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C-SUPPLEMENTED SUBGROUPS OF FINITE GROUPS

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Abstract. A subgroup H of a group G is said to be *c-supplemented in G* if there exists a subgroup K of G such that $HK = G$ and $H \cap K$ is contained in $Core_G(H)$. We follow Hall's ideas to characterize the structure of the finite groups in which every subgroup is c-supplemented. Properties of c-supplemented subgroups are also applied to determine the structure of some finite groups.

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1. Introduction. In this paper the word group always means finite group.

A subgroup H of a group G is said to be *complemented in G* (complemented if G is understood) if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. In this case we say that K is a complement of H in G.

In his well known series of papers about the structure of solvable groups, Hall proved that a group is solvable if and only if every Sylow subgroup is complemented [**5**]. He also characterized in [**6**] the groups in which every subgroup is complemented. He called these groups *complemented groups* and proved that these groups are exactly the supersolvable groups with elementary abelian Sylow subgroups. It is clear from these results that complementation of some families of subgroups of a group has a strong influence on its structure. This idea was strengthened in [**1**], where the complementation of minimal subgroups and maximal subgroups of the Sylow subgroups is studied. The main goal of the present paper is to study the c-supplemented subgroups. This concept arises naturally as an extension of the c-normality introduced in [**8**] and it is closely related to complementation. Following Hall's idea, we determine the structure of the groups in which every subgroup is c-supplemented and study the influence of the c-supplementation of some families of subgroups on the structure of the group.

Most of the notation is standard and can be found in [**4**] and [**7**].

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2. *C*-supplemented subgroups.

Definition. (a) A subgroup H of a group G is said to be *c-supplemented* (in G) if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G = Core_G(H)$. (Here H_G is the largest normal subgroup of G that is contained in H.) In this case, we say that K is a *c-supplement* of H in G.

(b) A group G is said to be *c-supplemented* if every subgroup of G is c-supplemented in G.

It is clear from the definition that a complemented subgroup is c-supplemented but the converse does not hold. For example, let G be a cyclic p-group, where p is a prime, with order greater than p. Then $\Phi(G)$ is the only maximal subgroup of G and is not complemented. However $\Phi(G)$ is c-supplemented because it is normal and G is a c-supplement of $\Phi(G)$ in G.

Recall that a subgroup H of G is c-normal in G if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$. (See [**8**].)

It is clear that normality implies c-normality and c-normality implies c-supplementation. However the Sylow 5-subgroups of A_5 are c-supplemented in A_5 (by A_4) but neither of them is c-normal because A_5 is a simple group.

In the next lemma we gather the basic properties of c-supplemented subgroups.

Lemma 2.1. *Let G be a group.*

(1) *If H is c-supplemented in G, $H \leq M \leq G$, then H is c-supplemented in M.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$. Then H is c-supplemented in G if and only if H/N is c-supplemented in G/N.*

(3) *Let π be a set of primes. Let N be a normal π'-subgroup of G and let H be a π-subgroup of G. If H is c-supplemented in G, then HN/N is c-supplemented in G/N. If, furthermore, N normalizes H, then the converse also holds.*

(4) *Let $H \leq G$ and $L \leq \Phi(H)$. If L is c-supplemented in G, then $L \lhd G$ and $L \leq \Phi(G)$.*

Proof. (1) If $HK = G$ with $H \cap K \leq H_G$, then $M = M \cap G = H(M \cap K)$ and $H \cap (K \cap M) \leq H_G \cap M \leq H_M$, so that H is c-supplemented in M.

(2) Suppose that H/N is c-supplemented in G/N. Then there exists a subgroup K/N of G/N such that $G/N = (H/N)(K/N)$ and $(H/N) \cap (K/N) \leq (H/N)_{G/N} = (H_G)/N$. Then $G = HK$, $H \cap K \leq H_G$ and H is c-supplemented in G.

Conversely, if H is c-supplemented in G, then there exists $K \leq G$ such that $G = HK$ and $H \cap K \leq H_G$. It is easy to check that KN/N is a c-supplement of H/N in G/N.

