_M(\varphi)$.

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

It follows that $PGL_2(q_0) \setminus PSL_2(q_0) \subseteq Nu$. Hence, every element in $PGL_2(q_0) \setminus PSL_2(q_0)$ is a 2-element. By [**11**, Lemma 2.1], both $q_0 - 1$ and $q_0 + 1$ are 2-power. By lemma 2.1, we get $q_0 = p = 3$ and $f = k$. Therefore, $N \cong PSL_2(3^f)$, where f is a 2-power, as required.

If $N \cong P\Omega^{\varepsilon}_{2n}(q)$, then either $(2,\, q-1) = 2$ or $(4,\, q^n - \varepsilon 1) = 2,\ 4$, forcing that q is odd. Assume first that $o(\delta) = 4$. By [**10**, Theorem 2.5.12], we see that $\delta\varphi$ is conjugate to $\delta^3\varphi$ in $\mathrm{Out}(N)$. It follows that $\omega(uN) = \omega(N\langle\delta\rangle\varphi \setminus (N\langle\delta^2\rangle\varphi))$. Assume first that $\varepsilon = +$. By [**12**, Lemma 1.2 and Fig. 2], we have that $\omega(uN) = k \cdot \omega(PCSO^{+}_{2n}(q_0) \setminus PSO^{+}_{2n}(q_0))$. Therefore, both $q_0^{n-1} - 1$ and $(q_0^{n-2} + 1)(q_0 - 1)$ are in $\omega(PCSO^{\varepsilon}_{2n}(q_0) \setminus PSO^{\varepsilon}_{2n}(q_0))$ by [**12**, Lemma 2.9]. Hence, both $q_0^{n-1} - 1$ and $(q_0^{n-2} + 1)(q_0 - 1)$ are r-power, which is a contradiction. Assume now that $\varepsilon = -$. Then $\delta^{\varphi} = \delta$ and n is odd. It follows that u is a 2-element. Let $W := PCSO^{-}_{2n}(q)$. Then $\mathbf{C}_W(\varphi) = PCSO^{-}_{2n}(q_0)$, where $q_0 = p^{f/k}$. Clearly, $\delta \in \mathbf{C}_W(\varphi)$. Therefore, we obtain that $\mathbf{C}_W(\varphi)\varphi \setminus \mathbf{C}_N(\varphi)\langle\delta^2\rangle\varphi \subseteq Nu$. This shows that $\mathbf{C}_W(\varphi) \setminus \mathbf{C}_N(\varphi)\langle\delta^2\rangle = PCSO^{-}_{2n}(q_0) \setminus PSO^{-}_{2n}(q_0)$ is a 2-element, contrary to [**12**, Lemma 2.9].

Assume now that $o(\delta) = 2$. By [**12**, Lemma 1.2 and Figs. 1, 2, 3], we have that $\omega(uN) = k \cdot \omega(PCSO^{+}_{2n}(q_0) \setminus PSO^{+}_{2n}(q_0))$ if $(4,\, q^n - 1) = 2$; $\omega(\varphi(PSO^{+}_{2n}(q) \setminus P\Omega^{+}_{2n}(q)))$ if n is odd and $(q^n - 1,\, 4) = 4$; $\omega(\varphi(PCSO^{+}_{2n}(q) \setminus PSO^{+}_{2n}(q)))$, or $\omega(\varphi(PSO^{+}_{2n}(q) \setminus P\Omega^{+}_{2n}(q)))$ if $n > 4$ is even. If $(4,\, q^n - 1) = 2$, by [**12**, Lemma 2.9], we have that $q_0^{n-1} - 1 \in \omega(PCSO^{+}_{2n}(q_0) \setminus PSO^{+}_{2n}(q_0))$. It is easy to get a contradiction by lemma 2.2. If n is odd and $(q^n - 1,\, 4) = 4$, then $\omega(\varphi(PSO^{+}_{2n}(q) \setminus P\Omega^{+}_{2n}(q))) \supseteq \omega(\varphi PSO^{+}_{2n}(q)) \setminus \omega(\varphi P\Omega^{+}_{2n}(q))$. By [**12**, Lemma 1.2], we have $\omega(\varphi(PSO^{+}_{2n}(q) \setminus P\Omega^{+}_{2n}(q))) \supseteq k \cdot (\omega(PSO^{+}_{2n}(q_0)) \setminus \omega(P\Omega^{+}_{2n}(q_0)))$. By [**12**, Lemmas 2.4 and 2.6], $(q_0^n - 1)/2 \in \omega(PSO^{+}_{2n}(q_0)) \setminus \omega(P\Omega^{+}_{2n}(q_0))$. It follows that $(q_0^n - 1)/2$ is an r-power. Since $n \geqslant 3$ and q_0 is odd, there exists some odd prime $p_1 \mid (q_0^n - 1)$ with $p_1 \neq r$ by lemma 2.2, a contradiction. By the same reason, we can also rule out the case $n \geqslant 4$ is even.

If $N \cong P\Omega_{2n+1}(q)$ with $n \geqslant 3$ and q odd, then $Nu = N\langle\delta\rangle\varphi \setminus N\varphi$. By [**10**, Lemma 2.8], we have that $\omega(Nu) \supseteq \omega(N\langle\delta\rangle\varphi) \setminus \omega(N\varphi) = k \cdot (\omega(\mathrm{Inndiag}(P\Omega_{2n+1}(q_0)) \setminus \omega(P\Omega_{2n+1}(q_0)))$. By [**10**, Lemma 2.1], we get that $p(q_0^{n-1} \pm 1) \in \omega(\mathrm{Inndiag}(P\Omega_{2n+1}(q_0)) \setminus \omega(P\Omega_{2n+1}(q_0))$. By hypothesis, $p(q_0^{n-1} \pm 1)$ is an r-power, a contradiction. By the same reason, we can also rule out the case $N \cong PSp_{2n}(q)$ with $n \geqslant 3$ and q odd.

If $N \cong E^{\varepsilon}_6(q)$, then $u^{\tau} = \delta^2\varphi$. It follows that $\omega(\mathrm{Inndiag}(N)\varphi) = \omega(M\varphi) \cup \omega(N\delta\varphi)$. Hence, $\omega(N\delta\varphi) = \omega(\mathrm{Inndiag}(N)\varphi) \setminus \omega(N\varphi)$. By [**30**, (3) and (5)], we have that $\omega(N\delta\varphi) = \omega(\mathrm{Inndiag}(N)\varphi) \setminus \omega(N\varphi) = k \cdot (\omega(\mathrm{Inndiag}(E^{\varepsilon}_6(q_0)) \setminus \omega(E^{\varepsilon}_6(q_0)))$. By [**30**, Lemmas 1 and 3], we see that $q^6 + \varepsilon q^3 + 1,\, (q^2 - 1)(q^4 + 1) \in \omega$ $(\mathrm{Inndiag}(E^{\varepsilon}_6(q_0)) \setminus \omega(E^{\varepsilon}_6(q_0))$. By hypothesis, we obtain that both $q^6 + \varepsilon q^3 + 1$ and $(q^2 - 1)(q^4 + 1)$ are r-power. Since $3 \mid (q^6 + \varepsilon q^3 + 1)$, we get that $r = 3$. As $(q^2 - 1)(q^4 + 1)$ is a 3-power, then both $q^2 - 1$ and $q^4 + 1$ are 3-power, a contradiction. Similarly, we may rule out the case $N \cong E_7(q)$.

Case 3. $u = \delta\tau$ with $\delta \neq 1$, $\tau \neq 1$.
By [**4**, Table 5], we obtain that $N \cong PSL_n(q)$ with $n \geqslant 3$, $P\Omega^{+}_{2n}(q)$ with $n \geqslant 4$, or $E_6(q)$. If $N \cong PSL_n(q)$, then $\delta^{\tau} = \delta^{-1}$ and thus $o(u) = o(\delta\tau) = 2$, yielding $r = 2$. Assume first that q is even. Then $(n,\, q-1)$ is odd. In this case, τ is conjugate to $\tau\delta_0^i$ for every i in $\mathrm{Out}(N)$, where δ_0 is the generator of the diagonal automorphism

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

group of N in $\mathrm{Out}(N)$. Therefore, $\omega(Nu)=\omega(N\tau)=\omega(PGL_n(q)\tau)$. By [1, 19.9], we have that $\mathbf{C}_N(\tau)$ is isomorphic to $PSp_n(q)$ if n is even, and to $P\Omega_n(q)$ if n is odd. Hence, there exists an element in Nu whose order is not a prime power, a contradiction. By the same reason, by applying [13], there is also a contradiction for the case q is odd.

If $N\cong P\Omega^+_{2n}(q)$, we have that $(2,\,q-1)=2$ or $(4,\,q^n-1)>1$, which follows that q is odd. If $\mathrm{Inndiag}(N)/N$ is cyclic, then $\delta^\tau=\delta^{-1}$. Therefore, $o(u)=o(\delta\tau)=2$, by the same reason as above, we can also get a contradiction. If $\mathrm{Inndiag}(N)/N\cong C_2\times C_2$, first we may consider the case $n>4$. By [9, Theorem 2.5.12] and [12, Fig. 3], we have $\omega(Nu)=\omega(PTO^+_{2n}(q_0)\setminus PSO^+_{2n}(q_0))$, against [12, Lemma 2.8]. Now, we consider the case $n=4$. According to [9, Theorem 2.5.12], we see that $o(\tau)=2$ since $o(Nu)\geqslant 4$ is a 2-power. By the same reason above, there is also a contradiction.

If $N\cong E_6(q)$, then $\delta^\tau=\delta^{-1}$. It follows that $o(u)=2$ and u is conjugate to τ in $\mathrm{Out}(N)$. Hence, $\omega(Nu)=\omega(N\tau)$. Since $\mathbf{C}_N(\tau)=F_4(q)$, we get that Nu contains a non-r-element, a contradiction.

Case 4. $u=\delta\varphi\tau$ with $\delta\neq 1$, $\varphi\neq 1$, and $\tau\neq 1$.
Let φ_0 be a generator of the field automorphism group of N such that $\varphi=\varphi_0^{f/k}$ and δ_0 a generator of the diagonal automorphism group of N such that $\delta=\delta_0^i$ for some positive integer i. Let $q=q_0^k$. By [4, Table 5], we have $N\cong PSL_n(q)$ with $n\geqslant 3$, $P\Omega^+_{2n}(q)$ with $n\geqslant 4$, or $E_6(q)$ with $3\mid(q-1)$.

