SOME REMARKS ON REGULAR BANACH SPACES

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A Banach space E is said to be *regular* if every bounded linear operator from E into E' is weakly compact. This property was studied in [7, 9] under the name *Property* (w). In [7], using James type spaces as constructed in [4], examples were given of regular Banach spaces which fail to have weakly sequentially complete duals, answering a question raised in [9]. In this paper, we present some more results concerning the regularity of James type spaces.

The study of polynomials, or more generally, analytic functions on Banach spaces leads one to consider symmetric operators on a Banach space. Recall that a bounded linear operator $T : E \to E'$, where E is a Banach space, is *symmetric* if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in E$. In [2], the class of Banach spaces E such that every symmetric operator from E into E' is weakly compact is found to play a useful role. Following [3], we call such spaces *symmetrically regular*. It is shown below that all James type spaces built on the basis of a reflexive Banach space, and all their duals, are symmetrically regular. Consequently, we obtain many examples of symmetrically regular Banach spaces which are not regular. This answers a question raised in [2, p. 83].

We use standard Banach space terminology as may be found in [8]. All subsequent results hold for both real or complex Banach spaces. For a sequence (e_i) in a Banach space, $[(e_i)]$ denotes the closed linear span of (e_i). If (f_i) is another sequence in a possibly different Banach space, we say that (e_i) *dominates* (f_i) if there is a constant C such that $\|\sum a_i f_i\| \leq C \|\sum a_i e_i\|$ for every finitely supported scalar sequence (a_i). Two sequences are *equivalent* if each dominates the other. A *subsymmetric* sequence is an unconditional sequence which is equivalent to each of its subsequences.

The author thanks Professor R. M. Aron for bringing to his attention the references [2, 3], and the question raised on p. 83 of [2].

1. Symmetric regularity of James type spaces. Let (e_i) be a normalized basis of a Banach space E. For $(a_i) \in c_{00}$, the space of all finitely supported scalar sequences, let

$$\left\| \sum a_i u_i \right\| = \sup \left\{ \left\| \sum_{i=1}^{k} \left(\sum_{j=p(i)}^{q(i)} a_j \right) e_{p(i)} \right\| : k \in \mathbb{N}, 1 \leq p(1) \leq q(1) < \ldots < p(k) \leq q(k) \right\}.$$

The completion of the linear span of the sequence (u_i) is denoted by $J(e_i)$. The biorthogonal sequences of (e_i) and (u_i) are denoted by (e_i') and (u_i') respectively. The functional S defined by $S(\sum a_i u_i) = \sum a_i$ is bounded; hence $S \in J(e_i)'$. *In the sequel, we always assume that E is reflexive.* In this case, by a combination of Theorems 2.2 and 4.1 of [4], we see that $J(e_i)$ is quasi-reflexive of order one, and that $J(e_i)' = [\{S\} \cup \{u_i'\}_{i=1}^{\infty}]$. Hence, if we let $L \in J(e_i)''$ be the functional which has the value 1 at S and annihilates $[(u_i')]$, then $J(e_i)'' = [\{L\} \cup \{u_i\}_{i=1}^{\infty}]$.

PROPOSITION 1. *A bounded linear operator $T : J(e_i) \to J(e_i)'$ is weakly compact if and only if* $\lim_i \lim_j \langle Tu_i, u_j \rangle = \lim_j \lim_i \langle Tu_i, u_j \rangle$.

Proof. Note that (u_i) converges to L in the $\sigma(J(e_i)'', J(e_i)')$-topology. If T is weakly compact, then (Tu_i) converges weakly to some $x' \in J(e_i)'$. Therefore,

$$\lim_i \lim_j \langle Tu_i, u_j \rangle = \lim_i \langle Tu_i, L \rangle = \langle x', L \rangle$$

$$= \lim_j \langle x', u_j \rangle = \lim_j \lim_i \langle Tu_i, u_j \rangle.$$

Conversely, assume that $\lim_i \lim_j \langle Tu_i, u_j \rangle = \lim_j \lim_i \langle Tu_i, u_j \rangle$. Since (Tu_i) is weakly Cauchy, it converges $\sigma(J(e_i)', J(e_i))$ to some $x' \in J(e_i)'$. Now

$$\lim_i \langle Tu_i, L \rangle = \lim_i \lim_j \langle Tu_i, u_j \rangle = \lim_j \lim_i \langle Tu_i, u_j \rangle$$

$$= \lim_j \langle x', u_j \rangle = \langle x', L \rangle.$$

Hence (Tu_i) converges to x' weakly. Thus $T''L = x' \in J(e_i)'$. Since $J(e_i)'' = [\{L\} \cup \{u_i\}_{i=1}^{\infty}]$, it follows that $T''J(e_i)'' \subseteq J(e_i)'$. Therefore, T is weakly compact. $\square$

For $k \in \mathbb{N}$, let $v_k' = \sum_{i=1}^{k} u_i'$. The proof of the following result is entirely similar to that of Proposition 1 and is left to the reader.

