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Čebyšev Sets in Hyperspaces over $\mathbb{R}^n$

Robert J. MacG. Dawson and Maria Moszyńska

Abstract. A set in a metric space is called a *Čebyšev set* if it has a unique "nearest neighbour" to each point of the space. In this paper we generalize this notion, defining a set to be *Čebyšev relative to* another set if every point in the second set has a unique "nearest neighbour" in the first. We are interested in Čebyšev sets in some hyperspaces over $\mathbb{R}^n$, endowed with the Hausdorff metric, mainly the hyperspaces of compact sets, compact convex sets, and strictly convex compact sets.

We present some new classes of Čebyšev and relatively Čebyšev sets in various hyperspaces. In particular, we show that certain nested families of sets are Čebyšev. As these families are characterized purely in terms of containment, without reference to the semi-linear structure of the underlying metric space, their properties differ markedly from those of known Čebyšev sets.

Introduction

Let (X, ρ) be a nonempty metric space and let $A \subseteq X$. The set A is a *Čebyšev set in* (X, ρ) if and only if for every $x \in X$ there is a unique nearest point in A.

The notion of a Čebyšev set has been studied by many authors, mainly for normed linear spaces (see [4, 10] and the literature cited there). The case of a metric space was considered in [2, 3, 7], and in [10, Appendix]. Some results on Čebyšev sets in the space $\mathcal{K}^n$ of nonempty compact convex subsets of $\mathbb{R}^n$ and in the space $\mathcal{K}^n_0$ of convex bodies in $\mathbb{R}^n$ (both endowed with the Hausdorff metric) can be found in [3]. In particular, the set of all balls in $\mathbb{R}^n$ and the set of all singletons were proved to be Čebyšev, the first one in $\mathcal{K}^n_0$ and the second in $\mathcal{K}^n$. Both are similarity invariant.

One of the central questions of the theory of Čebyšev sets concerns the relationship between this notion and convexity; this depends strongly on the ambient space. For instance, in any Minkowski space (*i.e.,* finite dimensional Banach space) with smooth balls every Čebyšev set is convex, while strict convexity of balls is sufficient for closed convex sets to be Čebyšev. Thus, in particular, these two classes of subsets coincide in Euclidean space.

For the space $\mathcal{K}^n$ this problem was considered in [3]. It was proved that a nonempty, closed, convex set in this space need not be a Čebyšev set, while strict convexity is sufficient.

In this paper we study subsets of $\mathcal{C}^n$ (the class of nonempty compact subsets of $\mathbb{R}^n$) or $\mathcal{K}^n$ which are either Čebyšev sets or are Čebyšev relative to some subspace of $\mathcal{K}^n$. This more general notion can be defined for an arbitrary metric space (X, ρ) as follows. Let $X_0 \subseteq X$; a subset A of X is a *Čebyšev set in* (X, ρ) *relative to* X_0 whenever every element of X_0 has a unique nearest point in A. This includes as a

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special case the notion of "reach" [5]. The reach of a set A is the supremum of the set $\{r \mid A \text{ is Čebyšev relative to } (A)_r\}$ where $(A)_r$ is the outer parallel body at distance r from A. (We define this notation more formally below.)

Our main results are Theorem 2.8 concerning strongly nested families and Theorems 3.3 and 3.8 concerning families of translates. Section 4 deals primarily with properties of Čebyšev sets in hyperspaces, some contrasting strongly with the properties of Čebyšev sets in $\mathbb{R}^n$. In particular, we disprove two conjectures (6.3 and 6.4) from [3] concerning relationships between convexity and Čebyšev sets in $\mathcal{K}^n$. Some new conjectures are made.

1 Preliminaries

In what follows we use the symbols $\subseteq$ and $\subset$ for inclusion and strict inclusion (that is, we write $X \subset Y$ to indicate $X \subsetneq Y$). We will denote by $\Delta(a, b)$ the affine segment $\{ta + (1-t)b : 0 \leq t \leq 1\}$, both in $\mathbb{R}^n$ and in hyperspaces.

We begin with a simple lemma concerning real functions. Let $I \subseteq \mathbb{R}$. As usual, for any $f\colon I \to \mathbb{R}$, the *support of* f is defined by

$$\operatorname{supp}(f) := \operatorname{cl}\{i \in I \mid f(i) \neq 0\}.$$

Lemma 1.1 *Let* $I = [0, 1]$ *or* $I = [0, \infty)$ *and let* $f, g\colon I \to \mathbb{R}_+$ *be continuous,* f *weakly increasing, and* g *weakly decreasing, both functions injective on their supports, and* $\operatorname{supp}(f) \cap \operatorname{supp}(g) \neq \varnothing$.

If either

(i) $I = [0, 1]$, *or*

(ii) $I = [0, \infty)$ *and* f *is unbounded,*

then the function $\max\{f, g\}$ *has a unique minimizer.*

Proof Evidently $g(0) > 0$, because otherwise $\operatorname{supp}(f) \cap \operatorname{supp}(g) \subseteq \operatorname{supp}(g) = \varnothing$.

(i) Suppose $f(0) \geq g(0)$ (Figure 1a). Then $f(i) \geq f(0) > 0$ for all i, $\max\{f, g\} = f$ is injective and 0 is the unique minimizer of $\max\{f, g\}$.

If $f(i) < g(i)$ for all i (Figure 1b), then $g(i) > f(i) \geq 0$, $\max\{f, g\} = g$ is injective, and 1 is the unique minimizer of $\max\{f, g\}$.

Finally, suppose $f(0) < g(0)$ and $f(i) \geq g(i)$ for some i (Figure 1c). Then by continuity and the Intermediate Value Theorem there exists j such that $f(j) = g(j)$. For $k < j$,

$$\max\{f, g\}(k) \geq g(k) \geq g(j) = \max\{f, g\}(j), \tag{1.1}$$

and for $k > j$,

$$\max\{f, g\}(k) \geq f(k) \geq f(j) = \max\{f, g\}(j) \tag{1.2}$$

so j minimizes $\max\{f, g\}$.

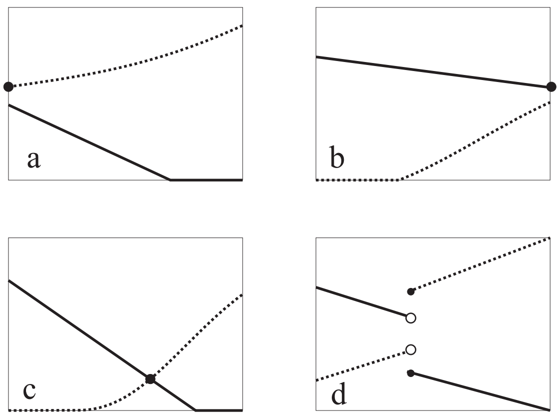

Figure 1: The functions f (dotted) and g (solid).