If $N\cong PSL_n(q)$, then, by [11, Lemma 3.3] and hypothesis, we see that k is an r-power and every element of $\omega(\delta(-q_0)PSU_n(q_0))$ is also an r-element. By [11, Lemma 2.1], we see that $\delta(-q_0)PSU_n(q_0)$ if $r=2$ or $\delta(q_0)PSL_n(q_0)$ if $r>2$, has an element of order $((q_0^n-(\varepsilon 1)^n)i)/((n,\,q_0+\varepsilon 1)(q_0-\varepsilon 1))$. By hypothesis, this order is an r-power. By a similar argument as in case 1, there is a contradiction.

If $P\Omega^+_{2n}(q)$ with $n\geqslant 4$, then q is odd. By [12], there is also a contradiction.

The remaining case is $N\cong E_6(q)$. In this case, $3\mid(q-1)$. Since $N\langle u\rangle/N$ is an r-group, we see that u is a field or graph, or a graph-field automorphism of N up to conjugation, the final contradiction. □

3. Proofs of theorem A and corollary 1

Theorem A. *Let G be a group and N be a non-trivial normal subgroup of G. Suppose that G and N satisfy property $(*)$. If N is non-solvable, then chief factor H/K of G satisfying $H\leqslant N$ is isomorphic to $PSL_2(3^f)$, where $f>1$ is a 2-power. In particular, G/N is solvable.*

Proof. Suppose on the contrary that G is a counter-example of minimal order. We first assert that chief factor H/K of G satisfying $H\leqslant N$ is also a non-solvable chief factor of $N\langle u\rangle$. Let M_1/N_1 be a non-solvable chief factor of G satisfying $M_1\leqslant N$. Write $M_1/N_1=F_1\times\cdots\times F_t$, where F_i are isomorphic non-abelian simple groups with integer $t\geqslant 1$. Assume that N/N_1 acts transitively on $\Omega=\{F_1,\ldots,F_t\}$. Then M_1/N_1 is also a chief factor of N and thus is a chief factor of $N\langle u\rangle$, we are done. Assume that N/N_1 does not act transitively on Ω. Therefore, $t>1$. Then there is an element $N_1w\in G/N_1\setminus N/N_1$ such that $N_1\langle w\rangle$ acts non-trivially on Ω. Otherwise, every element in $G/N_1\setminus N/N_1$ acts trivially on Ω, which indicates that

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

N/N_1 acts transitively on Ω, a contradiction. Hence, there is an orbit Ω_1 of $N_1\langle w\rangle$ on Ω having size greater than 1. Without loss of generality, we may assume that $\Omega_1 = \{F_1, \dots, F_s\}$ with $s > 1$. By hypothesis, we get that N_1w is an r_1-element for some prime r_1. Let $1 \neq f_1 \in F_1$ be an r_1'-element. Then there is an element $N_1w^{j_i}$ such that $f_1^{N_1w^{j_i}} \in F_i$ where j_i is a positive integer. Now, we get that $N_1w_0 := \prod_{i=1}^{s} f_1^{(N_1w)^{j_i}}$ is centralized by N_1w. In this case, N_1w_0w does not have prime power order and thus w_0w is not a prime power order element, contrary to the assumption of the theorem as $w_0w \in G \setminus N$. Consequently, we conclude that M_1/N_1 is a chief factor of N. So is of $N\langle u\rangle$, as required.

Take $1 \neq u \in G \setminus N$. If $N\langle u\rangle < G$, by assumption, every non-solvable chief factor H/K of $N\langle u\rangle$ satisfying $H \leqslant N$ is isomorphic to $PSL_2(3^f)$, where $f > 1$ is a 2-power. As chief factor H/K of G satisfying $H \leqslant N$ is also a chief factor of $N\langle u\rangle$, the theorem holds, against our assumption. As a result, $G = N\langle u\rangle$.

Let $M > 1$ be a minimal normal subgroup of G, which is contained in N. Clearly, each element in $G/M \setminus N/M$ has prime power order. This shows that G/M and N/M satisfy property $(*)$. Let $(M_1/M)/(M_2/M)$ be a chief factor of G/M such that $M_1/M \leqslant N/M$. Then M_1/M_2 is also a chief factor of G satisfying in $M_1 \leqslant N$. By induction, $(M_1/M)/(M_2/M)$ isomorphic to $PSL_2(3^f)$, where some $f > 1$ is a 2-power, so is M_1/M_2. In the following, we focus on the chief factors which are contained in M. We only need to consider the case that M is non-solvable.

In this case, M is a non-solvable minimal normal subgroup of G. Therefore, we may write $M = S_1 \times \cdots \times S_t$, where t is a positive integer and S_i are isomorphic non-abelian simple groups for all $i \in \{1, \dots, t\}$. Assume that $t > 1$. Easily, $\langle u\rangle$ acts transitively on $\{S_1, \dots, S_t\}$. Let $o(u) = m$ and $1 \neq x \in S_1$ be a q-element for some odd prime q distinct from p, where p is a prime divisor of m. Then $y = \prod_{i=1}^{m} x^{u^i}$ is centralized by u. That is, the order of $uy \in G\backslash N$ is divisible by pq. This contradiction deduces that $t = 1$. Consequently, M is a non-abelian simple group, and $u \in \mathrm{Aut}(M)$, where $\mathrm{Aut}(M)$ is the automorphism group of M.

Since $Mu \subseteq Nu \subseteq G \setminus N$, we see that the order of every element in Mu has prime power order. By theorem B, we get that $M \cong PSL_2(3^f)$, where f is a 2-power and u is the product of a field and a diagonal automorphism of $PSL_2(3^f)$. Therefore, chief factor H/K of G satisfying $H \leqslant N$ is isomorphic to $PSL_2(3^f)$, with $f > 1$ a 2-power, as required.

Let W be a maximal solvable normal subgroup of G contained in N and N_0/M be a chief factor of G with $N_0 \leqslant N$. Then $N_0/W \cong PSL_2(3^{f_1})$, where f_1 is a 2-power. Let $\widetilde{G} := G/W$. Take any prime power element $e \in G \setminus N$. Then $\widetilde{N_0}\widetilde{e} \subseteq \widetilde{N}\widetilde{e} \subseteq G \setminus N$. It follows that every element in $\widetilde{N_0}\widetilde{e}$ has prime power order. By theorem B, we get that the order of e is a 2-power. By the arbitrariness of e, we get that G/N is a 2-group, contrary our assumption that G/N is non-solvable. Consequently, G/N is solvable. $\square$

Corollary 1. *Let G be a group and $N \trianglelefteq G$. If every element $x \in G \setminus N$ has prime power order, and $\mathbf{C}_G(x)$ is maximal in G, then N is solvable.*

Proof. Suppose on the contrary that N is non-solvable. By theorem A, G has a non-solvable chief factor $M_2/M_1 \cong PSL_2(3^f)$, where f is a 2-power such that $M_2 \leqslant N$

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

and M_1 is solvable. Write $\overline{G} := G/M_1$. Since $\overline{M_2}\overline{u} \subseteq \overline{N}\overline{u} \subseteq \overline{G} \setminus \overline{N}$, we see that each element in $\overline{M_2}\overline{u}$ has prime power order. By theorem B, $\overline{u}$ is a product of a field automorphism and a diagonal automorphism of $\overline{M_2}$, and $\overline{M_2} \cong PSL_2(3^f)$ with $f > 1$ a 2-power. Easily, $\overline{u}$ is a 2-element. Without loss of generality, we may consider u is a 2-element satisfying $G = M_2\langle u\rangle$ and thus $\overline{G} = \overline{M_2}\langle\overline{u}\rangle$.

Now, consider $P := \mathbf{C}_G(u)$. Since every element of $G \setminus N$ is a 2-element, we see that P must be a 2-group. The maximality of P indicates that P is a Sylow 2-subgroup of G. If $M_1 \nleq P$, we have that $G = PM_1$, indicating that G is solvable, a contradiction. Hence, $M_1 \leqslant P$. Then $\overline{P}$ is a maximal subgroup of $\overline{G}$. Let $P_0 := P \cap M_2$. Then $P_0 \in \mathrm{Syl}_2(M_2)$, forcing $\overline{P_0} \in \mathrm{Syl}_2(\overline{M_2})$. Let $|\overline{P_0}| = 2^a$. If $\overline{P_0}$ is maximal in $\overline{M_2}$, then by [**17**, Theorem 2.8.27], $2^a = 3^f \pm 1$. By Lemma 2.1, it follows that $a = 3$ and $f = 2$. Therefore, $\overline{M_2} \cong PSL_2(3^2)$. However, according to [**4**], we see that $\overline{P_0}$ is not maximal in $\overline{M_2}$. This contradiction forces that $\overline{P_0}$ is not maximal in $\overline{M_2}$. Furthermore, as $\overline{M_2} \leqslant \overline{G} \leqslant \mathrm{Aut}(\overline{M_2})$ and $\overline{M_2} \cong PSL_2(3^f)$, we obtain that $\overline{G} \cong M_{10} \cong PSL_2(9) \cdot \langle\overline{u}\rangle$ and $\overline{P} \cong C_8 \rtimes C_2$ or D_{16} by [**7**, Theorem 1.1]. In this case, $\overline{u} \notin \mathbf{Z}(\overline{P})$, the final contradiction completes the proof. □

4. Theorem D and its proof

For the reader's convenience, we restate theorem D.

