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TRACE CLASS ELEMENTS AND CROSS-SECTIONS IN KAC-MOODY GROUPS

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ABSTRACT. Let G be an affine Kac-Moody group, $\pi_0, \ldots, \pi_r, \pi_\delta$ its fundamental irreducible representations and $\chi_0, \ldots, \chi_r, \chi_\delta$ their characters. We determine the set of all group elements x such that all $\pi_i(x)$ act as trace class operators, *i.e.*, such that $\chi_i(x)$ exists, then prove that the χ_i are class functions. Thus, $\chi := (\chi_0, \ldots, \chi_r, \chi_\delta)$ factors to an adjoint quotient $\bar{\chi}$ for G. In a second part, following Steinberg, we define a cross-section C for the potential regular classes in G. We prove that the restriction $\chi|_C$ behaves well algebraically. Moreover, we obtain an action of $\mathbb{C}^\times$ on C, which leads to a functional identity for $\chi|_C$ which shows that $\chi|_C$ is quasi-homogeneous.

Introduction. This work is on the adjoint quotient of affine Kac-Moody groups. The adjoint quotient is of relevance in singularity theory, because there is a correspondence between simple singularities and simple linear algebraic groups [S1] which extends to a correspondence between simple elliptic and cusp singularities and Kac-Moody groups [S3]. To elucidate this correspondence, one can use a group theoretic approach. We will give a short summary of the finite dimensional case [St1, St2].

Let G be a simply connected, semisimple linear algebraic group. G acts on itself by the adjoint action (conjugation)

$$G \times G \longrightarrow G, \quad g \cdot x := \mathrm{Ad}(g)(x) := gxg^{-1}.$$

The quotient G/Ad as well as the associated morphism (of varieties) $\chi \colon G \longrightarrow G/\mathrm{Ad}$ are called the *adjoint quotient of* G. There are several ways to obtain a realization of χ:

(i) Let T be a maximal torus, U the unipotent radical, $B = T \ltimes U$ a Borel subgroup and W the Weyl group of G. Each element $g \in G$ is conjugate to some element $b = tu \in B$. Let $\chi(g) := [\![t]\!] \in T/W \cong \mathbb{C}^r$.

(ii) Each element $g \in G$ may be decomposed into $g = su$, where s is semisimple and u is unipotent (Jordan-Chevalley decomposition). s is conjugate to some $t \in T$; let $\chi(g) := [\![t]\!] \in T/W \cong \mathbb{C}^r$.

(iii) If $\pi_i \colon G \longrightarrow \mathrm{GL}(V_i)$ ($i = 1, \ldots, r$) are the fundamental irreducible representations of G and $\chi_i := \mathrm{trace} \circ \pi_i$ their characters, then let $\chi := (\chi_1, \ldots \chi_r) \colon G \longrightarrow \mathbb{C}^r$.

Trying to generalize this to Kac-Moody groups, one runs into several difficulties: (1) Not all group elements are conjugate into a fixed Borel subgroup, (2) there is no Jordan-Chevalley decomposition, (3) the quotient T/W has no structure (in the sense of being a variety or such) and (4) the fundamental irreducible representations are infinite dimensional, $\chi_i(g)$ is not defined for all $g \in G$. Even if there are partial solutions to these

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problems, one considers the realization of an adjoint quotient for Kac-Moody groups as an open problem [S3, Section 11.7], [S4, section 10].

In this work we will generalize the group theoretic approach, as in (iii) above, to affine Kac-Moody groups. For that, let G be an affine Kac-Moody group, $\chi_i := \operatorname{trace} \circ \pi_i$ the characters of the fundamental irreducible representations ($i \in \{0, \ldots, r, \delta\}$) and $\chi := (\chi_0, \ldots, \chi_r, \chi_\delta)$, then $\chi(x)$ is defined iff all $\pi_i(x)$ are trace class operators.