If $f(j) = 0$, then $\operatorname{supp}(g) \subseteq [0, j]$ and $\operatorname{supp}(f) \subseteq [j, 1]$. Since $\operatorname{supp}(f) \cap \operatorname{supp}(g) \neq \varnothing$, it follows that j belongs to both supports. Thus g must be positive on $[0, j)$, while f is positive on $(j, 1]$. If $f(j) > 0$, then g is positive on $[0, j]$ (by (1.1)) and f is positive on $[j, 1]$ (by (1.2)). In either case, g is injective on $[0, j]$, f is injective on $[j, 1]$, and we conclude that j is the unique minimizer.

(ii) The proof is as above, except that as g must be bounded and f is by hypothesis unbounded, the case illustrated in Figure 1b does not occur. ∎

Remark 1.2. While continuity is not needed to prove that there is at most one minimizer, the existence of a minimizer does require continuity (see Figure 1d).

Let $\mathcal{C}^n$ be the space of nonempty compact subsets of R^n endowed with the Hausdorff metric induced by the Euclidean norm $\|\cdot\|$:

$$\varrho_H(A, B) := \max\{\vec{\varrho}_H(A, B), \vec{\varrho}_H(B, A)\},$$

where the *oriented Hausdorff metric* $\vec{\varrho}_H(A, B)$ is defined by the formula

$$\vec{\varrho}_H(A, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\|$$

for every $A, B \in \mathcal{C}^n$.

Further, for any $A \in \mathcal{C}^n$ and $\delta > 0$, let $(A)_\delta := A + \delta B^n$. It is well known that for every $A, B \in \mathcal{C}^n$, we have $\vec{\varrho}_H(A, B) = \inf\{\delta \mid A \subseteq (B)_\delta\}$.

Now let $\mathcal{K}^n$ be the subspace of $\mathcal{C}^n$ consisting of convex sets and let $\mathcal{K}_0^n$ consist of convex bodies: $\mathcal{K}_0^n := \{A \in \mathcal{K}^n \mid \operatorname{int} A \neq \varnothing\}$.

Recall the following consequence of the Minkowski additivity of the function $\operatorname{conv}\colon \mathcal{C}^n \to \mathcal{K}^n$: for any $A \in \mathcal{K}^n$ and $\epsilon > 0$, $\operatorname{conv}((A)_\epsilon) = (\operatorname{conv} A)_\epsilon$.

Let us also recall the following consequence of the cancellation law for Minkowski addition in $\mathcal{K}^n$: for any $A, B \in \mathcal{K}^n$ and $\epsilon > 0$,

$$(A)_\epsilon \subseteq (B)_\epsilon \Longrightarrow A \subseteq B. \tag{1.3}$$

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Minkowski subtraction will be denoted by $\ominus$.[1] It is defined by the formula

$$A \ominus B := \bigcap_{b \in B} (A - b). \tag{1.4}$$

Recall that a closed convex subset of $\mathbb{R}^n$ is said to be *strictly convex* if its boundary does not contain any segment. The following properties of strictly convex subsets of $\mathbb{R}^n$ are well known, but as they are not easily found in the literature, we provide proofs for some of them.

Proposition 1.3 *If A is closed and strictly convex, then for every $\alpha > 0$ the set $(A)_\alpha$ is strictly convex.*

Proof Suppose that $\mathrm{bd}((A)_\alpha)$ contains a segment $\Delta(x_0, x_1)$ and let y_i be the unique point of A nearest to x_i, for $i = 0, 1$. Then $y_0, y_1 \in \mathrm{bd}A$.

Let $x(t) := (1-t)x_0 + tx_1$ and $y(t) := (1-t)y_0 + ty_1$ for every $t \in [0, 1]$. Since A is strictly convex, $y(t_0) \in \mathrm{int}A$ for some $t_0 \in (0; 1)$. Thus

$$\min_{y \in A} \|x(t_0) - y\| < \|x(t_0) - y(t_0)\| = \|(1 - t_0)(x_0 - y_0) + t_0(x_1 - y_1)\| = \alpha,$$

whence $x(t_0) \notin \mathrm{bd}((A)_\alpha)$, a contradiction. ■

Proposition 1.4 *Let A and B be nonempty subsets of $\mathbb{R}^n$. If A is closed and strictly convex and B is compact, then $A \ominus B$ is strictly convex.*

Proof From (1.4) it follows that

$$x \in A \ominus B \iff B + x \subseteq A. \tag{1.5}$$

Hence, if A is a singleton, then $A \ominus B$ is a singleton (if B is), otherwise it is empty, and in either case $A \ominus B$ is strictly convex.

If A is not a singleton, then $\mathrm{int}A \neq \varnothing$. Let $x_0, x_1 \in A \ominus B$, $x_0 \neq x_1$, and $x(t) = (1-t)x_0 + tx_1$ for some $t \in (0; 1)$. By (1.10), it suffices to prove that $x(t) \in \mathrm{int}\{x \mid B + x \subseteq A\}$, that is, equivalently,

$$(\exists \epsilon > 0)\ (B + x(t))_\epsilon \subseteq A. \tag{1.6}$$

In particular, $B + x_i \subseteq A$ for $i = 0, 1$. Since $b + x(t) = (1-t)(b + x_0) + t(b + x_1)$ and A is strictly convex, it follows that $b + x(t) \in \mathrm{int}A$ for every $b \in B$.

Thus, for every $b \in B$ there is an $\epsilon > 0$ such that $b + x(t) + \epsilon B^n \subseteq A$. Since B is compact, it follows that we can choose the same ϵ for all $b \in B$, which proves (1.6). ■

The following is evident.

Proposition 1.5 *The intersection of two strictly convex subsets of $\mathbb{R}^n$ is strictly convex.*

[1] In [9] Schneider denotes it by $\sim$, while $A \ominus B$ is defined as $A \sim -B$.

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We shall use the symbol $\mathcal{O}^n$ to denote the family of strictly convex elements of $\mathcal{C}^n$. Thus, evidently, $\mathcal{O}^n$ consists of strictly convex bodies and singletons. Similarly, $\mathcal{O}^n_0$ will denote the family of strictly convex bodies.