Theorem 4.1. *Let G be a group and N be a proper normal subgroup of G such that $\mathbf{Z}(G) < N$. If $\mathbf{C}_G(x)$ is maximal in G for every element $x \in G \setminus N$, then G is solvable with G/N abelian. Furthermore,*

(I) *If G is nilpotent, then G/N is a p-group for some prime p. Moreover, $G = P \times \mathbf{Z}(G)_{p'}$ and $\mathbf{C}_G(x) \trianglelefteq G$ for every $x \in P \setminus N$;*

(II) *G is non-nilpotent, then $|G : \mathbf{C}_G(x)| = r^a$ with prime r and positive integer a. Suppose that R is a Sylow r-subgroup of G and K is a Hall r'-subgroup of G, then one of the statements holds:*
- (1) *If G/N is a p-group for some prime p, we have*
  - (1.1) *If $r = p$, write $\overline{G} := G/\mathbf{O}_p(G)\mathbf{Z}(G)$. Then $\overline{G} = \overline{K} \rtimes \overline{P}$ is a Frobenius group with abelian kernel $\overline{K}$ and complement $\overline{P}$ of order p.*
  - (1.2) *If $r \neq p$, write $\widetilde{G} := G/\mathbf{Z}(G)_r$. Then*
    - (1.2.1) *If $\mathbf{O}_r(\widetilde{G}) = \widetilde{1}$, then the Fitting length $h(\widetilde{G}) = 3$ and $\widetilde{G}$ has the following normal series:*
$$\widetilde{1} \trianglelefteq \mathbf{O}_{r'}(\widetilde{G}) \trianglelefteq \mathbf{O}_{r',r}(\widetilde{G}) \trianglelefteq \widetilde{G} = \mathbf{O}_{r',r,r'}(\widetilde{G}),$$
where $\mathbf{O}_{r',r}(\widetilde{G})/\mathbf{O}_{r'}(\widetilde{G}) \cong \widetilde{R}$ is elementary abelian and $\widetilde{G}/\mathbf{O}_{r',r}(\widetilde{G})$ is a p-group;
    - (1.2.2) *Assume that $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. If $\mathbf{C}_G(x)_{p'} \nleq \mathbf{Z}(G)$, then the Fitting length $h(\widetilde{G}) = 2$ and $\widetilde{G}$ has the following normal series:*
$$\widetilde{1} \trianglelefteq \widetilde{R} \trianglelefteq \widetilde{N} \trianglelefteq \widetilde{G},$$
where $\widetilde{R}$ is an elementary abelian r-group and G/R is nilpotent.

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

(1.2.3) *Write* $\widehat{G} := G/\mathbf{Z}(G)$. *Assume that* $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. *If* $\mathbf{C}_G(x)_{p'} \leqslant \mathbf{Z}(G)$, *then* $\widehat{G} = \widehat{R} \rtimes \widehat{P}$, *where* P *is a Sylow* p*-subgroup of* G *and* R *is a Sylow* r*-subgroup of* G *with* $\widehat{R}$ *elementary abelian. Let* $N_p = N \cap P$. *If* $\widehat{N_p} \trianglelefteq \widehat{G}$, *then* $\widehat{G}/\widehat{N_p}$ *is a Frobenius group; if* $\widehat{N_p} \ntrianglelefteq \widehat{G}$, *then* $\mathbf{N}_{\widehat{G}}(\widehat{N_p}) = \widehat{P}$.

(2) *If* $|\pi(G/N)| \geqslant 2$, *then one of the following statements holds:*

(2.1) *Let* $\overline{G} := G/\mathbf{O}_r(G)\mathbf{Z}(G)$. *Then* $\overline{G} = \overline{K} \rtimes \overline{R}$ *is a Frobenius group with abelian kernel* $\overline{K}$ *and complement* $\overline{R}$ *of order* r;

(2.2) *Let* $\overline{G} := G/\mathbf{Z}(G)$. *Then* $\overline{G} = \overline{R} \rtimes \overline{K}$ *is a Frobenius group with* $\overline{R}$ *a minimal normal subgroup of* $\overline{G}$ *and* $\overline{K}$ *cyclic. In particular,* $R \leqslant N$.

Proof. Let $x \in G \setminus N$ be an arbitrary element. Then there exists a component of x, say x_1, such that $x_1 \in G \setminus N$. Note that $\mathbf{C}_G(x) \leqslant \mathbf{C}_G(x_1)$ and both $\mathbf{C}_G(x)$ and $\mathbf{C}_G(x_1)$ are maximal in G. Then $\mathbf{C}_G(x) = \mathbf{C}_G(x_1)$. Without loss of generality, we may consider x as a p-element for some prime p. Furthermore,

Step 1. $\mathbf{C}_G(x) = \mathbf{C}_G(x)_p \times \mathbf{C}_G(x)_{p'}$, where $\mathbf{C}_G(x)_p$ is the Sylow p-subgroup of $\mathbf{C}_G(x)$ and $\mathbf{C}_G(x)_{p'}$ is the Hall p'-subgroup of $\mathbf{C}_G(x)$ with $\mathbf{C}_G(x)_{p'} \leqslant \mathbf{Z}(\mathbf{C}_G(x))$. In particular, $\mathbf{C}_G(x)$ is nilpotent.

For every p'-element $v \in \mathbf{C}_G(x)$, we always have $xv \in G \setminus N$. By lemma 5.1, it follows that $\mathbf{C}_G(vx) = \mathbf{C}_G(x) \cap \mathbf{C}_G(v)$. Note that $\mathbf{C}_G(x)$ and $\mathbf{C}_G(xv)$ are maximal subgroups of G. Then $\mathbf{C}_G(vx) = \mathbf{C}_G(x) \leqslant \mathbf{C}_G(v)$, yielding $v \in \mathbf{Z}(\mathbf{C}_G(x))$. As a result, $\mathbf{C}_G(x) = \mathbf{C}_G(x)_p \times \mathbf{C}_G(x)_{p'}$, where $\mathbf{C}_G(x)_p$ is the Sylow p-subgroup of $\mathbf{C}_G(x)$ and $\mathbf{C}_G(x)_{p'} \leqslant \mathbf{Z}(\mathbf{C}_G(x))$. Clearly, $\mathbf{C}_G(x)$ is nilpotent.

Step 2. G/N is abelian.
For every $y \in G \setminus N$, we see that $\mathbf{C}_G(y)$ is maximal in G and $\mathbf{C}_G(y)N/N \leqslant \mathbf{C}_{G/N}(yN)$. If $N \leqslant \mathbf{C}_G(y)$, then $y \in \mathbf{C}_G(N)$; if $N \nleqslant \mathbf{C}_G(y)$, then $G = \mathbf{C}_G(y)N$, forcing $\mathbf{C}_{G/N}(yN) = G/N$. Therefore, $yN \in \mathbf{Z}(G/N) := Z/N$, yielding to $y \in Z$. As a result, $G = Z \cup \mathbf{C}_G(N)$, which implies that $G = Z$ or $G = \mathbf{C}_G(N)$. Since $\mathbf{Z}(G) < N$, we obtain that $G = Z$. Hence, $G/N = Z/N = \mathbf{Z}(G/N)$. Consequently, G/N is abelian, as required.

Step 3. G is solvable.
Assume false. If there exists a $2'$-element $x_0 \in G \setminus N$ having prime power order, then by step 1, we see that $\mathbf{C}_G(x_0)$ is nilpotent. Write $\mathbf{C}_G(x_0) = T_0 \times U_0$, where T_0 is the Sylow 2-subgroup of $\mathbf{C}_G(x_0)$ and U_0 is the Hall $2'$-subgroup of $\mathbf{C}_G(x_0)$. By [**23**, Theorem 1], we obtain that $U_0 \trianglelefteq G$, $\mathbf{Z}(U_0) \leqslant \mathbf{Z}(G)$, and $G/\mathbf{Z}(U_0) \cong G/U_0 \times U_0/\mathbf{Z}(U_0)$.

Note that G is non-solvable. So is G/U_0. As $T_0U_0/U_0 = \mathbf{C}_G(x_0)/U_0$ is a maximal 2-subgroup of G/U_0, we assert that T_0U_0/U_0 must be a Sylow 2-subgroup of G/U_0, which indicates that T_0U_0/U_0 is not normal in G/U_0 as G/U_0 is non-solvable. On the contrary, by step 1, T_0 is abelian, so is T_0U_0/U_0. Hence, $\mathbf{N}_{G/U_0}(T_0U_0/U_0) \geqslant \mathbf{C}_{G/U_0}(T_0U_0/U_0) \geqslant T_0U_0/U_0$, which yields to $\mathbf{N}_{G/U_0}(T_0U_0/U_0) = \mathbf{C}_{G/U_0}(T_0U_0/U_0)$. By [**19**, Theorem 7.2.1], we see that G/U_0

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

has a normal 2-complement, against the fact that G/U_0 is non-solvable. Consequently, each element in $G \setminus N$ is a 2-element. By corollary 1, N is solvable, so is G by step 2, which is a contradiction.

Step 4. If G is nilpotent, then G/N is a p-group and $G = P \times \mathbf{Z}(G)_{p'}$ with $P \in \mathrm{Syl}_p(G)$. Furthermore, $\mathbf{C}_G(x)$ is a normal maximal subgroup of G for any $x \in P \setminus N$ satisfying $|G/\mathbf{C}_G(x)| = p$.
Assume that G is nilpotent. If $|\pi(G/N)| \geqslant 2$, then there exist two distinct primes p, $q \in \pi(G/N)$. Select $w \in P \setminus N$ a p-element and $v \in Q \setminus N$ a q-element, where P and Q are Sylow p-subgroup and Sylow q-subgroup of G, respectively. In this case, $wv = vw \in G \setminus N$, showing that $\mathbf{C}_G(v), \mathbf{C}_G(w)$, and $\mathbf{C}_G(vw)$ are all maximal subgroups of G. On the contrary, lemma 5.1 indicates that $\mathbf{C}_G(wv) = \mathbf{C}_G(w) \cap \mathbf{C}_G(v)$, which forces that $\mathbf{C}_G(wv) = \mathbf{C}_G(w) = \mathbf{C}_G(v)$. Write $G = P \times Q \times W$, where W is the Hall $\{p, q\}'$-subgroup of G. Clearly, $P \times W \leqslant \mathbf{C}_G(v)$ and $P \times Q \leqslant \mathbf{C}_G(w)$, forcing $v \in \mathbf{Z}(G) < N$. This contradiction deduces $|\pi(G/N)| = 1$. Moreover, G/N is a p-group. Let $N_{p'}$ be the Hall p'-subgroup of N. As $N_{p'}$ char $N \trianglelefteq G$, we obtain that $N_{p'}$ is a normal Hall p'-subgroup of G. Therefore, $G = P \times N_{p'}$, where P is the Sylow p-subgroup of G.