PROPOSITION 2. *A bounded linear operator $T : J(e_i)' \to J(e_i)''$ is weakly compact if and only if $\lim_i \lim_j \langle Tv_i', v_j' \rangle = \lim_j \lim_i \langle Tv_i', v_j' \rangle$.*

THEOREM 3. *If (e_i) is a normalized basis of a reflexive Banach space, then $J(e_i)$ and all its duals are symmetrically regular.*

Proof. This follows immediately from the preceding propositions and the fact that $J(e_i)$ is quasi-reflexive. $\square$

2. Regularity of James type spaces. In this section, we present more results on the regularity of spaces of the form $J(e_i)$ or $J(e_i)'$. In particular, we will see that if J is *the* James space [5], then J' is symmetrically regular but not regular. It is also shown that J itself is regular. As a corollary, we obtain a result of Andrew [1] that J is not isomorphic to any subspace of J'. Recall the standing assumption from §1 that (e_i) is a normalized basis of a reflexive Banach space.

PROPOSITION 4. *Suppose that (e_i) dominates each of its subsequences. Let (i_j) be a strictly increasing sequence in $\mathbb{N}$. For any $x' \in J(e_i)'$, the series $\sum_j x'(u_{i_j})u_j'$ is weak* convergent. Moreover, the operator $U : J(e_i)' \to J(e_i)'$, $Ux' = weak^*\text{-}\sum_j x'(u_{i_j})u_j'$, is bounded.*

Proof. It is easy to see that (u_i) dominates all of its subsequences. It follows easily that there is a constant K such that $\|\sum_j a_j u_{i_j}\| \leq K \|\sum_j a_j u_j\|$, for all subsequences (u_{i_j}) of

(u_i) and all $(a_j) \in c_{00}$. Let $x' \in J(e_i)'$. For any $k \in \mathbb{N}$, there exists $(a_j) \in c_{00}$ so that $\|\sum_j a_j u_j\| \leq 2$ and

$$\left\| \sum_{j=1}^{k} x'(u_{i_j})u_j' \right\| \leq \left| \left\langle \sum_j a_j u_j, \sum_{j=1}^{k} x'(u_{i_j})u_j' \right\rangle \right| = \left| \left\langle \sum_{j=1}^{k} a_j u_{i_j}, x' \right\rangle \right|.$$

As mentioned above, $\|\sum_j a_j u_{i_j}\| \leq K \|\sum_j a_j u_j\| \leq 2K$. Therefore, $\| \sum_{j=1}^{k} x'(u_{i_j})u_j' \| \leq 2K \|x'\|$ for every $k \in \mathbb{N}$. The series $\sum_j x'(u_{i_j})u_j'$ clearly converges on the linear span of (u_i). Since its partial sums are norm bounded, we conclude that it is weak* convergent. The boundedness of the operator U follows from the computation above. $\square$

For the following theorem, recall the sequence (v_i') as defined in § 1.

THEOREM 5. *Suppose that (e_i) dominates each of its subsequences. Then $J(e_i)$ is not regular if and only if the linear map T, which maps each u_i to v_i', is bounded from $J(e_i)$ into $J(e_i)'$.*

Proof. If T is bounded, then $J(e_i)$ fails to be regular because T is clearly not weakly compact. Conversely, assume that $J(e_i)$ is not regular. There is a non-weakly compact operator $R : J(e_i) \to J(e_i)'$. As $[(u_i')]$ has codimension 1 in $J(e_i)'$, we may as well assume that the range of R is contained in $[(u_i')]$. In particular, $\lim_j \langle Ru_i, u_j \rangle = 0$ for all i. By Proposition 1, $\lim_j \lim_i \langle Ru_i, u_j \rangle \neq \lim_i \lim_j \langle Ru_i, u_j \rangle = 0$. Note that the first double limit exists because (u_i) is weakly Cauchy. Without loss of generality, we may assume that $\lim_j \lim_i \langle Ru_i, u_j \rangle = 1$. Let $a(i,j) = \langle Ru_i, u_j \rangle$ and $a(j) = \lim_i \langle Ru_i, u_j \rangle$. One can find strictly increasing sequences (i_k) and (j_k) in $\mathbb{N}$ such that
   (1) $|a(i_k, j_l) - a(j_l)| \leq 2^{-k-l}$ if $k \geq l$,
   (2) $|a(i_k, j_l)| \leq 2^{-k-l}$ if $l > k$,
   (3) $\sum_l |a(j_l) - 1| \leq 1$.