To any nonempty subset X of $\mathbb{R}^n$ we assign the set $[X]$ of corresponding singletons:

$$[X] := \{\{x\} \mid x \in X\}.$$

In particular, the family $\mathcal{X}^n$ from [3] will be now denoted by $[\mathbb{R}^n]$.

2 Strictly Nested Families

Definition 2.1 Let $\mathcal{A}$ be a nonempty subset of $\mathcal{C}^n$.

- $\mathcal{A}$ is said to be a *nested family in* $\mathcal{C}^n$ if it is a proper chain under the inclusion order, that is, for every $A, A' \in \mathcal{A}$, either $A \subset A'$ or $A' \subset A$.
- $\mathcal{A}$ is said to be a *strongly nested family in* $\mathcal{C}^n$ if for distinct $A, A' \in \mathcal{A}$, either $A \subset \operatorname{int} A'$ or $A' \subset \operatorname{int} A$.
- The family $\mathcal{A}$ is said to be *bounded* if the set of diameters of its members is bounded, otherwise *unbounded*.
- The family $\mathcal{A}$ is said to be *dense* if whenever $A \subset A'$, there exists A'' such that $A \subset A'' \subset A'$.
- The nested family $\mathcal{A}$ is said to be *closed* if for every subfamily $\mathcal{S} \subset \mathcal{A}$, $\bigcap \mathcal{S} \in \mathcal{A}$ and, if $\mathcal{S}$ is bounded, $\operatorname{cl} \bigcup \mathcal{S} \in \mathcal{A}$.

The following proposition (for which we make no special claim of originality) allows us to use an indexing function to locate elements of certain nested families.

Proposition 2.2 *Let $\mathcal{A}$ be a strongly nested family in $\mathcal{C}^n$, and let*

$$I = \begin{cases} [0,1] & \text{if } \mathcal{A} \text{ is bounded;} \\ [0,\infty) & \text{if } \mathcal{A} \text{ is unbounded.} \end{cases}$$

Then $\mathcal{A}$ is closed and dense if and only if there exists a continuous indexing function $\phi \colon I \to \mathcal{A}$ such that

$$i < j \Longrightarrow \phi(i) \subset \operatorname{int}\phi(j). \tag{2.1}$$

Proof As $\mathcal{A}$ is closed, it has a smallest element $A_0 = \bigcap \mathcal{A}$ and, if bounded, a largest element $A_1 = \bigcup \mathcal{A}$. Define $\psi \colon \mathcal{A} \to I$ by the formula

$$\psi(A) = \begin{cases} \frac{\varrho_H(A_0,A)}{\varrho_H(A_0,A_1)} & \text{if } \mathcal{A} \text{ is bounded;} \\ \varrho_H(A_0,A) & \text{if } \mathcal{A} \text{ is unbounded.} \end{cases}$$

As $\mathcal{A}$ is strongly nested, if $A \subset A' \in \mathcal{A}$, then $A \subset \operatorname{int} A'$. As A is compact, then also $(A)_\epsilon \subset A'$ for some $\epsilon > 0$; and $\psi((A)_\epsilon) \leq \psi(A')$, so ψ is injective and increasing. We shall show that it is in fact bijective.

Suppose, to the contrary, that it is not surjective. Then there exists $i \in I \setminus \psi(\mathcal{A})$, and thus

$$\mathcal{A} = \mathcal{A}^- \cup \mathcal{A}^+ \tag{2.2}$$

where

$$\mathcal{A}^- = \{A \in \mathcal{A} \mid \psi(A) < i\}, \quad \mathcal{A}^+ = \{A \in \mathcal{A} \mid \psi(A) > i\}\,.$$

By continuity of ϱ_H, these subfamilies are closed. Moreover, A_0 belongs to $\mathcal{A}^-$, and either A_1 or arbitrarily large elements of $\mathcal{A}$ belong to $\mathcal{A}^+$; so both subfamilies are nonempty. Since $\mathcal{A}$ is closed and dense, there exists an element A with $\operatorname{cl}\bigcup \mathcal{A}^- \subset A \subset \bigcap \mathcal{A}^+$; but this contradicts (2.2). Hence ψ is bijective. Let us define $\phi\colon I \to \mathcal{A}$ to be the inverse of ψ.

As the inverse of an increasing function, ϕ is increasing; it remains to show that it is continuous. Suppose it is not; then for some $i \in I$, either $\phi(i) \subset \bigcap_{k>i} \phi(k)$ or $\phi(i) \supset \operatorname{cl}\bigcup_{k<i} \phi(k)$. In the former case, since $\mathcal{A}$ is closed and dense, there exists $A \in \mathcal{A}$ such that $\phi(i) \subset A \subset \bigcap_{k>i} \phi(k)$; but then $i < \psi(A) < k$ for all $k > i$ which is impossible. The other case leads to a contradiction in a similar fashion.

Finally, if a continuous increasing indexing function exists, then $\mathcal{A}$ is dense and closed by continuity and the corresponding properties of I. ∎

We shall use the notation A_i for $\phi(i)$.

Remark 2.3. While strong nesting is essential to Proposition 2.2, in the presence of other conditions a similar result can be obtained for arbitrary nested families. For instance, for a nested family of compact convex sets, not necessarily strongly nested, we could use (suitably scaled) mean width as the increasing bijection $\psi\colon \mathcal{A} \to I$.

Proposition 2.4 *Let $\mathcal{A}$ be a closed, dense, strongly nested family in $\mathcal{C}^n$ with indexing function $\phi\colon I \to \mathcal{A}$, and let $X \in \mathcal{K}^n$. If $f, g\colon I \to \mathrm{R}_+$ are defined by*

$$f(i) := \vec{\varrho}_H(A_i, X) \tag{2.3}$$

and

$$g(i) := \vec{\varrho}_H(X, A_i) \tag{2.4}$$

for every $i \in I$, then

(a) *f and g are continuous,*
(b) *f is weakly increasing and g is weakly decreasing,*
(c) *if $f \neq 0$, then $f|\operatorname{supp}(f)$ is increasing,*
(d) *if $g \neq 0$, then $g|\operatorname{supp}(g)$ is decreasing.*

Proof (a) The functions f and g are continuous because the indexing function and the directed Hausdorff metric are continuous.

(b) By (2.3), (2.4), and (1.3), for every $i \in I$

$$\psi(i) = \inf\{\delta > 0 \mid A_i \subseteq (X)_\delta\} \tag{2.5}$$

and

$$g(i) = \inf\{\delta > 0 \mid X \subseteq (A_i)_\delta\}. \tag{2.6}$$

Let $\Delta(i) := \{\delta > 0 \mid A_i \subseteq (X)_\delta\}$. Then

$$i < j \Longrightarrow \Delta(j) \subseteq \Delta(i) \Longrightarrow f(i) \leq f(j).$$

Thus f is weakly increasing. The proof that g is weakly decreasing is analogous.