We claim that $N_{p'} \leqslant \mathbf{Z}(G)$. If not, select $y \in N_{p'} \setminus \mathbf{Z}(G)$. Suppose that $x \in P \setminus N$. By lemma 5.1, we see that $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$. Since $\mathbf{C}_G(x)$ and $\mathbf{C}_G(xy)$ are maximal subgroups of G, we get that $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \leqslant \mathbf{C}_G(y)$. Moreover, $\mathbf{C}_G(x) = \mathbf{C}_G(y)$ as $y \notin \mathbf{Z}(G)$. Clearly, $N_{p'} \leqslant \mathbf{C}_G(x)$ and $P \leqslant \mathbf{C}_G(y)$, we get $y \in \mathbf{Z}(G)$, a contradiction. Consequently, $N_{p'} \leqslant \mathbf{Z}(G)$, leading that $G/\mathbf{Z}(G)$ is a p-group. Notice that $\mathbf{C}_G(x)/\mathbf{Z}(G)$ is a maximal subgroup of $G/\mathbf{Z}(G)$. Then $\mathbf{C}_G(x)$ is a maximal normal subgroup of G. Moreover, $|G/\mathbf{C}_G(x)| = |(G/\mathbf{Z}(G))/(\mathbf{C}_G(x)/\mathbf{Z}(G))| = p$, as required.

Step 5. The conclusion when G is non-nilpotent.
In the following, we consider the case that G is non-nilpotent. We will divide the proof into two cases depending on $|\pi(G/N)| = 1$ or not.

Case 1. $|\pi(G/N)| = 1$.
In this case, G/N is a p-group. Let $w \in G \setminus N$ be an arbitrary p-element. By assumption, $\mathbf{C}_G(w)$ is a maximal subgroup of G, implying $|G : \mathbf{C}_G(w)| = r^a$ as G is solvable, where r is a prime and a is a positive integer.

Subcase 1.1. $r = p$.
Then $|G : \mathbf{C}_G(w)| = p^a$. By step 1, $\mathbf{C}_G(w) = P_w \times K_w$ is nilpotent with Sylow p-subgroup P_w and abelian Hall p'-subgroup K_w. Clearly, $K := K_w$ is a Hall p'-subgroup of G. Let P be a Sylow p-subgroup of G containing P_w. Then $P_w \lneqq P$, leading $\mathbf{N}_G(P_w) > \mathbf{C}_G(w)$. Since $\mathbf{C}_G(w)$ is maximal in G, we have $P_w \trianglelefteq G$, and thus $\mathbf{C}_G(P_w) \trianglelefteq G$. Note that $K \leqslant \mathbf{C}_G(P_w) \leqslant \mathbf{C}_G(w)$. As $\mathbf{C}_G(w)$ is nilpotent, so is $\mathbf{C}_G(P_w)$. Hence, $K \trianglelefteq G$ because K char $\mathbf{C}_G(P_w) \trianglelefteq G$. As a result, $K = \mathbf{O}_{p'}(G)$ and $G = K \rtimes P$.

Clearly, $P \ntrianglelefteq G$, since otherwise, $G = P \times K$ with $K \leqslant \mathbf{Z}(G)$, implying that G is nilpotent, against our assumption. Hence, $\mathbf{O}_p(G) \lneqq P$. Along with the fact that $\mathbf{C}_G(w) = P_w \times K \leqslant \mathbf{O}_p(G) \times \mathbf{O}_{p'}(G) \lneqq G$. The maximality of $\mathbf{C}_G(w)$ indicates that $\mathbf{C}_G(w) = \mathbf{O}_p(G) \times \mathbf{O}_{p'}(G) \trianglelefteq G$. In particular, $|G : \mathbf{C}_G(w)| = p$ and $|P : \mathbf{O}_p(G)| = p$.

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

Let $N_0 := \mathbf{O}_p(G)\mathbf{Z}(G)$. Then $N_0 \trianglelefteq G$. Let further $\overline{G} := G/N_0$. Easily, $|\overline{P}| = p$. We show that $\overline{G} = \overline{K} \rtimes \overline{P}$ is a Frobenius group with abelian kernel $\overline{K}$ and complement $\overline{P}$. Otherwise, there must exist $k \in K \setminus N_0$ and $y \in P \setminus N_0$ such that $[k, y] \in N_0$. Since $K \trianglelefteq G$, we see that $[k, y]$ is a p'-element, forcing $[k, y] \in \mathbf{Z}(G)$. Then $1 = [k^{o(k)}, y] = [k, y]^{o(k)}$ and $1 = [k, y^{o(y)}] = [k, y]^{o(y)}$, forcing $[k, y] = 1$. Recall that $K = \mathbf{O}_{p'}(G)$ is abelian and $y \notin \mathbf{O}_p(G)$, we have $\mathbf{C}_G(k) \geqslant \langle \mathbf{C}_G(w), y\rangle > \mathbf{C}_G(w)$. The maximality of $\mathbf{C}_G(w)$ forces $k \in \mathbf{Z}(G) \leqslant N_0$, against the choice of k. Hence, statement (1.1) of the theorem holds.

Subcase 1.2. $r \neq p$.
In this case, $w \in G\backslash N$ is a p-element such that $|G : \mathbf{C}_G(w)| = r^a$ is a p'-number. By step 1, $\mathbf{C}_G(w) = P_w \times K_w$ is nilpotent with Sylow p-subgroup P_w and abelian Hall p'-subgroup K_w, showing that $P := P_w$ is a Sylow p-subgroup of G.

Assume first $P \trianglelefteq G$. Then $\mathbf{C}_G(P) \trianglelefteq G$. As $K_w \leqslant \mathbf{C}_G(P) \leqslant \mathbf{C}_G(w)$ and $\mathbf{C}_G(w)$ is nilpotent, we see that K_w char $\mathbf{C}_G(P) \trianglelefteq G$, yielding $K_w \trianglelefteq G$. Hence, $\mathbf{C}_G(w) = P \times K_w \leqslant P \times \mathbf{O}_{p'}(G)$. Let K be a Hall p'-subgroup of G containing K_w. If $K \trianglelefteq G$, then $G = P \times K$. Under this situation, $K \leqslant \mathbf{C}_G(w)$, forcing $G = \mathbf{C}_G(w)$. This contradiction indicates that $\mathbf{C}_G(w) = P \times \mathbf{O}_{p'}(G) \trianglelefteq G$ with $|K/\mathbf{O}_{p'}(G)| = r$.

Let $N_0 := \mathbf{O}_{p'}(G)(P \cap N)$. Then $N_0 \trianglelefteq G$. Write $\overline{G} := G/N_0$. Then $\overline{N} = \overline{K} \trianglelefteq \overline{G}$ as G/N is a p-group. Moreover, $\overline{G} = \overline{P} \times \overline{K}$ is a $\{p, r\}$-group. Take $z \in G \setminus N_0$ a $\{p, r\}$-element. Write $z = ab$, where $a \in P \setminus N_0$ is the p-part and $b \in K \setminus N_0$ is the r-part of z, respectively. By step 1, we see that $\mathbf{C}_G(a) = P_a \times K_a$, where P_a is the Sylow p-subgroup of $\mathbf{C}_G(a)$ and K_a is the Hall p'-subgroup of $\mathbf{C}_G(a)$. Note that $a \in \mathbf{C}_G(w)$. Then $K_w \leqslant K_a$. Analogously, $K_a \leqslant K_w$ since $w \in \mathbf{C}_G(a)$. This deduces that $b \in K_a = K_w = \mathbf{O}_{p'}(G) \leqslant N_0$, against our assumption. As a consequence, $P \ntrianglelefteq G$.

Recall that $\mathbf{C}_G(w) = P \times K_w$ with K_w abelian. Then $\mathbf{C}_G(w) \leqslant \mathbf{C}_G(K_w)$, leading to $\mathbf{C}_G(K_w) = \mathbf{C}_G(w)$ or $K_w \leqslant \mathbf{Z}(G)$ by the maximality of $\mathbf{C}_G(w)$. Assume first the former holds. Let R_0 be the Sylow r-subgroup of K_w and R be a Sylow r-subgroup of G such that $R_0 \lneqq R$. This indicates that $\mathbf{C}_G(w) < \mathbf{N}_G(R_0)$. Furthermore, the maximality of $\mathbf{C}_G(w)$ forces $R_0 \trianglelefteq G$, and thus $\mathbf{C}_G(R_0) \trianglelefteq G$.

On the contrary, K_w is abelian, implying $\mathbf{C}_G(w) \leqslant \mathbf{C}_G(R_0)$. Again by the maximality of $\mathbf{C}_G(w)$, we see that either $R_0 \leqslant \mathbf{Z}(G)$ or $\mathbf{C}_G(w) = \mathbf{C}_G(R_0)$. If the latter holds, then $\mathbf{C}_G(R_0) = \mathbf{C}_G(w) \trianglelefteq G$. Since $\mathbf{C}_G(w)$ is nilpotent, we obtain that $P \trianglelefteq G$, against our assumption.

As a result, $R_0 \leqslant \mathbf{Z}(G)$. Write $\widetilde{G} := G/R_0$. Then $\widetilde{G} = \widetilde{R}\mathbf{C}_{\widetilde{G}}(\widetilde{w})$. Since $\mathbf{C}_{\widetilde{G}}(\widetilde{w}) = \mathbf{C}_G(w)/R_0$ is maximal in $\widetilde{G}$, we have that $\mathbf{O}_r(\widetilde{G}) = \widetilde{1}$ or $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. Assume first that $\mathbf{O}_r(\widetilde{G}) = \widetilde{1}$. Since G is solvable, we have $\mathbf{O}_{r'}(\widetilde{G}) > \widetilde{1}$. We assert that $\widetilde{P} \ntrianglelefteq \widetilde{G}$. Since otherwise, $P \times R_0 \trianglelefteq G$, leading $P \trianglelefteq G$, contrary to our assumption.