(3) If H is c-supplemented in G, then there exists $K \leq G$ such that $G = HK$ and $H \cap K \leq H_G$. Since $|G|_{\pi'} = |K|_{\pi'} = |KN|_{\pi'}$, we have that $|K \cap N|_{\pi'} = |N|_{\pi'} = |N|$ and hence $N \leq K$. It is clear that $(HN/N)(K/N) = G/N$ and $(HN/N) \cap (K/N) = (H \cap K)N/N \leq (HN/N)_{G/N}$. Hence HN/N is c-supplemented in G/N.

Conversely, assume that HN/N is c-supplemented in G/N and N normalizes H. Let K/N be a c-supplement of HN/N. Then $HK = HNK = G$ and $(H \cap K)N/N \leq L/N = ((HN)/N)_{G/N}$. By hypothesis, $NH = N \times H$. Hence NH is both π-nilpotent and π-closed and so $L = H_1 \times N$ with $H_1 \leq H$ and $H_1 \lhd G$. Now we have $H \cap K \leq H_1 \leq H_G$ and H is c-supplemented in G.

(4) If L is c-supplemented in G with c-supplement K, then $LK = G$ and $L \cap K \leq L_G$. Now $H = H \cap G = L(H \cap K) = H \cap K$ since $L \leq \Phi(H)$. Therefore

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$L = L \cap K \leq L_G$ and hence $L \trianglelefteq G$. If $L \not\leq \Phi(G)$, then there exists a maximal subgroup M of G such that $LM = G$. Now $H = H \cap G = L(H \cap M) = H \cap M \leq M$. Therefore $G = LM \leq HM \leq M < G$, a contradiction. □

3. Theorems.

THEOREM 3.1. *Let G be a group. Then G is solvable if and only if every Sylow subgroup of G is c-supplemented in G.*

Proof. If G is solvable, then by [**4**, Theorem I.3.6] every Sylow subgroup of G is complemented and hence is c-supplemented.

Conversely, assume that every Sylow subgroup P of G is c-supplemented in G. By [**4**, Theorem I.3.6] we only need to prove that P is complemented in G. Let K_1 be a c-supplement of P in G. Then $PK_1 = G$ and $P \cap K_1 \leq P_G$. Let $K = P_G K_1$. We have $PK = G$ and $P \cap K = P_G(P \cap K_1) = P_G$. Note that $|G|_p = (|P||K|_p)/|P_G|$ and so P_G is a normal Sylow p-subgroup of K. By the Schur-Zassenhaus Theorem, [**7**, Theorem 9.1.10], we have that $K = P_G K_{p'}$ with $K_{p'}$ a Hall p'-subgroup of K. Now $G = PK = PK_{p'}$ and $P \cap K_{p'} = 1$. Hence P is complemented in G. The theorem is now proved. □

Using the same argument as in Theorem 3.1, we have the following corollary.

COROLLARY 3.2. *Let G be a group and let H be a Hall subgroup of G. Then H is complemented in G if and only if H is c-supplemented in G.*

THEOREM 3.3. *Let G be a group. Then the following statements are pairwise equivalent.*

(1) *G is c-supplemented.*
(2) *G is supersolvable. Let M be a subgroup of G and L be a subgroup M contained in $\Phi(M)$. Then $L \leq \Phi(G)$ and L is normal in G.*
(3) *G is supersolvable, every Sylow subgroup of $G/\Phi(G)$ is elementary abelian and every subgroup of $\Phi(G)$ is normal in G.*
(4) *$G/\Phi(G)$ is complemented and every subgroup of $\Phi(G)$ is normal in G.*

Proof. (1) $\Rightarrow$ (2). We prove that G is supersolvable by induction on the order of G.