(c) Let $i_0 := \inf \operatorname{supp}(f)$ and let $i_0 \leq i < j$. Suppose $f(i) = f(j) =: \alpha$. Then $\alpha > 0$ and by (2.5) and (2.1), for some $\epsilon > 0$

$$(A_i)_\epsilon \subseteq A_j \subseteq (X)_\alpha. \tag{2.7}$$

We may assume that $\epsilon < \alpha$.

Since X is convex, from (2.7) and (1.4) it follows that $(\operatorname{conv} A_i)_\epsilon \subseteq \operatorname{conv} A_j \subseteq (X)_\alpha$. Hence, by (1.5), $A_i \subseteq \operatorname{conv} A_i \subseteq (X)_{\alpha-\epsilon}$, a contradiction.

(d) Let now $j_0 := \sup \operatorname{supp}(g)$ and let $i < j \leq j_0$. Suppose $g(i) = g(j) =: \beta$. Then $\beta > 0$ and by (2.6),

$$X \subseteq (A_i)_\beta \cap (A_j)_\beta. \tag{2.8}$$

Since $A_i \subset \operatorname{int} A_j$ (by (2.1)) and A_i is compact, there exists an $\epsilon > 0$ such that $(A_i)_\epsilon \subseteq A_j$. We may assume that $\epsilon < \beta$. Then, from (2.8) it follows that $X \subseteq ((A_i)_\epsilon)_{\beta-\epsilon} \subset (A_j)_{\beta-\epsilon}$, a contradiction. ■

Theorem 2.5 *If $\mathcal{A}$ is a closed, dense, strongly nested family in $\mathcal{C}^n$, then for every $X \in \mathcal{K}^n$ there exists a unique nearest element in $\mathcal{A}$.*

Proof By Proposition 2.2, $\mathcal{A}$ has a continuous indexing function with domain I. Take a test set $X \in \mathcal{K}^n$ and let $f, g\colon I \to \mathbb{R}_+$ be defined by (2.3) and (2.4).

By Proposition 2.4, the functions f and g are continuous, f is weakly increasing and is increasing on its support, while g is weakly decreasing and is decreasing on its support.

By (1.3), $\varrho_H(X, A_i) = \max\{f(i), g(i)\}$.

Case 1: $f = 0$, *i.e.*, $\operatorname{supp}(f) = \varnothing$. Then $A_i \subseteq X$ for every $i \in I$, whence $I = [0, 1]$, and $\max\{f, g\} = g$. Thus there is at most one i_0 with $g(i_0) = 0$ (because there is at most one A_{i_0} which coincides with X). If such an i_0 exists, then $X = A_{i_0}$, so A_{i_0} is a unique element of the family nearest to X. Otherwise $g > 0$ and A_1 is a unique element of $\mathcal{A}$ nearest to X.

Case 2: $g = 0$, *i.e.*, $\operatorname{supp}(g) = \varnothing$. Then $X \subseteq A_i$ for every $i \in I$ and $\max\{f, g\} = f$, whence there is at most one i_0 with $f(i_0) = 0$. If such an i_0 exists, then $i_0 = 0$, whence A_0 is a unique element nearest to X. If not, then $f > 0$ and thus f attains its unique minimum at 0; so again A_0 is the unique nearest element.

Case 3: $f \neq 0 \neq g$, *i.e.*, both supports are nonempty. Then there exist $i_0, j_0 \in I$ such that

$$\operatorname{supp}(f) = [i_0, 1] \text{ (or } [i_0, \infty)), \ \operatorname{supp}(g) = [0, j_0].$$

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Let us note that if $i_0 > j_0$, then $\varrho_H(X, A_i) = 0$ (and thus $X = A_i$) for all $i \in [j_0, i_0]$, contrary to the assumption that $\mathcal{A}$ is strongly nested (Definition 2.1). Thus $i_0 \leq j_0$, whence the supports of f and g have a nonempty intersection. By Lemma 1.1, this completes the proof. ■

Theorem 2.5 may be reformulated as follows.

Corollary 2.6 *Every closed, dense, strongly nested family in $\mathcal{C}^n$ is a Čebyšev set relative to $\mathcal{K}^n$.*

Remark 2.7. In Proposition 2.2 as well as in Proposition 2.4, Theorem 2.5, and Corollary 2.6, the interval $[0, 1]$ may be replaced by an arbitrary $[\alpha, \beta]$. Similarly, $[0, \infty]$ may be replaced by $[\alpha, \infty)$. Thus, Theorem 2.5 is a generalization of [3, Theorem 4.3(i)].

We can combine several of the above results to obtain the following.

Theorem 2.8 *For a strongly nested family $\mathcal{A}$ in $\mathcal{K}^n$, the following are equivalent:*

(a) *$\mathcal{A}$ is a Čebyšev set;*
(b) *$\mathcal{A}$ is closed and dense;*
(c) *$\mathcal{A}$ has a continuous indexing function as defined in Proposition* 2.2 .

Proof The equivalence of (b) and (c) was established in Proposition 2.2, and Corollary 2.6 gives (b) $\Rightarrow$ (a); it remains to show that (a) implies (b). If $\mathcal{A}$ is not closed, it has a subfamily $\mathcal{S}$ such that $X := \bigcap \mathcal{S} \notin \mathcal{A}$ or a bounded subfamily $\mathcal{S}$ such that $X := \operatorname{cl} \bigcup \mathcal{S} \notin \mathcal{A}$. In either case, $\inf_{A \in \mathcal{A}} \varrho_H(X, A) = 0$, but this infimum is not attained; thus $\mathcal{A}$ is not a Čebyšev set.

Suppose $\mathcal{A}$ is not dense; then there exist $A, A' \in \mathcal{A}$ with $A \subset A'$, such that for every other $A'' \in \mathcal{A}$, either $A'' \subset \operatorname{int} A$ or $A' \subset \operatorname{int} A''$. Let $X = \frac{1}{2}(A + A')$; then $\varrho_H(X, A) = \varrho_H(X, A') < \varrho_H(X, A'')$, so the closest point of $\mathcal{A}$ to X is not unique and $\mathcal{A}$ is not a Čebyšev set. ■

The following proposition shows that in Theorem 2.5 the assumption that the test sets are convex is necessary; that is, nests (even of compact convex sets) are not Čebyšev sets in $\mathcal{C}^n$. This is also valid for the hyperspace $\mathcal{C}^n_0$ of compact bodies (a set $A \in \mathcal{C}^n$ is a body if $\operatorname{cl} \operatorname{int} A = A$).