Construct U as in Proposition 4 using the sequence (j_k). Then U is bounded. Also $V : J(e_i) \to J(e_i), V\left(\sum_i a_i u_i\right) = \sum_k a_k u_{i_k}$ is bounded since (u_i) dominates all of its subsequences. For $k \in \mathbb{N}$, let $Wu_k = \sum_{l=1}^{k} a(j_l)u_l'$. Extend W to the span of (u_i) by linearity. Then

$$\|Wu_k - URVu_k\| = \left\| \sum_{l=1}^{k} a(j_l)u_l' - \sum_{l=1}^{\infty} a(i_k, j_l)u_l' \right\|$$

$$\leq \sum_{l=1}^{k} |a(j_l) - a(i_k, j_l)| \, \|u_l'\| + \sum_{l=k+1}^{\infty} \|a(i_k, j_l)u_l'\|$$

$$\leq \sum_{l=1}^{\infty} \frac{\|u_l'\|}{2^{k+l}} \leq \frac{\sup \|u_l'\|}{2^k}.$$

Therefore

$$\left\| W\left(\sum c_k u_k \right) - URV\left(\sum c_k u_k \right) \right\| \le \sup_k |c_k| \sup_l \|u_l'\|,$$

for all $(c_k) \in c_{00}$. It follows that W is bounded. Finally, for all $(c_k) \in c_{00}$,

$$\left\| W\left(\sum c_k u_k \right) - T\left(\sum c_k u_k \right) \right\| = \left\| \sum_l \left(\sum_{k=l}^{\infty} c_k \right)(a(j_l) - 1)u_l' \right\|$$

$$\le \sum_l |a(j_l) - 1| \sup_l \left| \sum_{k=l}^{\infty} c_k \right| \sup_l \|u_l'\|$$

$$\le \sum_l |a(j_l) - 1| \left\| \sum_k c_k u_k \right\| \sup_l \|u_l'\|$$

$$\le \left\| \sum_k c_k u_k \right\| \sup_l \|u_l'\|.$$

Hence T is bounded, as required. □

THEOREM 6. *Suppose that (e_i) is subsymmetric. If (e_i) does not dominate (e_i'), then $J(e_i)$ is regular.*

Proof. Since (e_i) is subsymmetric, it is easy to see that (e_i) is equivalent to $(u_{2i} - u_{2i-1})$. Now, for sequences (a_i) and (b_i) in c_{00},

$$\left| \left\langle \sum b_i(u_{2i} - u_{2i-1}), \sum a_i u_{2i}' \right\rangle \right| = \left| \sum a_i b_i \right| = \left| \left\langle \sum b_i e_i, \sum a_i e_i' \right\rangle \right|.$$

It follows that (u_{2i}') is equivalent to (e_i'). If $J(e_i)$ is not regular, then the linear map $T : J(e_i) \rightarrow J(e_i)'$, $Tu_i = v_i'$, is bounded by Theorem 5. In particular, if $(a_i) \in c_{00}$, then

$$\left\| \sum a_i u_{2i}' \right\| = \left\| T \sum a_i(u_{2i} - u_{2i-1}) \right\| \le \|T\| \left\| \sum a_i(u_{2i} - u_{2i-1}) \right\|.$$

Thus $(u_{2i} - u_{2i-1})$ dominates (u_{2i}'). Hence (e_i) dominates (e_i'). □

The converse of Theorem 6 is false, as the next result shows. The James space is (isomorphic to) the space $J(e_i)$, where (e_i) is the usual basis of l^2.

PROPOSITION 7. *The James space is regular.*

Proof. Otherwise, the operator T given by Theorem 5 is bounded. Therefore

$$\left\| \sum b_i u_i' \right\| \le \|T\| \left\| \sum (b_i - b_{i+1})u_i \right\|, \tag{1}$$

for every $(b_i) \in c_{00}$. For any $(c_i) \in c_{00}$, and $1 \le p(1) < p(2) < \ldots < p(k)$, let

$$x = c_1 u_{p(1)} + (c_2 - c_1)u_{p(2)} + \ldots + (c_{k-1} - c_{k-2})u_{p(k-1)} - c_{k-1}u_{p(k)}.$$

Then $\|x\| \le 2 \|\sum c_i e_i\|$. Now

$$\left\| \sum b_i u_i' \right\| \ge \frac{|\langle x, \sum b_i u_i' \rangle|}{\|x\|}.$$

Taking the supremum over all (c_i) in the unit ball of l^2, we see that

$$\left\| \sum (b_{p(i)} - b_{p(i+1)}) e_i \right\| \le 2 \left\| \sum b_i u_i' \right\|.$$

Consequently, $\left\| \sum (b_i - b_{i+1}) u_i \right\| \le 2 \left\| \sum b_i u_i' \right\|$. Together with (1), we see that the map $R : [(u_i')] \to J$, $R(\sum b_i u_i') = \sum (b_i - b_{i+1}) u_i$ is an isomorphism. Since $[(u_i')]$ is isomorphic to J', this is a violation of a theorem of James [6]. $\square$

COROLLARY 8. (Andrew [1]). *There is no linear isomorphism from J into J'.*

Similar results hold for $J(e_i)'$. As the proofs are also similar, we only sketch the arguments.