Consequently, $\widetilde{P} \nleq \mathbf{O}_{r'}(\widetilde{G})$. Note that $\widetilde{P}\mathbf{O}_{r'}(\widetilde{G}) \leqslant \mathbf{C}_{\widetilde{G}}(\widetilde{w})$ and $\mathbf{C}_{\widetilde{G}}(\widetilde{w})$ is nilpotent. Then $\widetilde{J} = \widetilde{P} \times \widetilde{L}$ with $\widetilde{L}$ abelian, where L is a Hall $\{p, r\}'$-subgroup of $\mathbf{O}_{r'}(\widetilde{G})$. Easily, $\widetilde{L} \trianglelefteq \widetilde{G}$ and $\mathbf{C}_{\widetilde{G}}(\widetilde{L}) \geqslant \mathbf{C}_{\widetilde{G}}(\widetilde{w})$. By the maximality of $\mathbf{C}_{\widetilde{G}}(\widetilde{w})$, we have $\mathbf{C}_{\widetilde{G}}(\widetilde{L}) = \mathbf{C}_{\widetilde{G}}(\widetilde{w})$ or $\widetilde{L} \leqslant \mathbf{Z}(\widetilde{G})$. Assume first that $\mathbf{C}_{\widetilde{G}}(\widetilde{L}) = \mathbf{C}_{\widetilde{G}}(\widetilde{w})$ holds. In this situation, $\mathbf{C}_{\widetilde{G}}(\widetilde{L}) \trianglelefteq \widetilde{G}$, we get $\widetilde{P} \trianglelefteq \widetilde{G}$, against the argument in the previous paragraph. Hence, $\widetilde{L} \leqslant \mathbf{Z}(\widetilde{G})$. Therefore, $\mathbf{O}_{r'}(\widetilde{G}) = \mathbf{O}_p(\widetilde{G}) \times \mathbf{Z}(\widetilde{G})_{p'}$. Furthermore, $\mathbf{C}_{\widetilde{G}}(\mathbf{O}_{r'}(\widetilde{G})) \leqslant \mathbf{O}_{r'}(\widetilde{G})$, which indicates that the Hall $\{p, r\}'$-subgroup of $\mathbf{O}_{r'}(\widetilde{G})$

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

is also a Hall $\{p,\, r\}'$-subgroup of $\widetilde{G}$. Hence, $\mathbf{C}_{\widetilde{G}}(\widetilde{w}) = \widetilde{P} \times \mathbf{Z}(\widetilde{G})_{p'}$. Without loss of generality, we may assume that $\mathbf{Z}(\widetilde{G})_{p'} = \widetilde{1}$.

In this case, $\mathbf{C}_{\widetilde{G}}(\widetilde{w}) = \widetilde{P}$, and thus $\widetilde{G} = \widetilde{R}\widetilde{P}$, leading that $\widetilde{G}$ is a $\{p,\, r\}$-group. Recall that $\mathbf{O}_r(\widetilde{G}) = 1$, we have $\mathbf{O}_p(\widetilde{G}) \neq 1$. As a result, $\mathbf{O}_p(\widetilde{G}) < \widetilde{P}$ since $\widetilde{P} \ntrianglelefteq G$. Note that $\mathbf{O}_p(\widetilde{G}/\mathbf{O}_p(\widetilde{G})) = 1$. We see that $\mathbf{O}_r(\widetilde{G}/\mathbf{O}_p(\widetilde{G})) \neq 1$ as $\widetilde{G}$ is solvable. Therefore, $\widetilde{G}/\mathbf{O}_p(\widetilde{G}) = \mathbf{O}_{p,r}(\widetilde{G})/\mathbf{O}_p(\widetilde{G}) \rtimes \mathbf{C}_{\widetilde{G}}(\widetilde{w})/\mathbf{O}_p(\widetilde{G})$ by the maximality of $\mathbf{C}_{\widetilde{G}}(\widetilde{w})/\mathbf{O}_p(\widetilde{G})$. Furthermore, $\mathbf{C}_{\widetilde{G}}(\widetilde{w})/\mathbf{O}_p(\widetilde{G})$ acts irreducibly on $\mathbf{O}_{p,r}(\widetilde{G})/\mathbf{O}_p(\widetilde{G})$ and $\mathbf{O}_{p,r}(\widetilde{G})/\mathbf{O}_p(\widetilde{G})$ is an elementary abelian r-group. Since $\mathbf{O}_{p,r}(\widetilde{G})/\mathbf{O}_p(\widetilde{G}) \cong \widetilde{R}$, we get that $\widetilde{R}$ is elementary abelian. Clearly, $\widetilde{G}/\mathbf{O}_p(\widetilde{G})$ is not nilpotent. Hence, the Fitting length $h(\widetilde{G}) = 3$. In this case, $\widetilde{G}$ has the following normal series:

$$\widetilde{1} \trianglelefteq \mathbf{O}_p(\widetilde{G}) \trianglelefteq \mathbf{O}_{p,r}(\widetilde{G}) \trianglelefteq \widetilde{G} = \mathbf{O}_{p,r,p}(\widetilde{G}),$$

where $\mathbf{O}_{p,r}(\widetilde{G})/\mathbf{O}_p(\widetilde{G})$ is an elementary abelian Sylow r-group and $\widetilde{G}/\mathbf{O}_{p,r}(\widetilde{G})$ is a p-group, as required in Statement (1.2.1).

Assume now that $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. Then $\widetilde{G} = \mathbf{O}_r(\widetilde{G}) \rtimes \mathbf{C}_{\widetilde{G}}(\widetilde{w})$. By the same reason as above, $\mathbf{C}_{\widetilde{G}}(\widetilde{w})$ acts irreducibly on $\mathbf{O}_r(\widetilde{G})$ and $\mathbf{O}_r(\widetilde{G})$ is an elementary abelian r-group. In this case, $\widetilde{G}$ has the following normal series:

$$\widetilde{1} \trianglelefteq \widetilde{R} \trianglelefteq \widetilde{N} \trianglelefteq \widetilde{G}$$

where $\widetilde{R}$ is an elementary abelian r-group and G/R is nilpotent, statement (1.2.2) holds.

Now, we consider the case that $K_w \leqslant \mathbf{Z}(G)$. Let $l \in G \setminus N$ be an arbitrary primary element. As $\mathbf{Z}(G) < N$, then $l \notin \mathbf{Z}(G)$. By assumption, $\mathbf{C}_G(l)$ is maximal in G. Recall that G is solvable. Therefore, $|G : \mathbf{C}_G(l)|$ is a prime power. If $|G : \mathbf{C}_G(l)|$ is a p-power, we are done according to case 1. As a result, $|G : \mathbf{C}_G(l)|$ is a q-power with prime q distinct from p. Moreover, $\mathbf{C}_G(l)^g = P^g \times K_w \leqslant \mathbf{C}_G(l)$ for some $g \in G$, forcing $|G : \mathbf{C}_G(l)|$ is a power of r. By [**24**, Lemma 2.5], $l \in \mathbf{O}_{r,r'}(G)$, yielding $G \setminus N \subseteq \mathbf{O}_{r,r'}(G)$. Furthermore, $G = \mathbf{O}_{r,r'}(G)$, implying that G has a normal Sylow r-subgroup R.

Let $\widehat{G} := G/\mathbf{Z}(G)$. Since $\mathbf{C}_G(l)$ is maximal in G and $\mathbf{C}_{\widehat{G}}(\widehat{l}) \geqslant \mathbf{C}_G(l)/\mathbf{Z}(G)$, we obtain that $\mathbf{C}_{\widehat{G}}(\widehat{l}) = \widehat{G}$ or $\mathbf{C}_{\widehat{G}}(\widehat{l}) = \mathbf{C}_G(l)/\mathbf{Z}(G)$. Note that $\mathbf{C}_G(l) = P^g \times K_w$, we conclude that $\mathbf{C}_{\widehat{G}}(\widehat{l}) = \mathbf{C}_G(l)/\mathbf{Z}(G)$ and thus $\mathbf{C}_{\widehat{G}}(\widehat{l}) = \widehat{P}$ is maximal in $\widehat{G}$. Then $\pi(\widehat{G}) = \{p,\, r\}$, and $\widehat{P}$ acts on $\widehat{R}$ irreducibly. Moreover, $\widehat{R}$ is the minimal normal subgroup of $\widehat{G}$, and thus $\widehat{R}$ is elementary abelian. Let $N_p = N \cap P$. Then $\widehat{N_p} \trianglelefteq \widehat{P}$. If $\widehat{N_p} \trianglelefteq \widehat{G}$, then $\widehat{G}/\widehat{N_p}$ is a Frobenius group. If $\widehat{N_p} \ntrianglelefteq \widehat{G}$, then $\mathbf{N}_{\widehat{G}}(\widehat{N_p}) = \widehat{P}$, statement (1.2.3) holds.

Case 2. $|\pi(G/N)| \geqslant 2$.
Let $\pi := \pi(G/N)$. Recall that G/N is abelian. There must exist an element $wN \in G/N$ such that $\pi(wN) = \pi$. Without loss, we may consider $w \in G \setminus N$ is an element with $\pi(w) = \pi$. Suppose that $w_p,\, w_q \in G \setminus N$ is the p-part and the q-part of w, respectively. By lemma 5.1, we have $\mathbf{C}_G(w) \leqslant \mathbf{C}_G(w_p) \cap \mathbf{C}_G(w_q)$. Note that all of $\mathbf{C}_G(w)$, $\mathbf{C}_G(w_p)$, and $\mathbf{C}_G(w_q)$ are maximal subgroups of G. This indicates that

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

$\mathbf{C}_G(w) = \mathbf{C}_G(w_p) = \mathbf{C}_G(w_q)$. In particular, $\mathbf{C}_G(w)$ is abelian according to step 1. Recall that G is solvable and $\mathbf{C}_G(w)$ is a maximal subgroup of G, indicating that $|G : \mathbf{C}_G(w)| = r^a$, where r is a prime and $a > 0$ is an integer.

Write $\mathbf{C}_G(w) = K \times R_w$, where K is the abelian Hall r'-subgroup of G and R_w is the Sylow r-subgroup of $\mathbf{C}_G(w)$. Let R be a Sylow r-subgroup of G such that $R_w < R$. Easily, $\mathbf{N}_G(R_w) \gneq \mathbf{C}_G(w)$. The maximality of $\mathbf{C}_G(w)$ yields to $R_w \trianglelefteq G$. Moreover, $\mathbf{C}_G(w) \leqslant \mathbf{C}_G(R_w) \trianglelefteq G$. Again by the maximality of $\mathbf{C}_G(w)$, we get $\mathbf{C}_G(R_w) = \mathbf{C}_G(w)$ or $R_w \leqslant \mathbf{Z}(G)$.

Subcase 2.1. $\mathbf{C}_G(R_w) = \mathbf{C}_G(w)$.
In this case, $\mathbf{C}_G(w) \trianglelefteq G$ and $|G : \mathbf{C}_G(w)| = r$. Furthermore, $K \trianglelefteq G$ since K is the Hall r'-subgroup of abelian group $\mathbf{C}_G(w)$, implying $G = K \rtimes R$. Notice that $\mathbf{C}_G(w) = K \times R_w \leqslant \mathbf{O}_{r'}(G) \times \mathbf{O}_r(G)$. Since G is non-nilpotent and $\mathbf{C}_G(w)$ is maximal, we see that $R_w = \mathbf{O}_r(G)$ with $|R : R_w| = r$.