In the first part of our work we investigate the functional analytic properties of the operators $\pi_i(x)$. In particular, we determine the set $G^{\mathrm{tr}}_\circ$ of all $x \in G$ for which $\chi(x)$ is defined. Up to boundary points, this set coincides with $G^{>1} = \{(x, q) \in G = G' \rtimes \mathbb{C}^\times \mid |q| > 1\}$ and is invariant under conjugation by arbitrary group elements (Theorems 1 and 2). Moreover it was possible to prove that χ is invariant under the adjoint action, *i.e.*, invariant under conjugation by arbitrary group elements; this is nontrivial as it means conjugation by unbounded operators (Theorem 3)[1]. Thus, we have a class function $\chi\colon G^{>1} \longrightarrow \mathbb{C}^{r+1} \times \mathbb{C}^\times$ and a commutative diagram

$$\begin{tikzcd} G^{>1} \arrow[r, "\chi"] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathbb{C}^{r+1} \times \mathbb{C}^\times \\ G^{>1}/\operatorname{Ad} \arrow[ur, "\bar{\chi}"'] & \end{tikzcd}$$

As far as the fundamental characters can be considered to be algebraically independent (and this is essentially a question on $\chi_0, \ldots, \chi_r$, depending on the value χ_δ), χ may thus be viewed as a character-theoretic realization of an adjoint quotient of G (*cf.* [S4]).

The second part of this work is dedicated to the construction and investigation of an analogue of Steinberg's cross-section for regular classes in simply connected semisimple algebraic groups [St1, St2]. As in the finite dimensional case, we can define this cross-section C by using one-parameter subgroups,

$$\omega\colon \mathbb{C}^{r+1} \times \mathbb{C}^\times \longrightarrow G, \quad (c_0, \ldots, c_r, q) \longmapsto \Big(\prod_{i=0}^{r} x_i(c)n_i, q\Big),$$
$$C := \operatorname{im}(\omega) = \prod_{i=0}^{r} U_i n_i \times \mathbb{C}^\times.$$

In Steinberg's case, the restriction of the adjoint quotient $\chi\colon G \longrightarrow T/W$ induces an isomorphism of algebraic varieties $\chi|_C\colon C \longrightarrow T/W$. In our situation, we can show that $\chi|_C = \chi \circ \omega$ can be expressed as a power series in q with polynomial coefficients in $c_0, \ldots, c_r$ (Theorem 4), that the Jacobian determinant $\det(\chi \circ \omega)'(c_0, \ldots, c_r, q)$ is invertible in the ring of formal power series $\mathbb{C}[[q^{-1}]]$ (Theorem 5), and that there is a natural $\mathbb{C}^\times$-action on C rendering $\chi \circ \omega$ quasi-homogeneous with respect to the natural degrees (levels) of $\chi_0, \ldots, \chi_r$ (Theorem 6). As a consequence of Theorems 5 and 6, the map $\chi \circ \omega$ induces an isomorphism

$$\chi_q := \chi|_{C_q}\colon C_q = \operatorname{im}\Big(\omega\big(\mathbb{C}^{r+1} \times \{q\}\big)\Big) \longrightarrow \mathbb{C}^{r+1}$$

[1] Preliminary work on Theorems 1, 2, 3 by Stephen Slebarski [Sle] has to be acknowledged.

for all sufficiently large q (Theorem 7). The $\mathbb{C}^\times$-action on C has no counterpart in the finite-dimensional situation of semisimple algebraic groups. However, it may be viewed as an analogue of a $\mathbb{C}^\times$-action on transversal slices to nilpotent orbits in semisimple Lie algebras as originally introduced by Harish-Chandra and heavily exploited in [S1].

1. Kac-Moody algebras and Kac-Moody groups.

Generalized Cartan Matrices. Let A be a *generalized Cartan matrix* (GCM), *i.e.*, $A \in M_n(\mathbb{Z})$, $a_{ii} = 2$ for all i, $a_{ij} \leq 0$ for all $i \neq j$, and $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$. Such a matrix A is called *indecomposable* if there is no decomposition of $\{1, \ldots, n\}$ into nonempty disjoint subsets I_1, I_2 such that $a_{ij} = 0$ whenever $i \in I_1$, $j \in I_2$. Subsequently, we will consider only indecomposable GCM. The GCM roughly fall into three classes, called GCM of finite, affine or general type. A GCM is called *symmetrizable* (SCM), if there is an invertible diagonal matrix $D = (d_1, \ldots, d_n)$ and a symmetric matrix $B \in M_n(\mathbb{Z})$ with $A = DB$. All GCM of finite and affine type are symmetrizable.