Since every Sylow subgroup of G is c-supplemented in G, Theorem 3.1 implies that G is solvable. Let N be a minimal normal subgroup of G. Then N is an elementary abelian p-group for some prime p. By Lemma 2.1 (2) we know that G/N is c-complemented; hence G/N is supersolvable by induction. Let $x \in N$ with $|x| = p$. $<x>$ is c-supplemented in G implies that there exists $K \leq G$ such that $<x>K = G$ and $<x> \cap K \leq <x>_G$. $N = <x>(N \cap K)$ and $N \cap K \trianglelefteq G$ since N is abelian. N is a minimal normal subgroup implies that either $N \leq K$ or $N \cap K = 1$. If $N \leq K$, then $<x> = <x> \cap K \leq <x>_G$ and hence $N = <x>$. G/N is supersolvable implies that G is supersolvable. In the latter case we also have $N = <x> \cdot 1 = <x>$ and get the same conclusion.

Suppose that M is a subgroup of G and L is a subgroup of $\Phi(M)$. Then L is c-supplemented in G. Lemma 2.1 (4) implies that $L \triangleleft G$ and $L \leq \Phi(G)$.

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(2) $\Rightarrow$ (3). For every Sylow subgroup P of G, we have that $\Phi(P) \leq \Phi(G)$. Therefore every Sylow subgroup of $G/\Phi(G)$ is elementary abelian and every subgroup of $\Phi(G)$ is normal in G.

(3) $\Rightarrow$ (4). This follows from [**6**, Theorem 2].

(4) $\Rightarrow$ (1). Assume that $G/\Phi(G)$ is complemented and every subgroup of $\Phi(G)$ is normal in G. Let H be a subgroup of G. Then there exists a subgroup $K/\Phi(G)$ of $G/\Phi(G)$ such that $(H\Phi(G)/\Phi(G))(K/\Phi(G)) = G/\Phi(G)$ and $(H\Phi(G)/\Phi(G)) \cap (K/\Phi(G)) = ((H \cap K)\Phi(G))/\Phi(G) = 1$. It follows that $HK = G$ and $H \cap K \leq \Phi(G)$. Hence $H \cap K \leq H_G$. By definition, H is c-supplemented in G and so G is c-supplemented. The proof of the theorem is complete. □

4. Applications. In this section we investigate the influence of the existence of c-supplements for some families of subgroups on the structure of the group. We focus our attention to minimal subgroups of the group. Let us first introduce the following notation.

Let p be a prime and G a group. We write

$\mathcal{P}_p(G) = \{x | \, x \in G, |x| = p\}$,

$\mathcal{P}_4(G) = \{x | \, x \in G, |x| = 4\}$,

$\mathcal{P}(G) = \cup_{p \in \pi(G)} \mathcal{P}_p(G)$,

$\mathcal{P}^*(G) = \mathcal{P}_4(G) \cup \mathcal{P}(G)$.

Let x be an element of G. We say that x is *c-supplemented in G* if $< x >$ is c-supplemented in G.

THEOREM 4.1. *Let G be a group and let K be the supersolvable residual $G^{\mathcal{U}}$ of G. Suppose that every element of $\mathcal{P}^*(K)$ is c-supplemented in G. Then G is supersolvable.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order.

(1) Every proper subgroup of G is supersolvable. Furthermore

(a) there exists a normal Sylow p-subgroup of G such that $G = P \rtimes R$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$;

(b) if $p > 2$ then the exponent of P is p; when $p = 2$ the exponent of P is 2 or 4.

Let M be a maximal subgroup of G. It is clear that $M/M \cap K$ is supersolvable and hence $M^{\mathcal{U}} \leq M \cap K$. By Lemma 2.1, every element of $\mathcal{P}^*(M^{\mathcal{U}})$ is c-supplemented in M, so that M satisfies the hypotheses of G. The minimal choice of G yields that M is supersolvable. This holds for every maximal subgroup M of G. Hence we have that G is not supersolvable but every proper subgroup of G is supersolvable. [**3**, Satz 1] implies (1)(a) and (1)(b).

(2) $K = P$. Since G/P is supersolvable, we have that $K \leq P$. Then $K\Phi(P)/\Phi(P)$ is a normal subgroup of $G/\Phi(P)$ contained in $P/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, it follows that either $K\Phi(P) = P$ or $K \leq \Phi(P)$. If $K < \Phi(P)$, then K is actually contained in $\Phi(G)$ and $G/\Phi(G)$ is supersolvable. Hence G is supersolvable, a contradiction, and so we have that $P = K$.