Proposition 2.9 *No nest is a Čebyšev set in $\mathcal{C}^n$ or in $\mathcal{C}^n_0$.*

Proof Let the nest be $\{A_t \mid t \in [0, 1]\}$. Let $p \in A_0$, let r be large enough that $A_0 \subset \operatorname{int}(rB^n + p))$, and let $S = \operatorname{bd}(rB^n + p)$. Then $\varrho_{\overrightarrow{H}}(A_i, S) \geq r$, and equality is achieved for all $A_i \subset rB^n + p$, thus by continuity, for a nonsingleton collection. On the other hand, $\varrho_{\overrightarrow{H}}(S, A_i) \leq r$ for all A_i, whence $\varrho_H(A_i, S) = \varrho_{\overrightarrow{H}}(A_i, S)$. It follows that the nest is not a Čebyšev set.

For $\mathcal{C}^n_0$ we can assume the existence of an ϵ-ball within A_0 with centre p; the test set should then be a spherical shell of inner radius r, thickness $\epsilon/2$, and center p, and the result again follows (see Figure 2). ■

It is not hard to find examples of non-strongly nested families that are not Čebyšev sets.

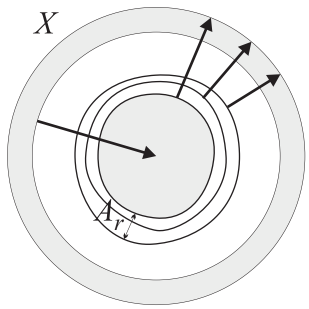

Figure 2: A nest of convex bodies equidistant from a nonconvex test body.

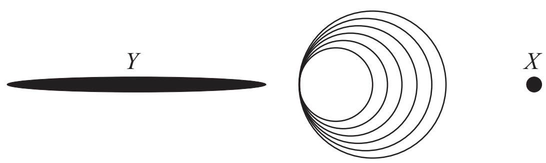

Figure 3: A non-Čebyšev nest.

Example 2.10 Let $A_t = t \cdot B^n + tu$, where $\|u\| = 1$. Then the "peacock eye" $\mathcal{A} = \{A_t \mid \frac{1}{2} \leq t \leq 1\}$ shown in Figure 3 is not a Čebyšev set. For instance, the singleton $X = \{3u\}$ and the set Y are each equidistant from all the members of $\mathcal{A}$.

3 Families of Translates

For a given $A \in \mathcal{K}^n$, let $\mathcal{A}$ be the family of all translates of A, *i.e.*, $\{A + x \mid x \in \mathrm{R}^n\}$.

We will consider under what assumptions on A the family $\mathcal{A}$ is a Čebyšev set in $\mathcal{K}^n$ or in a subspace of $\mathcal{K}^n$. For every $A, X \in \mathcal{C}^n$ and $\alpha \geq 0$, let

$$F(A, X; \alpha) := \{x \in \mathrm{R}^n \mid \varrho_H(X, A + x) \leq \alpha\}. \tag{3.1}$$

Directly from the definition (1.4) of Minkowski subtraction it follows that for $\alpha > 0$,

$$F(A, X; \alpha) = ((X)_\alpha \ominus A) \cap -((A)_\alpha \ominus X). \tag{3.2}$$

Lemma 3.1 (i) *Let $A, X \in \mathcal{C}^n$. If $0 < \alpha < \beta$, then $F(A, X; \alpha) \subseteq F(A, X; \beta)$.*
(ii) *For every $A, X \in \mathcal{C}^n$ and $\alpha > 0$, the set $F(A, X; \alpha)$ is compact.*
(iii) *If A, X are convex, then $F(A, X; \alpha)$ is convex.*
(iv) *If A, X are strictly convex, then so is $F(A, X; \alpha)$.*
(v) *For α large enough, the set $F(A, X; \alpha)$ is nonempty.*

(vi) *$F(A, X; 0)$ is a singleton if X is a translate of A and is empty otherwise.*

Proof (i) follows directly from (3.1). We will use (3.2) to prove (ii)–(iv).

(ii) The set $F(A, X; \alpha)$ is the intersection of compact subsets of $\mathbb{R}^n$, whence it is compact, because is bounded, closed in $\mathbb{R}^n$, and $\mathbb{R}^n$ is finitely compact.

(iii) If A, X are convex, then the set $F(A, X; \alpha)$ is the intersection of convex subsets of $\mathbb{R}^n$, whence it is convex.

(iv) By Prop. 1.3 the sets $(A)_\alpha$ and $(X)_\alpha$ are strictly convex, whence, by Prop. 1.4, both $(A)_\alpha \ominus X$ and $((X)_\alpha \ominus A)$ are strictly convex. Thus, by (3.2) combined with Prop. 1.5, the assertion follows.

(v) If $\alpha \geq \varrho_H(A, X)$, then $0 \in F(A, X; \alpha)$.

(vi) is obvious. ■

Lemma 3.2 *Let $A, X \in \mathcal{K}^n$ and let*

$$\alpha_0 := \inf\{\alpha > 0 \mid F(A, X; \alpha) \neq \varnothing\}. \tag{3.3}$$

Then the set $F(A, X; \alpha_0)$ is nonempty and has empty interior.

Proof By the assumption, $\alpha_0 = \lim \alpha_k$ for some decreasing sequence $(\alpha_k)_{k \in N}$ such that $F(A, X; \alpha_k) \neq \varnothing$ for every k. That is, for every k there exists $x_k \in \mathbb{R}^n$ such that $\varrho_H(X, A + x_k) \leq \alpha_k$. Since $(\alpha_k)_{k \in N}$ is bounded, so is $(x_k)_{k \in N}$ and thus there exists a convergent subsequence $(x_{n_k})_{k \in N}$. Let $x_0 = \lim x_{n_k}$. Then, by continuity of metric, $\varrho_H(X, A + x_0) \leq \alpha_0$, whence $x_0 \in F(A, X; \alpha_0)$.

It remains to prove that $\operatorname{int} F(A, X; \alpha_0) = \varnothing$. If $\alpha_0 = 0$, then $F(A, X; \alpha_0)$ has empty interior, because it is a singleton in view of Lemma 3.1 (vi).

