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A RESULT ON SEMI-ARTINIAN RINGS

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Abstract It was shown by Huynh and Rizvi that a ring R is semisimple artinian if and only if every continuous right R-module is injective. However, a characterization of rings, over which every *finitely generated* continuous right module is injective, has been left open. In this note we give a partial solution for this question. Namely, we show that for a right semi-artinian ring R, every finitely generated continuous right R-module is injective if and only if all simple right R-modules are injective.

Keywords: V ring; semi-artinian ring; CS module; injective module

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We consider associative rings with identity, and all modules are unitary modules. A module M is called a *CS module* (or an *extending module*) if every submodule of M is essential in a direct summand of M. Furthermore, a module M is called a *quasi-continuous* module if M is a CS module such that for any two direct summands U, V of M with $U \cap V = 0$, $U \oplus V$ is also a direct summand of M. If M is a CS module such that every submodule isomorphic to a direct summand of M is itself a direct summand of M, then M is called a *continuous* module. It is known that continuous modules are quasi-continuous.

A ring R is called a *right* V *ring* if every simple right R-module is injective. Further, a ring R is said to be *right semi-artinian* if every non-zero right R-module has a non-zero socle. Semi-artinian rings and modules were investigated, for example, in [**3**] and [**5**].

It was shown in [**10**] that if every continuous right R-module is injective, then R is semisimple artinian. Motivated by this, rings R with the following properties were discussed in [**9**].

(p) Every finitely generated continuous right R-module is injective.

(q) Every finitely generated CS right R-module is (quasi-) continuous.

The structure of rings satisfying either (p) or (q) is unknown. However, such a ring need not be semisimple artinian. It is shown in [**9**] that any simple right and left SI ring with zero socle is a ring with property (p) (but not with (q)). Moreover, [**7**, Example 3.2] shows that rings satisfying (p) and (q) need not be semisimple artinian. Note that a ring R is *right SI* if every singular right R-module is injective (cf. [**7**, Chapter 3]). In this note we show that all right semi-artinian right V rings satisfy both (p) and (q). For examples of such rings see [**5**].

Theorem 1. *For a right semi-artinian ring R the following conditions are equivalent:*

(a) *R is a right V ring;*

(b) *every finitely generated CS right R-module is injective;*

(c) *every 2-generated CS right R-module is quasi-continuous; and*

(d) *every finitely generated continuous right R-module is injective.*

To prove Theorem 1 we give a sufficient condition for a finitely generated CS module to have finite uniform dimension, in a general setting. A module A is defined to be a *QFD module* if every factor module of A has finite uniform dimension (cf. [**1**, p. 294] and [**2**]). Note that any noetherian or artinian module is QFD, and, more generally, so too is any module with Krull dimension by [**6**, 6.1(1) and 6.2(2)]. It is easy to check that if Y is a submodule of a module X such that Y and X/Y are both QFD, then X is also QFD. Consequently, any finite sum of QFD submodules of a module is itself QFD. Note further that the sum $S_1(M)$ of all QFD submodules of a given module M is a fully invariant submodule of M. Therefore, if $M = M_1 \oplus M_2$, then $S_1(M) = S_1(M_1) \oplus S_1(M_2)$.

Proposition 2. *Let M be a finitely generated CS module such that for every proper submodule $K \subset M$, the factor module M/K contains a non-zero QFD submodule. Then M is a direct sum of finitely many uniform submodules.*

Proof. As above, let $S_1(M)$ be the sum of all QFD submodules of M. By assumption, $S_1(M)$ is non-zero. Inductively we can define a *QFD series* $\{S_\alpha(M)\}$ of M as follows: $S_\alpha(M)$ is a submodule of M containing $S_{\alpha-1}(M)$ such that $S_\alpha(M)/S_{\alpha-1}(M) = S_1(M/S_{\alpha-1}(M))$. If α is a limit ordinal, then $S_\alpha(M) = \cup_{\beta<\alpha} S_\beta(M)$. By the assumption on M, there exists an ordinal γ such that $S_\gamma(M) = M$. We call the least ordinal γ with this property the *QFD length* of M.

We prove Proposition 2 by induction on γ. The statement is true for $\gamma = 1$ because a finitely generated module M with $S_1(M) = M$ is QFD, and hence of finite uniform dimension. Suppose that $\gamma > 1$ and the result holds for all ordinals less than γ.