LEMMA 9. *If (e_i) is dominated by all of this subsequences, then (v_i') dominates all of its subsequences.*

Sketch of Proof. There is a constant K such that for every $\sum c_i u_i \in J(e_i)$ and every sequence $0 = p_0 < p_1 < \ldots < p_k$, the norm of

$$x = \sum_{j=1}^{k} \left(\sum_{i=p_{j-1}+1}^{p_j} c_i \right) u_j$$

is no more than $K \| \sum c_i u_i \|$. Since

$$\left\langle \sum c_i u_i, \sum_{i=1}^{k} a_i v_{p_i}' \right\rangle = \left\langle x, \sum_{i=1}^{k} a_i v_i' \right\rangle,$$

the result follows. $\square$

For the following theorem, note that (v_i') is a basis of $[(u_i')]$.

THEOREM 10. *Suppose that (e_i) is dominated by all of its subsequences. Then $J(e_i)'$ is not regular if and only if the linear map T which maps each v_i' to u_i is bounded from $[(u_i')]$ into $J(e_i)$.*

Sketch of Proof. If T is bounded, then $J(e_i)'$ is clearly not regular. Conversely, assume that $J(e_i)'$ is not regular. Then there exists a non-weakly compact operator $R : [(u_i')] \to J(e_i)$. Now (Rv_i') is weak* convergent to $x + aL$ for some $x \in J(e_i)$ and some scalar a. We may take x to be 0. If $a = 0$, R is weakly compact by Proposition 2. Thus we may assume that $a = 1$. By a perturbation argument similar to the one used in the proof of Theorem 5, we may assume that there is a subsequence (v_{i_k}') of (v_i') such that (Rv_{i_k}') is a block basis of (u_i) satisfying $\langle Rv_{i_k}', S \rangle = 1$, for all k. By Lemma 9, (v_i') dominates (v_{i_k}'), and hence (Rv_{ik}'). Since (e_i) is dominated by all of its subsequences, and $\langle Rv_{i_k}', S \rangle = 1$, for all k, we also see that (Rv_{i_k}') dominates (u_i). Therefore (v_i') dominates (u_i), which means that T is bounded. $\square$

THEOREM 11. *Suppose that (e_i) is subsymmetric. Then $J(e_i)'$ is regular if and only if (e_i') does not dominate (e_i).*

Sketch of Proof. Assume that $J(e_i)'$ is not regular. By Theorem 10, the linear map $T : [(u_i')] \to J(e_i)$, $Tv_i' = u_i$, is bounded. Hence $(u_{2i} - u_{2i-1})$, being the image under T of the sequence (u_{2i}'), is dominated by the latter sequence. Using the assumption on (e_i), we

see that (e_i) is equivalent to $(u_{2i} - u_{2i-1})$, and (u'_{2i}) is equivalent to (e'_i). Combining the above, we deduce that (e'_i) dominates (e_i). Conversely, assume that (e'_i) dominates (e_i). For any $1 \leq p(1) \leq q(1) < p(2) \leq q(2) < \ldots < p(k) \leq q(k)$, and $(c_i) \in c_{00}$, let $x = \sum_{i=1}^{k} c_i(u_{p(i)} - u_{q(i)+1})$. There is a constant K, depending only on the sequence (e_i), such that $\|x\| \leq K \|\sum c_i e_i\|$. Computing the norm of any $\sum a_i u'_i$ on the elements of the form x, and using the fact that (e'_i) dominates (e_i), we obtain a constant M such that

$$\left\| \sum_i (a_{p(i)} - a_{q(i)+1})e_i \right\| \leq M \left\| \sum a_i u'_i \right\|$$

for any $(a_i) \in c_{00}$, and $1 \leq p(1) \leq q(1) < p(2) \leq q(2) < \ldots < p(k) \leq q(k)$. Consequently,

$$\left\| \sum (a_i - a_{i+1})u_i \right\| \leq M \left\| \sum a_i u'_i \right\|.$$

It follows immediately that the operator T is bounded. □

The following result, which follows by combining the theorem above with Theorem 3, answers a question raised on p. 83 of [2].

THEOREM 12. *Let (e_i) be the unit vector basis of l^p for some p with $2 \leq p < \infty$. Then $J(e_i)'$ is symmetrically regular but not regular.*

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