Let $N_0 := \mathbf{Z}(G)\mathbf{O}_r(G)$ and $\overline{G} := G/N_0$. Then $\overline{G} = \overline{K} \rtimes \overline{R}$ with $|\overline{R}| = r$. Assume that there exists an $\{r, t\}$-element $\overline{e} \in \overline{G}$ for some prime $t \neq r$. We may assume that $e \in G \setminus N_0$ is an $\{r, t\}$-element. Write $e = e_1e_2$, where $e_1 \in R \setminus N_0$ and $e_2 \in K \setminus N_0$ are the r-part and the t-part of e, respectively. In this case, $e_1 \in \mathbf{C}_G(e_2) = \mathbf{C}_G(w)$, forcing $e_1 \in \mathbf{O}_r(G)$. This contradiction shows that $\overline{G}$ is a Frobenius group with abelian kernel $\overline{K}$ and a complement $\overline{R}$ of order r, statement (2.1) of the theorem holds.

Subcase 2.2. $R_w \leqslant \mathbf{Z}(G)$.
Recall that $w \in G \setminus N$ with $\pi(w) = \pi$. We assert that $r \notin \pi$. If not, assume that $w_r \in G \setminus N$ is the r-part of w. Easily, $w_r \in \mathbf{C}_G(w) = K \times R_w$, forcing $w_r \in R_w \leqslant \mathbf{Z}(G) < N$, which is a contradiction. As a result, $r \nmid |G/N|$, leading $R \leqslant N$. Moreover, $G = \mathbf{C}_G(w)N = \mathbf{C}_G(w)R$.

Write $\widetilde{G} := G/\mathbf{Z}(G)$. Then $\widetilde{G} = \widetilde{\mathbf{C}_G(w)}\widetilde{R}$. Easily, $\widetilde{\mathbf{C}_G(w)}$ is a maximal r'-subgroup of $\widetilde{G}$, implying $\mathbf{O}_r(\widetilde{G}) = \widetilde{1}$ or $\widetilde{R}$. Assume first $\mathbf{O}_r(\widetilde{G}) = \widetilde{1}$. Then $\mathbf{O}_{r'}(\widetilde{G}) > \widetilde{1}$ since G is solvable. In particular, $\widetilde{K} \leqslant \mathbf{C}_{\widetilde{G}}(\mathbf{O}_{r'}(\widetilde{G})) \leqslant \mathbf{O}_{r'}(\widetilde{G}) \leqslant \widetilde{K}$ since $\widetilde{K}$ is an abelian Hall r'-subgroup of $\widetilde{G}$, yielding $\mathbf{C}_{\widetilde{G}}(\mathbf{O}_{r'}(\widetilde{G})) = \mathbf{O}_{r'}(\widetilde{G}) = \widetilde{K}$. As a result, $K \trianglelefteq G$ and $\mathbf{C}_G(w) \trianglelefteq G$.

Consequently, $\widetilde{G} = \widetilde{\mathbf{C}_G(w)} \rtimes \widetilde{R}$. We prove that $\widetilde{G}$ is a Frobenius group. Suppose false, there exists an $\{r, t\}$-element $\widetilde{e} \in \widetilde{G}$ for some prime $t \neq r$. We may assume that e is a $\{r, t\}$-element. Write $e = e_1e_2$, where $e_1 \in R \setminus \mathbf{Z}(G)$ and $e_2 \in K \setminus \mathbf{Z}(G)$ are the r-part and the t-part of e, respectively. Note that $e_1 \in \mathbf{C}_G(e_2) = K \times R_w = \mathbf{C}_G(w)$. This contradiction shows that $\widetilde{G}$ is a Frobenius group with abelian kernel $\widetilde{\mathbf{C}_G(w)}$. Moreover, $\widetilde{\mathbf{C}_G(w)}$ is maximal in $\widetilde{G}$ indicates that $\widetilde{R}$ is of order r, statement (2.1) of the theorem holds.

Now, we consider $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. Write $\widetilde{G} = \widetilde{R} \rtimes \widetilde{\mathbf{C}_G(w)}$. Since $\widetilde{\mathbf{C}_G(w)}$ is maximal in $\widetilde{G}$, we see that $\widetilde{\mathbf{C}_G(w)}$ acts irreducibly on $\widetilde{R}$. By the same argument in the previous paragraph, we conclude that $\widetilde{G}$ is a Frobenius group with kernel $\widetilde{R}$ and complement $\widetilde{\mathbf{C}_G(w)}$. Furthermore, $\widetilde{R}$ is a minimal normal subgroup of $\widetilde{G}$ and thus $\widetilde{R}$ is elementary abelian by the maximality of $\widetilde{\mathbf{C}_G(w)}$. Also $\widetilde{\mathbf{C}_G(w)}$ is cyclic as $\widetilde{\mathbf{C}_G(w)}$ is abelian, implying that G/N is cyclic, statement (2.2) of the theorem holds. $\square$

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

5. Proofs of theorems E and C

To prove theorem E and corollary 2, here we list several lemmas, which will be used in the sequel.

LEMMA 5.1. *Let G be a group. If $x, y \in G$ such that $[x, y] = 1$ and $(o(x), o(y)) = 1$, then $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$.*

LEMMA 5.2 ([**15**, Proposition 2]). *Let G be a non-solvable CP-group. Then there exist normal subgroups B and C of G such that $1 \trianglelefteq B \trianglelefteq C \trianglelefteq G$, where B is a 2-subgroup of G, C/B is a non-abelian simple group, and G/C is a p-group for some prime p. In particular, G/C is either cyclic or a generalized quaternion group.*

LEMMA 5.3 ([**15**, Proposition 3]). *If G is a non-abelian simple CP-group, then G is isomorphic to one of the following groups: $PSL_2(q)$, for $q = 5, 7, 8, 9, 17$, $PSL_3(4)$, $Sz(8)$, or $Sz(32)$.*

Proof of theorem E. First consider $\overline{N}$ is nilpotent. Then N is also nilpotent. If there exist two distinct primes $p_1, p_2 \in \pi(\overline{N})$, we may take $a_1 \in P_1 \setminus \mathbf{Z}(N)$ and $a_2 \in P_2 \setminus \mathbf{Z}(N)$, where P_1 and P_2 are Sylow p_1 and p_2-subgroups of N, respectively. By lemma 5.1, $\mathbf{C}_G(a_1a_2) = \mathbf{C}_G(a_1) \cap \mathbf{C}_G(a_2) \leqslant \mathbf{C}_G(a_i)$ for $i = 1, 2$. As all of $\mathbf{C}_G(a_1a_2)$, $\mathbf{C}_G(a_1)$, $\mathbf{C}_G(a_2)$ are maximal in G, we have $\mathbf{C}_G(a_1a_2) = \mathbf{C}_G(a_1) = \mathbf{C}_G(a_2)$. This indicates that $P_1 \leqslant \mathbf{C}_G(a_2) = \mathbf{C}_G(a_1)$, forcing $a_1 \in \mathbf{Z}(N)$. This contradiction deduces that $\overline{N}$ is a p-group.

Let $P \in \mathrm{Syl}_p(N)$. Then $\overline{N} = \overline{P}$. On the contrary, P char $N \trianglelefteq G$, implying $P \trianglelefteq G$ and thus $\Phi(P) \leqslant \Phi(G) \cap N$. Along with the fact that $\Phi(G) \leqslant \mathbf{C}_G(u)$ for any $u \in N \setminus \mathbf{Z}(N)$, we obtain that $\Phi(P) \leqslant \mathbf{C}_N(u)$, yielding $\Phi(P) \leqslant \mathbf{Z}(N)$ by the choice of u. In this case, $\overline{N} = \overline{P} = P/\mathbf{Z}(N) \cong (P/\Phi(P)/(\mathbf{Z}(N)/\Phi(P))$ is elementary abelian, statement (1) of the theorem holds.

Now, we consider that $\overline{N}$ is non-nilpotent. Let $x \in N \setminus \mathbf{Z}(N)$ be an arbitrary element. Write $x = x_1 \cdots x_s$, where $x_1, \ldots, x_s$ are distinct components of x. Since $x \notin \mathbf{Z}(N)$, without loss of generality, we may consider $x_1 \notin \mathbf{Z}(N)$. By lemma 5.1, we see that $\mathbf{C}_G(x) = \mathbf{C}_G(x_1) \cap \cdots \cap \mathbf{C}_G(x_s) \leqslant \mathbf{C}_G(x_1)$. Since both $\mathbf{C}_G(x)$ and $\mathbf{C}_G(x_1)$ are maximal subgroups of G, we have $\mathbf{C}_G(x) = \mathbf{C}_G(x_1)$. Consequently, x can be assumed to be a p-element with prime p.

In the following, we distinguish the proof into two cases:

Case 1. $\overline{\mathbf{C}_N(v)}$ is an r-group, for any r-element $v \in N \setminus \mathbf{Z}(N)$ with prime r.

Step 1. $\overline{N}$ is a CP-group.
Assume false. Then there exists an element $\overline{z} \in \overline{N}$ of order $q^a r^b$, where $q, r \in \pi(\overline{N})$ and $a, b > 0$ are positive integers. By assumption, $\overline{\mathbf{C}_N(z^{q^a})}$ is an r-group. Notice that $\overline{z} \in \overline{\mathbf{C}_N(z^{q^a})}$, which is a contradiction because $qr \mid o(\overline{z})$.

Step 2. $\overline{N}$ is solvable.
Assume on the contrary that $\overline{N}$ is non-solvable. By lemma 5.2, $\overline{N}$ has normal subgroups $\overline{B}$, $\overline{C}$ such that $\overline{1} \trianglelefteq \overline{B} \trianglelefteq \overline{C} \trianglelefteq \overline{N}$, where $\overline{B}$ is a 2-group, $\overline{C}/\overline{B}$ is non-abelian and simple, and $\overline{N}/\overline{C}$ is a q_1-group for some prime q_1, which is cyclic or generalized quaternion.