Realizations. A *(minimal) realization of* A is a triple $(\mathbf{h}, \Delta, \nabla)$, consisting of a complex vector space $\mathbf{h}$ of dimension $2n - \operatorname{rank} A$, a linearly independent subset $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbf{h}^*$ and a linearly independent subset $\nabla = \{h_1, \ldots, h_n\} \subset \mathbf{h}$, with $\alpha_j(h_i) = a_{ij}$ for all i, j. Elements of Δ are called *simple roots*, elements of ∇ are called *simple coroots*. Furthermore, $\langle \Delta \rangle_{\mathbb{Z}}$ is called the *root lattice* and $\langle \nabla \rangle_{\mathbb{Z}}$ is called the *coroot lattice*. There is an order relation (reflexive, antisymmetric, transitive) on $\mathbf{h}^*$ defined by $\lambda \leq \mu :\Leftrightarrow \mu - \lambda \in \langle \Delta \rangle_{\mathbb{N}_0}$.

Weyl group. One introduces linear maps $w_i\colon \mathbf{h}^* \longrightarrow \mathbf{h}^*$, $\lambda \longmapsto \lambda - \lambda(h_i)\alpha_i$ $(i = 1, \ldots, n)$; these maps generate a subgroup of $\mathrm{GL}(\mathbf{h}^*)$, called the *Weyl group* $W = W(A)$. In the same way, one defines linear maps $s_i\colon \mathbf{h} \longrightarrow \mathbf{h}$, $h \longmapsto h - \alpha_i(h)h_i$; the s_i generate a subgroup of $\mathrm{GL}(\mathbf{h})$ which is isomorphic to the Weyl group. The Weyl group is a Coxeter group, *i.e.*, is a group with generators $w_1, \ldots, w_n$ and relations $(w_i w_j)^{m_{ij}} = \mathbf{1}$, where $m_{ij} \in \mathbb{Z} \cup \{\infty\}$, $m_{ii} = 1$, $m_{ji} = m_{ij} \geq 2$ for $i \neq j$ [K, Proposition 3.13]. The m_{ij} depend on A as follows:

$$\begin{array}{c|ccccc} a_{ij}a_{ji} & 0 & 1 & 2 & 3 & \geq 4 \\ m_{ij} & 2 & 3 & 4 & 6 & \infty. \end{array}$$

If $A = DB$ is a SCM, then there is a nondegenerate, symmetric, $\mathbb{C}$-bilinear form $(\ \mid\ )\colon \mathbf{h} \times \mathbf{h} \longrightarrow \mathbb{C}$ satisfying $(h_i \mid h_j) = a_{ij}d_j$. Furthermore, there is an isomorphism $\nu\colon \mathbf{h} \longrightarrow \mathbf{h}^*$ defined by $\nu(h)(h') = (h \mid h')$, $h, h' \in \mathbf{h}$; ν induces a nondegenerate, symmetric, $\mathbb{C}$-bilinear form $(\ \mid\ )\colon \mathbf{h}^* \times \mathbf{h}^* \longrightarrow \mathbb{C}$.

Kac-Moody algebras. If A is a SCM, then the Kac-Moody algebra $\mathbf{g}(A)$ associated with A can be obtained as the Lie algebra with generators e_i, f_i, h_κ and Serre relations

$$[h_\kappa, h_{\kappa'}] = 0, \quad [h_\kappa, e_i] = \alpha_i(h_\kappa)e_i, \quad [h_\kappa, f_i] = -\alpha_i(h_\kappa)f_i,$$
$$[e_i, f_j] = \delta_{ij}h_i, \quad (\operatorname{ad} e_i)^{1-a_{ij}}e_j = 0, \quad (\operatorname{ad} f_i)^{1-a_{ij}}f_j = 0.$$

The main point of symmetrizability is, that $\mathbf{g}(A)$ admits a nondegenerate invariant symmetric $\mathbb{C}$-bilinear form $(\ \mid\ )\colon \mathbf{g}(A) \times \mathbf{g}(A) \longrightarrow \mathbb{C}$ if and only if A is a SCM [K, Theorem 2.2, Exercise 2.3].