(3) $\Phi(P) \neq 1$. Otherwise P is elementary abelian and hence, by (1)(b), every element of P lies in $\mathcal{P}(K)$. Our hypotheses claim that every element of P is c-supplemented in G. Let $1 \neq x \in P$. Then there exists $M \leq G$ such that $< x > M = G$ and $< x > \cap M \leq < x >_G$. Then $P = < x > (P \cap M)$. Since P is abelian, it follows that $P \cap M \trianglelefteq G$. By (1)(a), P is a minimal normal subgroup of G when $\Phi(P) = 1$.

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Therefore $P \cap M = 1$ or $P \leq M$. In both cases, we have that $<x>= P$ and therefore G is supersolvable, a contradiction.

(4) $p = 2$. Assume that $p > 2$. Then by (1)(b) every element of P is c-supplemented in G. Moreover, by Lemma 2.1 (4), $\Phi(P)$ is contained in $\Phi(G)$ and $\Phi(P) \lhd G$. Next we see that the hypotheses of the theorem hold in $G/\Phi(P)$. Let $x \in P - \Phi(P)$. By hypotheses there exists a subgroup M of G such that $G =< x > M$ and $< x > \cap M \leq < x >_G$. If $< x >=< x >_G$, then (1)(a) implies that $P =< x > \Phi(P) = < x >$. Then G is supersolvable, a contradiction, and so we have that $< x > \cap M = 1$. Hence M is a maximal subgroup of G because $o(x) = p$. This implies that $G/\Phi(P) =< x > \Phi(P)/\Phi(P) \cdot M/\Phi(P)$ and $(< x > \Phi(P)/\Phi(P)) \cap (M/\Phi(P)) = 1$ and $< x > \Phi(P)/\Phi(P)$ is c-supplemented in $G/\Phi(P)$. The minimal choice of G (notice that $\Phi(P) \neq 1$ by (3)) implies that $G/\Phi(P)$ is supersolvable. Since $\Phi(P) \leq \Phi(G)$, we have that $G/\Phi(G)$ is supersolvable and so is G, a contradiction.

(5) $exp(P) = 4$, $\Phi(P) \lhd G$, $\Phi(P) \leq \Phi(G)$ and $G/\Phi(P)$ satisfies the hypotheses of the theorem.

If $exp(P) = 2$, then P is elementary abelian, contrary to (3). Note that every element of $\Phi(P)$ is c-supplemented and hence, by Lemma 2.1 (4), $\Phi(P) \lhd G$ and $\Phi(P) \leq \Phi(G)$. For any element $x \in P - \Phi(P)$ with $|x| = 2$, the same argument of (4) shows that $< x > \Phi(P)/\Phi(P)$ is c-supplemented in $G/\Phi(P)$. Now assume that $|x| = 4$. If $< x > \unlhd G$, nothing remains to be proved. Note that $< x^2 > \leq \Phi(P)$ and hence $< x^2 > \unlhd G$, by Lemma 2.1 (4). Let K_1 be a c-supplement of $< x >$. Then $< x > K_1 = G$ and $< x > \cap K_1 \leq < x >_G =< x^2 >$. Let $K =< x^2 > K_1$. Then $< x > K = G$ and $< x > \cap K =< x^2 >$. Now $|G : K| = 2$ implies that K is a maximal subgroup of G. Hence $\Phi(P) \leq K$ and

$$(< x > \Phi(P))/\Phi(P))(K/\Phi(P)) = G/\Phi(P)$$

and

$$(< x > \Phi(P))/\Phi(P)) \cap (K/\Phi(P)) \leq ((< x > \Phi(P))/\Phi(P))_{G/\Phi(P)}.$$

Therefore (5) holds.