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

Suppose that $\overline{B} \neq \overline{1}$. Let $v \in N \setminus \mathbf{Z}(N)$ be a q-element for some odd prime q. Then $\mathbf{C}_G(v)$ is maximal in G such that $B \nleq \mathbf{C}_G(v)$ by step 1. Therefore, $G = B\mathbf{C}_G(v)$ as $\mathbf{C}_G(v)$ is maximal in G, yielding $N = B\mathbf{C}_N(v)$. In this case, $|\overline{N} : \overline{\mathbf{C}_N(v)}| = |N : \mathbf{C}_N(v)| = |B : \mathbf{C}_B(v)|$ is a 2-number. This shows that $|\overline{N}|$ has exactly two prime divisors, against the fact that $\overline{N}$ is non-solvable.

As a result, $\overline{B} = \overline{1}$, and $\overline{C}$ is a non-abelian simple CP-group. By lemma 5.3, $\overline{C}$ is one of the following groups: $PSL_2(q)$, for $q = 5, 7, 8, 9, 17$, $PSL_3(4), Sz(8)$, or $Sz(32)$. Recall that $\overline{G}/\mathbf{C}_{\overline{G}}(\overline{C}) \leqslant \mathrm{Aut}(\overline{C})$. As $\overline{C} \cap \mathbf{C}_{\overline{G}}(\overline{C}) = \overline{1}$ and $\overline{C} \times \mathbf{C}_{\overline{G}}(\overline{C}) \trianglelefteq \overline{G}$, we see that $\mathbf{C}_{\overline{G}}(\overline{C}) = \overline{1}$ by step 1 and thus $\overline{C} \leqslant \overline{G} \leqslant \mathrm{Aut}(\overline{C})$. Moreover, for any $z \in C \setminus \mathbf{Z}(N)$, we see that $\mathbf{C}_{\overline{G}}(\overline{z}) \geqslant \overline{\mathbf{C}_G(z)}$ and $\overline{z} \notin \mathbf{Z}(\overline{N})$. Since $\mathbf{C}_G(z)$ is maximal in G, we obtain that $\mathbf{C}_{\overline{G}}(\overline{z}) = \overline{\mathbf{C}_G(z)}$ is also maximal in $\overline{G}$.

If $\overline{C} \cong PSL_2(5)$, then $\overline{G} \leqslant S_5$. However, by [4], $\mathbf{C}_{\overline{G}}(\overline{v})$ is not maximal in $\overline{G}$ for a 5-element $v \in C \setminus \mathbf{Z}(N)$, a contradiction. By the same reason, we can rule out the cases $\overline{C} \cong PSL_2(q)$, when $q = 7, 8, 9, 17$, and $Sz(8)$ or $Sz(32)$. For the remaining case $\overline{C} \cong PSL_3(4)$, we can find an element $u \in C \setminus \mathbf{Z}(N)$ with order 2 such that $\overline{\mathbf{C}_G(v)}$ is not maximal in $\overline{G}$ according to [4], also a contradiction.

Step 3. The conclusion of case 1.
Let $\overline{S}$ be a minimal normal subgroup of $\overline{G}$ contained in $\overline{N}$. Then $\overline{S}$ is an elementary abelian s-group for some prime s. Since $\overline{N}$ is non-nilpotent, there must exist an s_1-element $a \in N \setminus \mathbf{Z}(N)$ with $s_1 \neq s$. By assumption, $\overline{\mathbf{C}_N(a)}$ is an s_1-subgroup, indicating that $\overline{S} \nleq \overline{\mathbf{C}_G(a)}$, and thus $S \nleq \mathbf{C}_G(a)$. Hence, $G = S\mathbf{C}_G(a)$ by the maximality of $\mathbf{C}_G(a)$, yielding $\overline{N} = \overline{S}\overline{\mathbf{C}_N(a)}$. In particular, $\overline{N} = \overline{S} \rtimes \overline{\mathbf{C}_N(a)}$ is a Frobenius group with complement $\overline{\mathbf{C}_N(a)}$ by step 1.

We show that $|\overline{\mathbf{C}_N(a)}| = s_1$. For every $a_1 \in \mathbf{C}_N(a) \setminus \mathbf{Z}(N)$, the similar argument in the previous paragraph deduces that $\overline{N} = \overline{S} \rtimes \overline{\mathbf{C}_N(a_1)}$ is also a Frobenius group. Since both $\overline{\mathbf{C}_N(a)}$ and $\overline{\mathbf{C}_N(a_1)}$ are Frobenius complement of $\overline{N}$, and $\overline{a_1} \in \overline{\mathbf{C}_N(a)} \cap \overline{\mathbf{C}_N(a_1)} \neq \overline{1}$, we have $\overline{\mathbf{C}_N(a)} = \overline{\mathbf{C}_N(a_1)}$, forcing $\overline{\mathbf{C}_N(a)}$ is abelian. By [17, Theorem 5.8.7], $\overline{\mathbf{C}_N(a)}$ is cyclic.

Let $1 \neq \overline{d} \in \overline{S}$. Without loss, we assume that $d \in S \setminus \mathbf{Z}(N)$. Then $\mathbf{C}_G(d)$ is maximal in G by hypothesis, forcing that $\overline{\mathbf{C}_G(d)}$ is maximal in $\overline{G}$. In particular, $\overline{G} = \overline{N}\overline{\mathbf{C}_G(d)}$. On the contrary, $\overline{\mathbf{C}_G(d)} \leqslant \mathbf{C}_{\overline{G}}(\overline{d})$ and $\overline{d} \notin \mathbf{Z}(\overline{N})$, we obtain that $\overline{S} \leqslant \mathbf{C}_{\overline{G}}(\overline{d}) = \overline{\mathbf{C}_G(d)}$. In this case, $\overline{G}/\overline{S} = \overline{N}/\overline{S} \rtimes \overline{\mathbf{C}_G(d)}/\overline{S}$. The maximality of $\overline{\mathbf{C}_G(d)}/\overline{S}$ indicates that $\overline{N}/\overline{S}$ is a minimal normal subgroup of $\overline{G}/\overline{S}$, so $\overline{N}/\overline{S}$ has prime power order. Recall that $\overline{N}/\overline{S} \cong \overline{\mathbf{C}_N(a)}$ is cyclic. Then $|\overline{\mathbf{C}_N(a)}| = s_1$, as required.

Case 2. There exists a p-element $x \in N \setminus \mathbf{Z}(N)$ such that $\overline{\mathbf{C}_N(x)}$ is not a p-group.

Step 4. $\mathbf{C}_N(x) = P_x \times H_x$, where $P_x \in \mathrm{Syl}_p(\mathbf{C}_N(x))$ and H_x is an abelian Hall p'-subgroup of $\mathbf{C}_N(x)$.
Let $y \in \mathbf{C}_N(x) \setminus \mathbf{Z}(N)$ be an arbitrary p'-element. By lemma 5.1, we have $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \cap \mathbf{C}_G(y) \leqslant \mathbf{C}_G(x)$. Notice that $\mathbf{C}_G(xy)$, $\mathbf{C}_G(x)$, $\mathbf{C}_G(y)$ are all maximal in G. We have $\mathbf{C}_G(xy) = \mathbf{C}_G(x) = \mathbf{C}_G(y)$, yielding $y \in \mathbf{Z}(\mathbf{C}_G(x))$. Moreover, $y \in \mathbf{Z}(\mathbf{C}_N(x))$. As a result, $\mathbf{C}_N(x) = P_x \times H_x$, where $P_x \in \mathrm{Syl}_p(\mathbf{C}_N(x))$ and H_x is an abelian Hall p'-subgroup of $\mathbf{C}_N(x)$.

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

Step 5. $\mathbf{C}_N(x) \leqslant \mathbf{Z}(\mathbf{C}_G(x))$.
By the assumption of case 2, we see that $H_x \nleq \mathbf{Z}(N)$. Take a q-element $v \in H_x \setminus \mathbf{Z}(N)$ with prime q. A similar argument as in the previous paragraph deduces that $\mathbf{C}_N(v) = Q_v \times K_v$, where $Q_v \in \mathrm{Syl}_q(\mathbf{C}_N(v))$ and K_v is an abelian Hall q'-subgroup of $\mathbf{C}_N(v)$. Notice that $x \in K_v$ and $v \in H_x$. Then $\mathbf{C}_N(v) \leqslant \mathbf{C}_N(x)$ and $\mathbf{C}_N(x) \leqslant \mathbf{C}_N(v)$. In particular, $\mathbf{C}_N(v) = \mathbf{C}_N(x) \leqslant \mathbf{Z}(\mathbf{C}_G(x))$.

Step 6. $\mathbf{C}_N(x) \cap \mathbf{C}_N(y) = \mathbf{Z}(N)$ for any $y \in N \setminus \mathbf{C}_N(x)$.
Assume false. Then there exists an element $z \in (\mathbf{C}_N(x) \cap \mathbf{C}_N(y)) \setminus \mathbf{Z}(N)$. Since $\mathbf{C}_N(x) \leqslant \mathbf{Z}(\mathbf{C}_G(x))$, we see that $\mathbf{C}_G(z) \geqslant \langle \mathbf{C}_G(x), y\rangle > \mathbf{C}_G(x)$. As $\mathbf{C}_G(x)$ is maximal in G, it follows that $\mathbf{C}_G(z) = G$, that is, $z \in \mathbf{Z}(N)$, a contradiction.

Step 7. The contradiction of case 2.
By step 5, $\mathbf{C}_N(x) \trianglelefteq \mathbf{C}_G(x)$, which implies that $\mathbf{N}_G(\mathbf{C}_N(x)) \geqslant \mathbf{C}_G(x)$. Consequently, $\mathbf{N}_G(\mathbf{C}_N(x)) = \mathbf{C}_G(x)$ or $\mathbf{C}_N(x) \trianglelefteq G$ as $\mathbf{C}_G(x)$ is maximal in G.

Assume first that $\mathbf{C}_N(x) \trianglelefteq G$. Then P_x is a normal subgroup of G according to step 4. Let $z \in N \setminus \mathbf{C}_N(x)$ be a primary element. Then $\mathbf{C}_G(z)$ is maximal in G. Moreover, $P_x \nleq \mathbf{C}_G(z)$ by step 6. As a result, $G = P_x\mathbf{C}_G(z)$, implying $N = P_x\mathbf{C}_N(z)$. Furthermore, by step 6, $\mathbf{C}_N(x) = \mathbf{Z}(N)P_x$, showing that $\overline{\mathbf{C}_N(x)}$ is a p-group, contrary to our assumption.