Assume on the contrary that M does not have finite uniform dimension. Since M is finitely generated, γ cannot be a limit ordinal. Hence $\gamma - 1$ exists and $M/S_{\gamma-1}(M)$ is the sum of its QFD submodules. Moreover, as $M/S_{\gamma-1}(M)$ is finitely generated, $M/S_{\gamma-1}(M)$ is a sum of finitely many QFD submodules. Therefore, $M/S_{\gamma-1}(M)$ has finite uniform dimension. Let k denote the uniform dimension of $M/S_{\gamma-1}(M)$, and let n be any integer greater than k. Since M is CS (with infinite uniform dimension), we can decompose it as $M = M_1 \oplus \cdots \oplus M_n$, where each M_i does not have finite uniform dimension. Now, our above remark shows that $S_{\gamma-1}(M) = S_{\gamma-1}(M_1) \oplus \cdots \oplus S_{\gamma-1}(M_n)$.

Hence $M/S_{\gamma-1}(M) \cong M_1/S_{\gamma-1}(M_1) \oplus \cdots \oplus M_n/S_{\gamma-1}(M_n)$. As the uniform dimension of $M/S_{\gamma-1}(M)$ is k, there must be an M_i in the decomposition of M that is contained in $S_{\gamma-1}(M_i)$. On the other hand, as a direct summand of M, M_i is finitely generated and CS. Moreover, M_i satisfies the other assumption about M. By the induction hypothesis, M_i has finite uniform dimension. But this is a contradiction. Thus M must have finite uniform dimension. As M is CS, M is a direct sum of finitely many uniform modules. □

Corollary 3. *Let R be a ring such that for every proper right ideal A there exists a right ideal B containing A such that B/A is a non-zero QFD R-module. Then every finitely generated CS right R-module has finite uniform dimension.*

Proof. Let M be a finitely generated CS module. By hypothesis, every non-zero homomorphic image of M contains a non-zero QFD submodule. Apply Proposition 2. □

Corollary 4 (see Theorem 4.2 in [4]). *A finitely generated CS right module over a right semi-artinian ring R has finite uniform dimension.*

Proof. By Corollary 3. □

Proof of Theorem 1.

(a) $\Rightarrow$ (b). By Corollary 4, any finitely generated CS right R-module X has finite uniform dimension. Hence by (a), $X = \mathrm{Soc}(X)$. This proves that X is injective.

(b) $\Rightarrow$ (c). Clear.

(c) $\Rightarrow$ (a). Let S be a simple right R-module with injective hull S^*. Suppose that $S^* \neq S$. Then there is a submodule $X \subseteq S^*$ containing S such that X/S is simple because R is a right semi-artinian ring. Hence X is cyclic and the composition length of X is 2. Let $Y = S \oplus X$. By [**6**, 8.14], Y is a CS module. But Y is 2-generated, hence quasi-continuous by (c). Therefore, S is X-injective (cf. [**11**, Proposition 2.10]). Thus S splits in X, a contradiction. Hence we must have $S = S^*$, proving that R is a right V ring.

(b) $\Rightarrow$ (d) $\Rightarrow$ (a). Clear. □

Remark 5. Proposition 2 was motivated by [**4**, Theorem 4.2] and its proof. Moreover, in [**4**, Theorem 4.5], it was shown that if R is a right semi-artinian ring such that $R_R^{(\mathbb{N})}$ is CS, then $R_R^{(A)}$ is CS for any set A, and hence R is right and left artinian, and the injective hull of R_R is projective.

Remark 6. Let R be a right semi-artinian right V ring. Then R is von Neumann regular (see, for example, [**6**, 3.13(3)]). In particular, R is semiprime. Therefore, each minimal right (left) ideal of R is generated by an idempotent, and for each idempotent $e \in R$, eR is a minimal right ideal if and only if Re is a minimal left ideal. This implies that $\mathrm{Soc}(R_R) = \mathrm{Soc}({}_R R)$. Let S_α^l denote the αth left socle of R. As R/S_α^l is von Neumann regular and right semi-artinian, we can inductively prove that S_α^l equals the αth right socle of R for each ordinal α. Hence R is the union of its left socle series, and so R is left semi-artinian (cf. [**6**, 3.12]). However, in general, R is not left V (see [**8**, Example 6.19]). This shows that the conditions (b) and (c) in our theorem are not left–right symmetric.

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