Explicit construction of affine Kac-Moody algebras. Kac-Moody algebras of affine type can also be obtained via the following construction: Start with the *loop algebra* $\mathring{\mathbf{g}} \otimes \mathbb{C}[t, t^{-1}]$ of a finite dimensional simple Lie algebra $\mathring{\mathbf{g}}$; the loop algebra is then centrally extended to give the *derived algebra* $\mathbf{g}' = \mathring{\mathbf{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ (direct sum of vector spaces); finally, we take the semidirect product $\mathbf{g} := \mathbf{g}' \rtimes \mathbb{C}d$, where $d\colon \mathbf{g}' \longrightarrow \mathbf{g}'$ is the derivation defined by $x \otimes t^m \longmapsto mx \otimes t^m$, $c \longmapsto 0$ (*cf.* [K, Chapter 7], [FLM, Section 1.6]). The Lie bracket in $\mathbf{g}$ is given by

$$[x\otimes t^m + \xi c + \eta d, y \otimes t^n + \xi' c + \eta' d] = \big([x,y]\otimes t^{m+n} + \eta n y \otimes t^n - \eta' m x \otimes t^m\big) + m\delta_{m+n,0}(x \mid y)c,$$

where $(\ \mid\ )$ is the Killing form on $\mathring{\mathbf{g}}$. This construction gives the so-called *untwisted* affine Kac-Moody algebras; a slight modification of this construction gives the *twisted* affine algebras [K, Chapter 8], [FLM, Section 1.6].

Roots. For arbitrary GCM, $\mathbf{g} = \mathbf{g}(A)$ is a $\mathbf{h}^*$-graded Lie algebra via

$$\mathbf{g} = \bigoplus_{\alpha \in \mathbf{h}^*} \mathbf{g}_\alpha, \quad \text{where} \quad \mathbf{g}_\alpha := \big\{x \in \mathbf{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathbf{h}\big\} \tag{1}$$

and $\dim \mathbf{g}_\alpha < \infty$. Those $\alpha \neq 0$ with $\mathbf{g}_\alpha \neq \{0\}$ are called *roots*, the corresponding $\mathbf{g}_\alpha$ are called *root spaces*, and the set $\mathcal{R}$ of all roots is called the *root system*. We have $\Delta \subset \mathcal{R} \subset \langle \Delta \rangle_{\mathbb{Z}}$. The root system $\mathcal{R}$ is invariant under the action of the Weyl group and consists of *real roots* $\mathcal{R}^{\mathrm{re}} := \mathcal{W} \cdot \Delta$ and *imaginary roots* $\mathcal{R}^{\mathrm{im}} := \mathcal{R} \setminus \mathcal{R}^{\mathrm{re}}$. Furthermore, the root system decomposes into *positive* and *negative* roots, $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_- = \mathcal{R}_+ \cup -\mathcal{R}_+$, where $\mathcal{R}_+ := \{\alpha \in \mathcal{R} \mid \alpha > 0\}$.

If A is of affine type, then $r := \operatorname{rank}(A) = n - 1$ and we use $\{0, \ldots, r\}$ as indices instead of $\{1, \ldots, n\}$. $\mathbf{g} = \mathbf{g}(A)$ is no longer simple, but has exactly two nontrivial ideals: The one-dimensional center $\mathbf{c}$ and the commutator subalgebra $\mathbf{g}' = [\mathbf{g}, \mathbf{g}]$. The center $\mathbf{c}$ is spanned by $c = \sum_{i=0}^r a_i^\vee h_i$; the coefficients $a_i^\vee \in \mathbb{N}$ defined by this equation are called the *dual labels* or *dual marks* of A. Furthermore, $\dim \mathbf{h} = r + 2$ and ∇ is completed to a basis $\{h_0, \ldots, h_r, d\}$ of $\mathbf{h}$. The imaginary roots are $\mathcal{R}^{\mathrm{im}} = \mathbb{Z}^\times \delta$, where $\delta = \sum_{i=0}^r a_i \alpha_i$ (the numbers $a_i \in \mathbb{N}$ are called the *labels* or *marks* of A).