Hence, $\mathbf{N}_G(\mathbf{C}_N(x)) = \mathbf{C}_G(x)$, forcing $\mathbf{N}_N(\mathbf{C}_N(x)) = \mathbf{C}_N(x)$. Since $\mathbf{N}_{\overline{G}}(\overline{\mathbf{C}_N(x)}) \geqslant \overline{\mathbf{N}_G(\mathbf{C}_N(x))} = \overline{\mathbf{C}_G(x)}$, we get that $\mathbf{N}_{\overline{G}}(\overline{\mathbf{C}_N(x)}) = \overline{G}$ or $\mathbf{N}_{\overline{G}}(\overline{\mathbf{C}_N(x)}) = \overline{\mathbf{C}_G(x)}$. If the former holds, then $\mathbf{C}_N(x) \trianglelefteq G$, against our assumption. Hence, $\mathbf{N}_{\overline{G}}(\overline{\mathbf{C}_N(x)}) = \overline{\mathbf{C}_G(x)}$, yielding $\mathbf{N}_{\overline{N}}(\overline{\mathbf{C}_N(x)}) = \overline{\mathbf{C}_N(x)}$.

We claim that for any $g \in N \setminus \mathbf{C}_N(x)$, we always have $\mathbf{C}_N(x)^g \cap \mathbf{C}_N(x) = \mathbf{Z}(N)$. Let $d \in (\mathbf{C}_N(x)^g \cap \mathbf{C}_N(x)) \setminus \mathbf{Z}(N)$. Note that $\mathbf{C}_N(x) \leqslant \mathbf{Z}(\mathbf{C}_G(x))$. Then $\mathbf{C}_G(d) \geqslant \langle \mathbf{C}_G(x)^g, \mathbf{C}_G(x)\rangle$. Since $\mathbf{C}_G(x)$ is maximal and $\mathbf{C}_G(x)^g \neq \mathbf{C}_G(x)$, we have that $d \in \mathbf{Z}(G)$ and thus $d \in \mathbf{Z}(N)$, a contradiction. By [17, Theorem 5.7.6], $\overline{N}$ is a Frobenius group with a complement $\overline{\mathbf{C}_N(x)}$. Write $\overline{N} = \overline{T_x} \rtimes \overline{\mathbf{C}_N(x)}$, where $\overline{T_x}$ is the Frobenius kernel of $\overline{N}$. Let $\overline{Q} \in \mathrm{Syl}_q(\overline{T_x})$. Note that $\overline{\mathbf{C}_G(x)}$ is maximal in $\overline{G}$. Then $\overline{G} = \overline{Q} \rtimes \overline{\mathbf{C}_G(x)}$ by step 6. The maximality of $\overline{\mathbf{C}_G(x)}$ indicates that $\overline{T_x} = \overline{Q}$ is a minimal normal subgroup of $\overline{G}$. In particular, $\overline{T_x}$ is abelian.

Take $y \in T_x \setminus \mathbf{Z}(N)$. Then $N \nleq \mathbf{C}_G(y)$. The maximality of $\mathbf{C}_G(y)$ implies that $G = N\mathbf{C}_G(y)$, and thus $\overline{G}/\overline{T_x} \cong \overline{N}/\overline{T_x}\ \overline{\mathbf{C}_G(y)}/\overline{T_x}$. Notice that $\overline{\mathbf{C}_G(y)}/\overline{T_x}$ is a maximal subgroup of $\overline{G}/\overline{T_x}$. Then $\overline{N}/\overline{T_x}$ must be a minimal normal subgroup of $\overline{G}/\overline{T_x}$, forcing that $\overline{N}/\overline{T_x}$ has prime power order. However, $\overline{N}/\overline{T_x} \cong \overline{\mathbf{C}_N(x)}$ does not have prime power order, the final contradiction completes the proof. $\square$

As an application of theorem E, here we give a new proof of theorem C.

Proof of theorem C. Take $G = N$ in theorem E. Then $\overline{G}$ is either an elementary abelian p-group for some prime p, or $\overline{G} = \overline{P} \rtimes \overline{Q}$ is a Frobenius group, with Frobenius kernel $\overline{P}$ and Frobenius complement $\overline{Q}$. In particular, $\overline{P}$ is the minimal normal subgroup of $\overline{G}$ and $\overline{Q}$ is of prime order.

For any $1 \neq x \in P \setminus \mathbf{Z}(G)$, $\mathbf{C}_G(x)$ is maximal in G, which implies that $\overline{\mathbf{C}_G(x)}$ is maximal in $\overline{G}$. Note that $\overline{G}$ is a Frobenius group and $\overline{P} \leqslant \overline{\mathbf{C}_G(x)}$, it follows that $\overline{P} = \overline{\mathbf{C}_G(x)}$ and $|\overline{Q}| = q$ is a prime. Let $v \in Q \setminus \mathbf{Z}(G)$. Then $\mathbf{C}_G(v)$ is maximal in

https://doi.org/10.1017/prm.2024.71 Published online by Cambridge University Press

G. Note that Q is abelian, we get that $\overline{\mathbf{C}_G(v)} = \overline{Q}$ is maximal in $\overline{G}$. As a result, each subgroup of $\overline{G}$ is contained in $\overline{P}$ or $\overline{Q}^{\overline{g}}$ for some $\overline{g} \in \overline{G}$, showing that G is a minimal non-abelian group. $\square$

Acknowledgements

We thank the referee for his or her careful reading of the manuscript and for the constructive comments which substantially helped to improve the quality of the paper. The authors are supported by the National Natural Science Foundation of China (No. 12071181) and Natural Science Research Start-up Foundation of Recruiting Talents of Nanjing University of Posts and Telecommunications (Grant Nos. NY222090 and NY222091).

References

1 M. Aschbacher and G. M. Seitz. Involutions in Chevalley groups over fields of even order. *Nagoya Math. J.* **63** (1976), 1–91.

2 A. R. Ashrafi. On finite groups with a given number of centralizers. *Algebra Colloq.* **7** (2000), 139–146.

3 R. A. Bryce. The subgroups of Baer and Hughes. *Arch. Math. (Basel)* **61** (1993), 305–312.

4 J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson. *Atlas of Finite Groups* (London: Oxford University Press, 1985).

5 A. Espuelas. The Fitting length of the Hughes subgroup. *J. Algebra* **105** (1987), 365–371.

6 W. Feit, M. Hall and J. G. Thompson. Finite groups in which the centralizer of any non-identity element is nilpotent. *Math. Z.* (1960), 1–17.

7 M. Giudici, Maximal subgroups of almost simple groups with socle $PSL(2, q)$, eprint, arXiv: math/0703685.

8 D. Gorenstein and R. Lyons. The local structure of finite groups of characteristic 2 type. *Mem. Am. Math. Soc.* **42** (1983), vii+731.

9 D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups. Number 3. Part I. Chapter A.* Mathematical Surveys and Monographs, Vol. 40.3 (American Mathematical Society, Providence, RI, 1998).

10 M. A. Grechkoseeva. On the spectra of almost simple groups with a symplectic or orthogonal socle. *Sib. Math. J.* **57** (2016), 582–588.

11 M. A. Grechkoseeva. On orders of elements of finite almost simple groups with linear or unitary socle. *J. Group Theory* **20** (2017), 1191–1222.

12 M. A. Grechkoseeva. On spectra of almost simple extensions of even-dimensional orthogonal groups. *Sib. Math. J.* **59** (2018), 623–640.

13 M. E. Harris. Finite groups containing an intrinsic 2-component of Chevalley type over a field of odd order. *Trans. Am. Math. Soc.* **272** (1982), 1–65.

14 D. R. Hughes and J. G. Thompson. The H_p-problem and the structure of H_p-groups. *Pacific J. Math.* **9** (1959), 1097–1101.

15 H. Heineken. On groups all of whose elements have prime power order. *Math. Proc. R. Ir. Acad.* **106A** (2006), 191–198.

16 G. Higman. Finite groups in which every element has prime order. *J. London Math. Soc.* **32** (1957), 335–342.

17 B. Huppert. *Endliche Gruppen I* (Berlin-Heidelberg-New York: Springer-Verlag, 1967).

18 S. M. Jararian Amiri, M. Amiri and H. Rostami. Finite groups determined by the number of element centralizers. *Commun. Algebra* **45** (2017), 3792–3797.

19 H. Kurzweil and B. Stellmacher. *The Theory of Finite Groups An introduction* (Berlin-Heidelberg-New York: Springer-Verlag, 2004).

20 M. L. Lewis. Groups having all elements off a normal subgroup with prime power order. *Vietnam J. Math.* **51** (2023), 577–587.

21 C. H. Li and L. Wang. Finite REA-groups are solvable. *J. Algebra* **522** (2019), 195–217.

22 A. R. Moghaddamfar and W. J. Shi. The number of finite groups whose element orders is given. *Contrib. Algebra Geom.* **47** (2006), 463–479.

23 J. S. Rose. On finite insoluble groups with nilpotent maximal subgroups. *J. Algebra* **48** (1977), 182–196.

24 C. G. Shao and Q. H. Jiang. Structure of finite groups with four conjugacy class sizes of certain elements. *Commun. Algebra* **46** (2018), 1484–1491.

25 M. Suzuki. Finite groups with nilpotent centralizers. *Trans. Am. Math. Soc.* **99** (1961), 425–470.

26 X. H. Zhao, R. F. Chen and X. Y. Guo. Groups in which the centralizer of any non-central elements is maximal. *J. Group Theory* **23** (2020), 871–878.

27 X. H. Zhao, Y. Y. Zhou, R. F. Chen, H. L. Zuo and Q. Huang. On the normal subgroup with minimal G-conjugacy class sizes. *Mediterr. J. Math.* **18** (2021), 266.

28 A. Kh. Zhurtov, D. V. Lytkina and V. D. Mazurov. Primary cosets in groups. *Algebra Logic* **59** (2020), 216–221.

29 K. Zsigmondy. Zur Theorie der Potenzreste. *Monatsh. Math. Phys.* **3** (1892), 265–284.

30 M. A. Zvezdina. On the spectra of automorphic extensions of finite simple exceptional groups of Lie type. *Algebra Logic* **55** (2016), 354–366.