BULL. AUSTRAL. MATH. SOC. 34C25, 34D05
VOL. 16 (1977), 61-65.

Pseudo-autonomous linear systems

W.A. Coppel

Pseudo-autonomous linear differential equations are defined. A linear differential equation with bounded coefficient matrix is pseudo-autonomous if and only if it is *almost reducible*. A linear differential equation with recurrent coefficient matrix is pseudo-autonomous if and only if it has *pure point spectrum*.

Let $X(t)$ be the fundamental matrix for the linear differential equation

$$x' = A(t)x \tag{1}$$

such that $X(0) = I$, where the $n \times n$ coefficient matrix $A(t)$ is continuous on the whole line $-\infty < t < \infty$. The equation (1) will be said to be *pseudo-autonomous* if there exist real numbers $\lambda_1, \ldots, \lambda_m$ and supplementary projections $P_1, \ldots, P_m$ such that for each $\varepsilon > 0$ there is a constant $K_\varepsilon > 0$ satisfying

$$\left|X(t)P_jX^{-1}(s)\right| \le K_\varepsilon e^{\lambda_j(t-s)} e^{\varepsilon|t-s|}$$

for $-\infty < s, t < \infty$, and $j = 1, \ldots, m$.

It follows that, for any non-trivial solution $x(t)$ of (1), $P_jx(0) = x(0)$ implies

$$\lim_{t\to\pm\infty} t^{-1} \log|x(t)| = \lambda_j .$$

Hence the numbers λ_j and the projections P_j are uniquely determined. Moreover, the Lyapunov characteristic exponents of (1) are $\lambda_1, \ldots, \lambda_m$

Received 23 August 1976.

https://doi.org/10.1017/S0004972700023005 Published online by Cambridge University Press

with multiplicity tr P_1, ..., tr P_m respectively.

If (1) is pseudo-autonomous then by [2, Lemma 2], it is kinematically similar to a block diagonal system

$$y_j' = B_j(t)y_j \quad (j = 1, \ldots, m)$$

whose fundamental matrices $Y_j(t)$ satisfy

$$\left|Y_j(t)Y_j^{-1}(s)\right| \le L_\varepsilon e^{\lambda_j(t-s)} e^{\varepsilon|t-s|}$$

for $-\infty < s, t < \infty$, and $j = 1, \ldots, m$. Moreover, if $A(t)$ is bounded then $B_1(t), \ldots, B_m(t)$ are also bounded.

Pseudo-autonomous equations are closely connected to the *almost reducible* equations of Bylov [1]. In fact, if (1) is almost reducible then by [1, Corollary 2, p. 344] there exist real numbers $\lambda_1, \ldots, \lambda_m$ and positive integers $n_1, \ldots, n_m$ with sum n such that, for every $\varepsilon > 0$, (1) is kinematically similar to an equation

$$z' = [C+D(t)]z \ ,$$

where $C = \operatorname{diag}\left[\lambda_1 I_{n_1}, \ldots, \lambda_m I_{n_m}\right]$ and $|D(t)| \le \varepsilon$ for $-\infty < t < \infty$. Since by the proof of [2, Theorem 2] we can assume further that $D(t) = \operatorname{diag}\left[D_1(t), \ldots, D_m(t)\right]$, it follows that (1) is pseudo-autonomous.

Conversely, if (1) is pseudo-autonomous and its coefficient matrix $A(t)$ is bounded then it is almost reducible. To see this it is sufficient to show that if an equation

$$y' = B(t)y$$

with bounded coefficient matrix $B(t)$ has a fundamental matrix $Y(t)$ satisfying

$$|Y(t)Y^{-1}(s)| \le L_\varepsilon e^{\varepsilon|t-s|} \ ,$$

then it is almost reducible to 0. We have

$$|\det Y(t)Y^{-1}(s)| \le M_\varepsilon e^{n\varepsilon|t-s|}$$

https://doi.org/10.1017/S0004972700023005 Published online by Cambridge University Press

and hence, by Liouville's formula,

$$\left| R \int_s^t \operatorname{tr} B(\tau)d\tau \right| \le \log M_\varepsilon + n\varepsilon|t-s| \ .$$

Thus

$$\lim_{t-s\to\infty} (t-s)^{-1} \int_s^t R \operatorname{tr} B(\tau)d\tau = 0 \ ,$$

and the result follows from [1, Theorem 7].