By our minimal order choice, (5) implies that $G/\Phi(P)$ is supersolvable and so is G. This final contradiction completes our proof. □

Since $G^{\mathcal{U}} \leq G'$ for every group G, we have the following corollary.

COROLLARY 4.2. *Let G be a group. If every element of $\mathcal{P}^*(G')$ is c-supplemented in G, then G is supersolvable.*

THEOREM 4.3. *Let G be a group and let $K = G^{\mathcal{N}}$ be the nilpotent residual of G. Suppose that every element of $\mathcal{P}_4(K)$ is c-supplemented in G. Then G is nilpotent if and only if $< x >$ lies in the hypercenter $Z_\infty(G)$ of G, for every element x of $\mathcal{P}(K)$.*

Proof. If G is nilpotent, then $G = Z_\infty(G)$. Certainly $\mathcal{P}(K) \subset G$ and so we only need to prove that the converse is true.

Assume that the statement is false and let G be a counterexample of minimal order. The following statements hold.

(1) Every proper subgroup M of G is nilpotent.

In fact, if $M < G$, then we have $M/M \cap K \cong MK/K \leq G/K$ which is nilpotent. We have that $M^{\mathcal{N}} \leq M \cap K \leq K$. By Lemma 2.1 and $Z_\infty(G) \cap M \leq Z_\infty(M)$, M satisfies the hypotheses of G. By the minimal choice of G, M is nilpotent.

(2) G is a minimal non-nilpotent group so that G has the following properties.

(2)(a) There exists a normal Sylow p-subgroup of G such that $G = P \rtimes Q$. $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, and $\Phi(P) \leq Z(G)$.

(2)(b) If $p > 2$, then the exponent of P is p. When $p = 2$ the exponent of P is 2 or 4. See [**3**, Satz A].

(3) $K = P$.

Since G/P is nilpotent, we have that $K \leq P$. If $K < P$, then we have that $K\Phi(P)/\Phi(P) \lhd G/\Phi(P)$ which implies that $P = K \leq \Phi(P) \leq \Phi(G)$. Therefore $G/\Phi(G)$ is nilpotent and so is G, a contradiction.

(4) $p = 2$ and P has element of order 4.

Otherwise, by our hypotheses, every element of P lies in $\mathcal{P}(K) \subset Z_\infty(G)$. Now $G/Z_\infty(G)$ is nilpotent yields G is nilpotent, a contradiction.

(5) There is an element y of order 4 such that $y \notin \Phi(P)$.

If all the elements of order 4 lie in $\Phi(P)$, then again $P \leq Z(G) \leq Z_\infty(G)$, a contradiction.

(6) G is 2-nilpotent.

By (5), there exists an element $x \in P - \Phi(P)$ with $|x| = 4$. Since $< x >$ is c-supplemented in G, there exists a subgroup K_1 of G such that $G = < x > K_1$ and $< x > \cap K \leq < x >_G$. If $< x > = < x >_G$, then $< x > \Phi(P)/\Phi(P)$ is a non-trivial normal subgroup of $G/\Phi(G)$. (2)(a) implies that $P = < x > \Phi(P) = < x >$ is cyclic. By [**7**, 10.1.9] we have that G is 2-nilpotent.

Assume that $< x >_G$ is a proper subgroup of $< x >$. Then $< x >_G = < x^2 >$ because $x^2 \in Z(G)$. Let $K = < x^2 > K_1$. Then $G = < x > K$ and $< x > \cap K = < x^2 >$. Therefore $|G : K| = 2$ and K is normal in G. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, it follows that either $P \leq K$ or $P \cap K \leq \Phi(P)$. If $P \leq K$, then $< x > = < x^2 >$, a contradiction. Hence $P \cap K \leq \Phi(P)$ and $P = < x >$ is cyclic. Again we have that G is 2-nilpotent.

By step (6), Q is normal in G and so G is nilpotent, a contradiction. □

COROLLARY 4.4. *Let G be a group and suppose that every element of order 4 of G' is c-supplemented in G. Then G is nilpotent if and only if every element of $\mathcal{P}(G')$ lies in $Z_\infty(G)$.*

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