Compact involution. Defining $e_i \longmapsto -f_i$, $f_i \longmapsto -e_i$, $h \longmapsto -h$ ($h \in \mathbf{h}_\mathbb{R}$), one obtains an antilinear involution ω_0 on $\mathbf{g} = \mathbf{g}(A)$, which is called the *compact involution*. The fixed point set $\mathbf{k} := \mathbf{k}(A) := \{x \in \mathbf{g} \mid \omega_0(x) = x\}$ is called the *compact form* of $\mathbf{g} = \mathbf{g}(A)$. Since ω_0 is an antilinear involution, it is easily verified that $\mathbf{k}$ is a real Lie algebra and $\mathbf{g}$ is the complexification of $\mathbf{k}$.

Weights. Let A be a GCM and $\mathbf{h}$ the Cartan subalgebra of $\mathbf{g} = \mathbf{g}(A)$. An element $\Lambda \in \mathbf{h}^*$ is called *integral* or a *weight (dominant, integral dominant)* if $\Lambda(h_i) \in \mathbb{Z}$ ($\in \mathbb{R}_0^+$, $\in \mathbb{N}_0$) for all $i = 1, \ldots, n$. There is a very close relationship between roots and weights, and we will recall a few facts for the finite and affine case.

Finite case. Let A be a GCM of finite type and $X := \langle \nabla \rangle_{\mathbb{Z}}$. The elements defined by $\lambda_i(h_j) = \delta_{ij}$ ($i = 1, \ldots, r$) are called *fundamental dominant weights*. Since $\mathbf{h} = \langle \nabla \rangle_{\mathbb{C}}$, the

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set of weights is just $X^* := \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) = \langle \lambda_1, \ldots, \lambda_r \rangle_{\mathbb{Z}}$. Thus, the relationship between roots and weights is essentially summarized in the chain of inclusions

$$\Delta \subset \mathcal{R} \subset \langle \Delta \rangle_{\mathbb{Z}} \subset X^* = \langle \lambda_1, \ldots, \lambda_r \rangle_{\mathbb{Z}} \subset \mathbf{h}^* = X^* \otimes_{\mathbb{Z}} \mathbb{C}.$$

In particular, every simple root α_i is a $\mathbb{Z}$-linear combination of λ_i's and the set D of all integral dominant elements coincides with the $\mathbb{N}_0$-linear combinations of λ_i's:

$$\alpha_i = \sum_{j=1}^{r} a_{ji} \lambda_j \qquad (i = 1, \ldots, r)$$
$$D = \langle \lambda_1, \ldots, \lambda_r \rangle_{\mathbb{N}_0}.$$

Affine case. In the affine case, $X := \langle h_0, \ldots, h_r, d \rangle_{\mathbb{Z}}$ and the fundamental dominant weights λ_i $(i = 0, \ldots, r)$ are defined by $\lambda_i(h_j) = \delta_{ij}$, $\lambda_i(d) = 0$. Following the conventions as in [K],

$$\alpha_0(d) = 1, \quad \alpha_i(d) = 0 \ (i = 1, \ldots, r), \quad \delta(h_j) = 0, \quad \delta(d) = a_0,$$

we get a chain of inclusions

$$\Delta \subset \mathcal{R} \subset \langle \Delta \rangle_{\mathbb{Z}} \subset X^* = \langle \lambda_0, \ldots, \lambda_r \rangle_{\mathbb{Z}} \oplus \tfrac{1}{a_0} \mathbb{Z}\delta \subset \mathbf{h}^* = X^* \otimes_{\mathbb{Z}} \mathbb{C}$$

and the simple roots resp. integral dominant elements can be written as

$$\alpha_i = \sum_{j=0}^{r} a_{ji} \lambda_j + n_i \delta \quad (i = 0, \ldots, r), \text{ where } n_0 = \frac{1}{a_0}, n_1 = \cdots = n_r = 0, \tag{2}$$

$$D = \langle \lambda_0, \ldots, \lambda_r \rangle_{\mathbb{N}_0} \oplus \mathbb{C}\delta. \tag{3}$$

Because of (3) it is sometimes convenient to consider δ as the $r + 2$-nd fundamental dominant element.