Let $\alpha_0 > 0$ and suppose that $\operatorname{int} F(A, X; \alpha_0) \neq \varnothing$. Then, for every $x \in F(A, X; \alpha_0)$ there exists an $\epsilon > 0$ with $x + \epsilon B^n \subseteq F(A, X; \alpha_0)$.

Of course, we may assume that $\epsilon < \alpha_0$. Then by (3.2),

$$x + \epsilon B^n \subseteq (-((A)_{\alpha_0} \ominus X)) \cap ((X)_{\alpha_0} \ominus A),$$

whence, by (1.4) and the cancellation law,

$$x \in \bigcap_{a \in A} ((X)_{\alpha_0 - \epsilon} - a) \cap - \bigcap_{b \in X} ((A)_{\alpha_0 - \epsilon} - b) = F(A, X; \alpha_0 - \epsilon).$$

Thus $F(A, X; \alpha_0 - \epsilon) \neq \varnothing$, contrary to (3.3). ■

Theorem 3.3 *For every $A \in \mathcal{O}^n$, the family $\mathcal{A}$ of all translates of A is a Čebyšev set in the space $\mathcal{O}^n$.*

Proof Take a strictly convex $X \in \mathcal{C}^n$. By Lemma 3.1(ii–iv), the set $F(A, X; \alpha)$ is compact and strictly convex for every $\alpha > 0$. By Lemma 3.2, if α_0 is defined by (3.3), then $F(A, X; \alpha_0)$ is nonempty and has empty interior. Hence, as a nonempty strictly convex set with empty interior, the set $F(A, X; \alpha_0)$ is a singleton. This means that there is a unique point x such that $\varrho_H(X, A + x)$ is minimal, *i.e,* X has a unique nearest element in $\mathcal{A}$. This completes the proof. ■

Remark 3.4. Let us note that if A is a singleton, then the family $\mathcal{A}$ coincides with $[\mathrm{R}^n]$, the set of all singletons in R^n, (compare (1.12)). Thus, as a consequence of Theorem 3.3 we obtain a weaker version of [3, Proposition 3.2], which states that the set of singletons is a Čebyšev set in $\mathcal{K}^n$.

The following proposition shows that in Theorem 3.3 the assumption of strict convexity is essential and the hyperspace $\mathcal{O}^n$ cannot be replaced by $\mathcal{K}^n$.

Proposition 3.5 *If $\{A + x \mid x \in \mathrm{R}^n\}$ is a Čebyšev set in $\mathcal{K}^n$ for $n > 1$, then A is a singleton.*

Proof Suppose that $A \in \mathcal{K}^n$ is not a singleton and $\operatorname{diam} A = \alpha$. Choose $a_1, a_2 \in A$ with $||a_1 - a_2|| = \alpha$ and let $a_0 = \frac{1}{2}(a_1 + a_2)$. Take a test body

$$X := (A)_{\frac{\alpha}{8}} \cap \{x \in \mathrm{R}^n \mid ||x - a_0|| \leq ||x - a_i|| \text{ for } i = 1, 2\}$$

(see Figure 4). Let $\pi\colon \mathrm{R}^n \to \operatorname{aff}\{a_1 a_2\}$ be the orthogonal projection. Then, for every $x \in \mathrm{R}^n$ the set $\pi(A + x)$ is a segment of length α while $\pi(X)$ is a segment of length $\frac{\alpha}{2}$ and thus

$$\rho_{\overrightarrow{H}}(A + x, X) \geq \rho_{\overrightarrow{H}}(\pi(A + x), \pi(X)) \geq \frac{\alpha}{4}.$$

On the other hand, if $x \perp a_1 - a_2$ and $||x|| < \frac{\alpha}{8}$, then

$$\rho_{\overrightarrow{H}}(X, A + x) < \frac{\alpha}{4} = \rho_{\overrightarrow{H}}(A + x, X).$$

Hence, the minimal distance $\rho_H(A + x, X) = \frac{\alpha}{4}$ is achieved for many translates of A. ∎

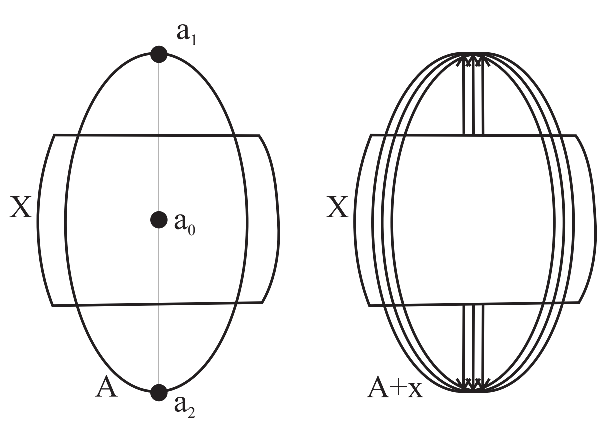

Figure 4: Equidistant nearest translates of A.

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Remark 3.6. Note that even when A is strictly convex, $\{A + x \mid x \in \mathrm{R}^n\}$ is not a Čebyšev set in $\mathcal{K}^n$. Interchanging A and X shows, furthermore, that families of translates in $\mathcal{K}^n$ are not in general Čebyšev sets relative to $\mathcal{O}^n$.

Remark 3.7. While $\mathcal{O}^n$ seems to be the most natural setting for Theorem 3.3, further generalizations are still possible at cost of some artificiality. The reader may verify, for instance, that if for some fixed u_0 and α we let $\mathcal{D}^n$ be the subspace of $\mathcal{K}^n$ consisting of all compact "D-shaped" sets, obtained by intersecting elements of $\mathcal{O}^n$ with halfspaces of the form $\{x \mid x \circ u_0 \geq \alpha\}$, we may substitute $\mathcal{D}^n$ for $\mathcal{O}^n$ in Theorem 3.3 and the proof is more or less unchanged.

We are now going to derive a stronger version of Theorem 3.3, replacing the family of all translates of a given A by the subfamily of translates by vectors in an arbitrary nonempty closed convex set.

For every $T \subseteq \mathrm{R}^n$, every $A, X \in \mathcal{C}^n$, and $\alpha \geq 0$, let

$$F_T(A, X; \alpha) := \{x \in T \mid \varrho_H(X, A + x) \leq \alpha\}$$

and

$$\alpha_0(T) := \inf\{\alpha \mid F_T(A, X; \alpha) \neq \varnothing\}. \tag{3.4}$$

Of course, $F_T(A, X; \alpha) = T \cap F(A, X; \alpha)$.