Assume now that the coefficient matrix $A(t)$ *is bounded and uniformly continuous.* Then by Ascoli's Theorem any sequence $\{h_\nu\}$ of real numbers contains a subsequence $\{k_\nu\}$ such that

$$\tilde{A}(t) = \lim_{\nu\to\infty} A\left(t+k_\nu\right)$$

exists uniformly on compact intervals. The collection of all equations

$$(2) \qquad x' = \tilde{A}(t)x$$

is called the *hull* of the equation (1). The fundamental matrix $\tilde{X}(t)$ of (2) such that $\tilde{X}(0) = I$ is given by

$$\tilde{X}(t) = \lim_{\nu\to\infty} X\left(t+k_\nu\right)X^{-1}\left(k_\nu\right) \ .$$

Suppose (1) is pseudo-autonomous. Then, by restricting attention in the first instance to a subsequence, $X\left(k_\nu\right)P_jX^{-1}\left(k_\nu\right) \to \tilde{P}_j$, where $\tilde{P}_1, \ldots, \tilde{P}_m$ are supplementary projections. It follows that

$$\left|\tilde{X}(t)\tilde{P}_j\tilde{X}^{-1}(s)\right| \le K_\varepsilon e^{\lambda_j(t-s)} e^{\varepsilon|t-s|}$$

for $-\infty < s, t < \infty$, and $j = 1, \ldots, m$. Thus every equation (2) in the hull of (1) is also pseudo-autonomous, with the same numbers λ_j and similar projections $\tilde{P}_j$. Hence all equations in the hull of a pseudo-autonomous equation have the same characteristic exponents, counting multiplicities. Moreover, if (2) has a non-trivial solution $\tilde{x}(t)$ such that $e^{-\lambda t}|\tilde{x}(t)|$ is bounded on $-\infty < t < \infty$ for some real λ , then

https://doi.org/10.1017/S0004972700023005 Published online by Cambridge University Press

$\lambda = \lambda_j$ for some j. In fact, j is the greatest integer k for which $\tilde{P}_k \tilde{x}(0) \neq 0$. Therefore a pseudo-autonomous equation has *pure point spectrum*, in the terminology of Sacker and Sell [6].

If $A(t)$ *is recurrent* and (1) has pure point spectrum then conversely (1) is pseudo-autonomous, by Sacker and Sell [6, Theorem 2].

Suppose finally that $A(t)$ *is almost periodic* and all equations (2) in the hull of (1) have the same characteristic exponents, counting multiplicities. Then the sum of the characteristic exponents is the same for all equations (2). It follows that if some equation (2) is regular, in the sense of Lyapunov, then every equation (2) is regular, since $\operatorname{tr} \tilde{A}(t)$ has the same mean value as $\operatorname{tr} A(t)$. But according to Millionščikov [4, Theorem 3], at least one equation in the hull is regular. Hence all equations in the hull of (1) are regular. Therefore (1) is almost reducible, by Millionščikov [3, Theorem 1 and Lemma] (*cf.* the proof of Theorem 2).

The preceding results establish in particular the

THEOREM. *If the coefficient matrix* $A(t)$ *is almost periodic then the following assertions are equivalent:*

(i) *the equation* (1) *is pseudo-autonomous;*

(ii) *the equation* (1) *is almost reducible;*

(iii) *the equation* (1) *has pure point spectrum;*

(iv) *all equations in the hull of* (1) *have the same characteristic exponents.*

Millionščikov [5] has given an example of an equation (1) which is not almost reducible, even though $A(t)$ *is quasi-periodic.* In conjunction with the theorem, this disproves Sacker and Sell's conjecture that every almost periodic linear differential system has pure point spectrum.

https://doi.org/10.1017/S0004972700023005 Published online by Cambridge University Press

References

[1] Б.Ф. Былов [B.F. Bylov], "Почти приводимые системы дифференциальных уравнений" [Almost reducible systems of differential equations], *Sibirsk. Mat. Ž.* 3 (1962), 333-359.

[2] W.A. Coppel, "Dichotomies and reducibility", *J. Differential Equations* 3 (1967), 500-521.

[3] В.М. Миллионщиков [V.M. Millionščikov], "О связи между устойчивостью характеристических показателей и почти приводимостью систем с почти периодическими коэффициентами" [The connection between the stability of characteristic exponents and almost reducibility of systems with almost periodic coefficients], *Differencial'nye Uravnenija* 3 (1967), 2127-2134.

[4] В.М. Миллионщиков [V.M. Millionščikov], "Критерий устойчивости вероятного спектра линейных систем дифференциальных уравнений с рекуррентными коэффициентами и критерий почти приводимости систем с почти периодическими коэффициентами" [A criterion for the stability of the probable spectrum of linear systems of differential equations with recurrent coefficients and a criterion for the almost reducibility of systems with almost periodic coefficients], *Mat. Sb. (N.S.)* 78 (1969), 179-201.

[5] В.М. Миллионщиков [V.M. Millionščikov]. "Доказательство существования неправильных систем линейных дифференциальных уравнений с квазипериодическнмн коэффициентами" [A proof of the existence of irregular systems of linear differential equations with quasi-periodic coefficients], *Differencial'nye Uravnenija* 5 (1969). 1979-1983.

[6] Robert J. Sacker and George R. Sell, "A spectral theory for linear almost periodic differential equations", *International Conference on Differential Equations*, 698-708 (Proc. Internat. Conf. Differential Equations, Los Angeles, California, 1974. Academic Press [Harcourt Brace Jovanovich], New York, London, 1975).

Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT.

https://doi.org/10.1017/S0004972700023005 Published online by Cambridge University Press