The category $\mathcal{O}$. A $\mathbf{g}(A)$-module V is called *diagonalizable* if

$$V = \bigoplus_{\lambda \in \mathbf{h}^*} V_\lambda, \qquad \text{where } V_\lambda = \{ v \mid h \cdot v = \lambda(h) v \, \forall h \in \mathbf{h} \}. \tag{4}$$

If $V_\lambda \neq \{0\}$, then λ is called a *weight of* V and V_λ is called a *weight space*. The set of all weights of V will be denoted by $P(V)$. A diagonalizable $\mathbf{g}(A)$-module V is said to belong to the category $\mathcal{O}$ if all the weight spaces are finite dimensional and $P(V) \subset \bigcup_{i=1}^{N} \{ \mu \in \mathbf{h}^* \mid \mu \leq \Lambda_i \}$ for some $\Lambda_1, \ldots, \Lambda_N \in \mathbf{h}^*$. Submodules, quotient modules, finite sums and finite tensor products of modules in $\mathcal{O}$ are again in $\mathcal{O}$. The morphisms in $\mathcal{O}$ are simply the $\mathbf{g}(A)$-module homomorphisms.

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Highest weight modules. Examples of modules, which are in $\mathcal{O}$, are the *highest weight modules*. These are $\mathbf{g}(A)$-modules, such that there is a $\Lambda \in \mathbf{h}^*$ and a *highest weight vector* $v_\Lambda \in V$ satisfying

$$\begin{aligned} \mathbf{n}_+ \cdot v_\Lambda &= 0, \\ h \cdot v_\Lambda &= \Lambda(h) v_\Lambda \quad \text{for all } h \in \mathbf{h}, \\ U\big(\mathbf{g}(A)\big) \cdot v_\Lambda &= V. \end{aligned} \tag{5}$$

For each $\Lambda \in \mathbf{h}^*$, there is a unique irreducible highest weight module with highest weight Λ, denoted by $L(\Lambda)$. Conversely, any irreducible representation in $\mathcal{O}$ is isomorphic to some $L(\Lambda)$. Thus, $\Lambda \longmapsto L(\Lambda)$ is a bijection between $\mathbf{h}^*$ and the set of all (isomorphism classes of) irreducible representations in $\mathcal{O}$. For any $L(\Lambda)$, we have

$$\dim L(\Lambda)_\Lambda = 1 \quad \text{and} \quad P(\Lambda) := P\big(L(\Lambda)\big) \subset \{\mu \mid \mu \leq \Lambda\}.$$

Integrable modules. A $\mathbf{g}(A)$-module V of the category $\mathcal{O}$ is called *integrable*, if all e_i, f_i act locally nilpotent on V. This local nilpotency is one of the reasons why such representations of $\mathbf{g}$ can be lifted to a representation of the associated group. Of particular interest are the highest weight modules $L(\Lambda)$ with integral dominant Λ, since $L(\Lambda)$ is integrable if and only if Λ is integral dominant [K, Lemma 10.1]. Thus, $\Lambda \longmapsto L(\Lambda)$ is a bijection between D and the set of all integrable irreducible representations in $\mathcal{O}$.