Theorem 3.8 *Let $A \in \mathcal{O}^n$ and let T be a nonempty, closed, and convex subset of* R^n*. Let*

$$\mathcal{A}_T := \{A + x \mid x \in T\}. \tag{3.5}$$

Then $\mathcal{A}_T$ is a Čebyšev set in $\mathcal{O}^n$.

Proof Let $X \in \mathcal{O}^n$. Then, by Lemma 3.1(iv), the set $F(A, X; \alpha_0(T))$ is strictly convex. It suffices to prove that $F_T(A, X; \alpha_0(T))$ is a singleton. Suppose, to the contrary, that $x_1, x_2 \in F_T(A, X; \alpha_0(T))$ and $x_1 \neq x_2$.

Then $\frac{1}{2}(x_1 + x_2) \in T \cap \mathrm{int} F(A, X; \alpha_0(T))$, because T is convex and $F(A, X, \alpha_0(T))$ is strictly convex. But then there exists $\epsilon > 0$ such that

$$\frac{1}{2}(x_1 + x_2) \in T \cap \mathrm{int} F(A, X; \alpha_0(T) - \epsilon),$$

contrary to (3.4). ■

Note that 3.8 is a generalization of a weaker version of Theorem 4.5 in [3], in which $\mathcal{K}^n$ is replaced by $\mathcal{O}^n$. However, no such generalization is possible for $\mathcal{K}^n$.

Proposition 3.9 *If the family $\mathcal{A}_T$, as defined by* (3.5)*, is a Čebyšev set in $\mathcal{K}^n$, then A or T is a singleton.*

Proof Suppose that A and T are closed, convex, and nonsingletons. Unless A and T are parallel segments, we may choose a segment $S \subset T$ such that A has nonzero width in a direction normal to S and follow the construction in the proof of Proposition 3.5 to obtain a test set X equidistant from various $A + s$, $s \in S$, and which is no closer to any other translate of A.

If on the other hand A and T are parallel segments, A having length l and (without loss of generality) T centered at the origin, take X to be a segment of length $2l$, with the midpoint common with A, and perpendicular to A. It is clear that if $x \in T$ and $\|x\|$ is small enough, then $\varrho_H(X, A) = \varrho_H(X, A+x) = l$ and no translate of A is nearer to X. ■

4 Properties of Čebyšev Sets in Hyperspaces

Conjecture 6.4 of [3] states that if a pair $\{A, B\}$ in $\mathcal{K}^n$ has more than one metric midpoint, $\Delta(A, B)$ is not a Čebyšev set in $\mathcal{K}^n$. The following example disproves this conjecture.

Example 4.1 Let A and B be convex bodies, $A \subset \operatorname{int} B$, such that B is not a parallel body of A. By Theorem 2.8, the affine segment $\Delta(A, B)$ is a Čebyšev set; but by [9, p.59, note 10] there is more than one metric segment joining A and B, and by [6] this implies that the metric midpoint of $\{A, B\}$ is not unique.

As is well known, a subset of R^n is a Čebyšev set if and only if it is nonempty, closed, and convex. Examples of nonempty, compact, affine convex subsets of $\mathcal{K}^n$ which are not Čebyšev sets in $\mathcal{K}^n$ were given in [3] (see 4.1, 4.3, and 4.7). Conversely, we shall prove, using Theorem 2.8, that a Čebyšev set in $\mathcal{K}^n$ need not be affine convex, that is, Conjecture 6.2 in [3] is false. Moreover, the class of Čebyšev sets in $\mathcal{K}^n$ is not closed under intersection.

Example 4.2 Let $f(t) = t(1 - t)$; let $A_t = t \cdot B^n$ and let $A'_t = A_t + f(t)u$ where $\|u\| = 1$. Let $\mathcal{A} = \{A_t \mid 0 \le t \le 1\}$, and let $\mathcal{A}' = \{A'_t \mid 0 \le t \le 1\}$ (see Figure 5). The family $\mathcal{A}$ is strongly nested and affine convex.

As the balls $\{A'_t : t \in [0, 1]\}$ are centered on the line through the origin and u, the family $\mathcal{A}$ is strongly nested if the family of their intersections with that line is strongly nested. The intersection of $A'(t)$ with the line extends from $-t^2 u$ to $(2t - t^2)u$, and for $0 \le s < t \le 1$ we have $-t^2 < -s^2$ and $2s - s^2 < 2t - t^2$. Thus, by Corollary 2.6, both families are Čebyšev sets.

However, $\mathcal{A} \cap \mathcal{A}' = \{A_0, A_1\}$, which is not affine convex. But $\mathcal{A}$ is affine convex and the class of affine convex sets is closed under intersection, so $\mathcal{A}'$ is not affine convex.

We also note that $\{A_0, A_1\}$, the intersection of Čebyšev sets, is not a Čebyšev set, since the two bodies are equidistant from $A_{1/2}$.

Since the space $\mathcal{K}^n$ has infinite dimension, it is reasonable to ask about possible dimensions of Čebyšev sets in $\mathcal{K}^n$.

Proposition 4.3 *For every $k \in \{0, \dots, n + 1\}$ there exists a Čebyšev set in $\mathcal{K}^n$ with*

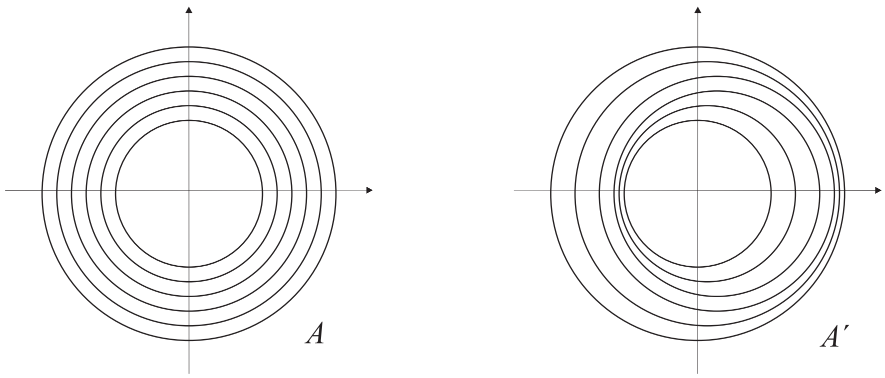

Figure 5: Two Čebyšev nests with exactly two common elements.

dimension k.

Proof Let $0 \leq k \leq n$ and let A be a closed convex subset of R^n with $\dim A = k$. By [3, Theorem 4.5], the set of singletons $\mathcal{A} := \{\{a\} \mid a \in A\}$, is a Čebyšev set in $\mathcal{K}^n$. Since $\mathcal{A}$ is homeomorphic (even isometric) to A, its dimension is equal to k.