Integrable modules for affine Kac-Moody algebras. In chapter 2 we will focus on the *fundamental representations* $L(\lambda_0), \ldots, L(\lambda_r)$, $L(\delta)$ of affine Kac-Moody algebras. The module $L(\lambda_0)$ is also known as the *basic representation* which is used in string theory (*cf.* chapters III and IV of the introduction in [FLM]). The $L(\lambda_i)$ are infinite dimensional, but the $L(n\delta)$, $n \in \mathbb{C}$, are one-dimensional: $\mathbf{g} = \mathbf{g}(A)$ acts on $L(n\delta) = \mathbb{C}$ via

$$\pi_{n\delta}(e_i)(1) := 0, \quad \pi_{n\delta}(f_i)(1) := 0, \quad \pi_{n\delta}(h_i)(1) := 0, \quad \pi_{n\delta}(d)(1) := na_0.$$

Obviously, $v_{n\delta} := 1 \in \mathbb{C}$ is a highest weight vector which satisfies the conditions (5). After identifying $\mathrm{End}(\mathbb{C})$ with $\mathbb{C}$, we see that $\pi_{n\delta}$ can also be thought of as being $n\delta \colon \mathbf{h} \longrightarrow \mathbb{C}$, trivially extended to a linear map (denoted by the same letter) $n\delta \colon \mathbf{g} \longrightarrow \mathbb{C}$.

Later, we would like to restrict ourselves "without loss of generality" to $L(\Lambda)$ with $\Lambda \in \langle \lambda_0, \ldots, \lambda_r \rangle_{\mathbb{N}_0}$. Thus, if $\Lambda + n\delta = \sum n_i \lambda_i + n\delta \in D$, we need to know how the representations $L(\Lambda + n\delta)$, $L(\Lambda)$ and $L(n\delta)$ relate to each other. Using the definition of a highest weight module, it is easy to see that $L(\Lambda + n\delta) = L(\Lambda) \otimes L(n\delta)$. Furthermore, after identifying the vector spaces $L(\Lambda + n\delta)$ and $L(\Lambda)$, the actions $\pi_{\Lambda + n\delta}$, π_Λ and $\pi_{n\delta}$ are related to each other by

$$\pi_{\Lambda + n\delta}(x) = \pi_\Lambda(x) + n\delta(x)\, \mathrm{id}_{L(\Lambda)}. \tag{6}$$

If $\Lambda \in D$, then $\Lambda(c) \in \mathbb{N}_0$ is called the *level of* Λ. The weight system $P(\Lambda)$ can be described as follows [K, Proposition 12.5]: $P(\Lambda)$ lies in the hyperplane $\{\lambda \in \mathbf{h}^* \mid \lambda(c) = \Lambda(c)\}$ and is the intersection of the parabola given by $(\lambda \mid \lambda) \leq (\Lambda \mid \Lambda)$ and the "lattice" $\Lambda - \langle \Delta \rangle_{\mathbb{N}_0}$. From this, it is straightforward to prove some technical details needed later, collected in the following lemma.

LEMMA 1. *The weight system $P(\Lambda)$ of an integrable representation $L(\Lambda)$ of an affine Kac-Moody algebra has the following properties.*

(1) $P(\Lambda) \subset X^* \Leftrightarrow \Lambda \in X^*$;
(2) *If* $\Lambda \notin \mathbb{C}\delta$, *then for each i there exists a* $\mu \in P(\Lambda)$ *such that also* $\mu - \alpha_i \in P(\Lambda)$;
(3) $\lambda_0 - \alpha_0 - \delta \in P(\lambda_0)$;
(4) *if* $\delta(h) \neq 0$ *and* $R \in \mathbb{R}$, *then* $\{\mu \in P(\Lambda) \mid \mu(h) = R\}$ *is finite.* ∎

The hermitian form $\langle\ |\ \rangle$ on $L(\Lambda)$. As a direct sum of countably many finite dimensional weight spaces, the $L(\Lambda)$, $\Lambda \in \mathbf{h}^*$, are themselves of countably infinite dimension. The following lemma is essential for using functional analytic techniques in chapter 2, as it ensures the existence of an inner product on $L(\Lambda)$ with additional 'nice' properties [K, Proposition 9.4, Lemma 11.5, Theorem 11.7]:

LEMMA 2. *For every $\Lambda \in D$, there is an inner product $\langle\ |\ \rangle : L(\Lambda) \times L(\Lambda) \longrightarrow \mathbb{C}$ such that*

$$\langle x \cdot v \mid w \rangle = -\langle v \mid \omega_0(x) \cdot w \rangle \quad \textit{(contravariance)} \tag{7}$$

$$\langle v_\Lambda \mid v_\Lambda \rangle = 1. \tag{8}$$

Together with this inner product, $L(\Lambda)$ is a unitary space of countably infinite dimension; the completion $H(\Lambda)$ of $L(\Lambda)$ is a separable Hilbert space.