By [3, Theorem 3.3], the closure of the family $\mathcal{B}^n$ of balls is a Čebyšev set in $\mathcal{K}^n$. This family is homeomorphic to $\mathrm{R}^n \times [0, \infty)$, whence its dimension is equal to $n + 1$. ■

The space $\mathcal{K}^n$ itself is, trivially, a Čebyšev set of infinite dimension. However, this is not the only example.

Proposition 4.4 *There exists a nontrivial Čebyšev set in $\mathcal{K}^n$ of infinite dimension.*

Proof Let $\mathcal{A}$ be the ball with centre $\{0\}$ and radius α in the space $\mathcal{K}^n$ endowed with the metric ρ_2 defined by $\rho_2(A, B) := \|h_A - h_B\|_2$. In view of [3], 5.3 combined with 5.2, the ball $\mathcal{A}$ is a Čebyšev set in $\mathcal{K}^n$. By [11], it is homeomorphic to the ball with centre $\{0\}$ and radius α in the space $(\mathcal{K}^n, \rho_H)$. By [1], the subspace of $\mathcal{K}^n$ consisting of bodies with a constant width is a Q-manifold (where Q is the Hilbert cube). Thus Q can be topologically embedded into $\mathcal{A}$. ■

Problem 4.5 Does there exist a Čebyšev set in $\mathcal{K}^n$ with finite dimension greater than $n + 1$?

Remark 4.6. Let us note that while sets in R^n that are isometric to Čebyšev sets are themselves Čebyšev sets, this is not the case in $\mathcal{K}^n$. There are many and varied counterexamples.

For instance, let $A_r = r \cdot B^n$ and $A'_r = \frac{r}{2} \cdot B^n + \frac{r}{2}u$, where $\|u\| = 1$. Take $\mathcal{A} = \{A_r \mid 0 \leq r \leq 1\}$ and $\mathcal{A}' = \{A'_r \mid 0 \leq r \leq 1\}$ (see Figure 6). Then $\varrho_H(A_r, A_s) = \varrho_H(A'_r, A'_s) = |r - s|$, so there is an isometry of $\mathcal{A}$ onto $\mathcal{A}'$. However, it is evident that $\mathcal{A}$ is a Čebyšev set, while $\mathcal{A}'$ is not. Specifically, if X is the singleton $\{2u\}$, then $\varrho_H(X, A'_r) = 2$ for all r.

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Obviously, an isometry of $\mathcal{K}^n$ onto $\mathcal{K}^n$ must preserve the class of Čebyšev sets. However, an isometric embedding which is not surjective, such as Minkowski translation (see [8, Example 4.1.2]) need not do so. Consider, for instance, the set $[A]$ of singletons for some convex body A. This is, as mentioned above, a Čebyšev set in $\mathcal{K}^n$, but for any convex body B, the family $\{B + x \mid x \in A\}$ is not (this follows from Proposition 3.5).

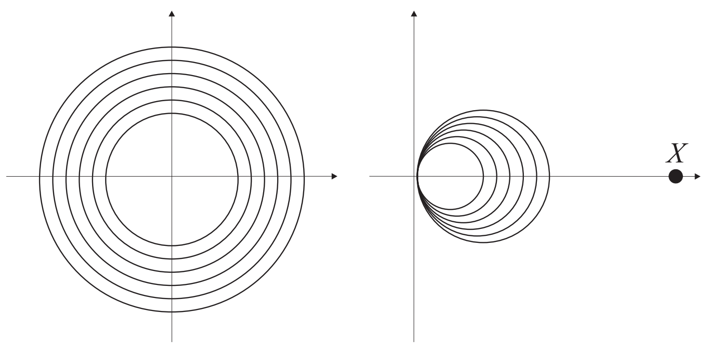

Figure 6: Isometric nests, only one of which is a Čebyšev set.

Remark 4.7. Let A be a segment. Then $[A]$ is a Čebyšev arc in $\mathcal{K}^n$ that is not nested. Similarly, for any strictly convex body B, the family $\{B + x \mid x \in A\}$ is (by Theorem 3.8) a Čebyšev set in $\mathcal{O}^n$ but is not nested. There is, however, some evidence for the following conjecture.

Conjecture 4.8 A Čebyšev arc in $\mathcal{K}_0^n$ must be nested .

In Example 2.10 we observed that a non-strongly nested family need not be a Čebyšev set.

Conjecture 4.9 A nested family in $\mathcal{K}^n$, $\mathcal{K}_0^n$, $\mathcal{O}^n$ or $\mathcal{O}_0^n$ is a Čebyšev set if and only if it is strongly nested.

The situation is not as straightforward as it might at first seem. The "peacock eye" of Figure 3 may be extended, as shown in Figure 7, to include nearer neighbours of both X and Y, thereby hiding from these sets the common boundary point. The conjecture is that this cannot be done simultaneously for all test sets.

5 Conclusion

It was made apparent in [3] that there are several diverse types of Čebyšev sets in hyperspaces. In this paper, we have generalized one of these, showing that not only families of parallel bodies but all closed, continuous, and strongly nested families

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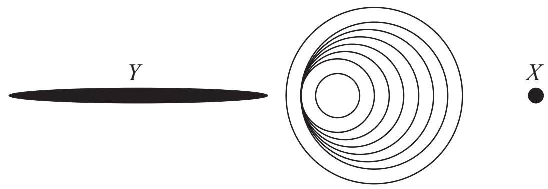

Figure 7: X and Y have unique nearest neighbours.

are Čebyšev sets. This class of examples is surprising because it depends more on the topology and the containment order of the hyperspace than on its semi-linear structure.

We have also introduced a new class of Čebyšev sets in $\mathcal{O}^n$, those sets consisting of the translates of a fixed strictly convex body by a closed convex set of vectors. For these sets, as for the sets of balls and singletons described in [3], the semi-linear structure of the hyperspace does appear to be essential.

The infinite-dimensional Čebyšev sets, described in detail in [3] and to which we have briefly alluded here, appear to have a completely different nature again. The main question, then, in the theory of Čebyšev sets in hyperspaces appears to be this: is there any hidden structure uniting these various families of Čebyšev sets? Is there, in the image of the folk tale, some single elephant of which these very different-seeming objects are all in fact parts?

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Dept. of Mathematics and, Computing Science, Saint Mary's University, Halifax, Nova Scotia, Canada
e-mail: rdawson@cs.stmarys.ca

Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland
e-mail: mariamos@mimuw.edu.pl