By the contravariance (7), the adjoint map $\pi_\Lambda(x)^*$ of $\pi_\Lambda(x)$ exists for any $x \in \mathbf{g}$, and equals $\pi_\Lambda\big(-\omega_0(x)\big)$. Furthermore, one can prove that (7) implies that the weight spaces are orthogonal:

$$\big\langle L(\Lambda)_\lambda \mid L(\Lambda)_\mu \big\rangle = 0 \quad \text{if } \lambda \neq \mu. \tag{9}$$

If, for each weight μ, one chooses an orthonormal basis of $L(\Lambda)_\mu$, then the union of all these bases is, by (9), an orthonormal basis of $L(\Lambda)$; this is called an *adapted basis*.

Kac-Moody groups. There are several ways of associating a group $G(A)$ to a Kac-Moody algebra $\mathbf{g}(A)$. Kac and Peterson exploit the properties of the modules $L(\Lambda)$ to introduce a "minimal derived" Kac-Moody group [PK, K2, G, MP]. Alternatively, Kac-Moody groups can be defined via an amalgamation process, *cf.* [S3] and the works of Tits cited there. Or, one gives a definition via generators and relations, without referring to the algebra $\mathbf{g}(A)$ [KP]. In the present work I will use the "minimal" version of a Kac-Moody group, which is slightly larger than the "minimal derived" version; we give a short introduction and will supply some information which is needed later.

Let $\mathbf{g}$ be an arbitrary Lie algebra and (π, V) be a $\mathbf{g}$-module. An element $x \in \mathbf{g}$ is called *π-locally finite* if $\pi(x)$ acts locally finitely on V; denote the set of all π-locally finite elements of $\mathbf{g}$ by F_π. If $F \subset F_\pi$, then one can consider the subgroup of $\mathrm{GL}(V)$ generated by the $\exp\big(\pi(x)\big)$, $x \in F$.

Let $\mathbf{g} = \mathbf{g}(A)$ be an affine Kac-Moody algebra. If we choose

$$(\pi, V) := \bigoplus_{\Lambda \in D} \big(\pi_\Lambda, L(\Lambda)\big) \tag{10}$$

and

$$F := \mathbf{h} \cup \bigcup_{i=0}^{r} \mathbf{g}_{\alpha_i} \cup \bigcup_{i=0}^{r} \mathbf{g}_{-\alpha_i} \quad \text{or} \quad \mathbb{C}d \cup \bigcup_{i=0}^{r} \mathbf{g}_{\alpha_i} \cup \bigcup_{i=0}^{r} \mathbf{g}_{-\alpha_i}, \tag{11}$$

then we get the *(minimal) Kac-Moody group $G = G(A)$ associated to* $\mathbf{g} = \mathbf{g}(A)$, which is discussed in the present work.

The torus T. Since all $\pi(h)$, $h \in \mathbf{h}$, act diagonally on V, the subgroup $T := \big\{\exp\big(\pi(h)\big) \mid h \in \mathbf{h}\big\}$ is commutative. As a $\mathbb{Z}$-module, it can be identified with $X \otimes_{\mathbb{Z}} \mathbb{C}^\times$ via $\exp\big(z_i\pi(h_i)\big) \leftrightarrow h_i \otimes e^{2\pi z_i}$. To avoid confusion, we list the different $\mathbb{Z}$-module actions:

$$\begin{aligned}
\mathbb{C}: &\quad n \cdot z := nz,\\
\mathbb{C}^\times: &\quad n \cdot q := q^n,\\
\mathbf{h} = X \otimes_{\mathbb{Z}} \mathbb{C}: &\quad n \cdot (x \otimes z) := nx \otimes z = x \otimes nz,\\
T = X \otimes_{\mathbb{Z}} \mathbb{C}^\times: &\quad n \cdot (x \otimes q) := nx \otimes q = x \otimes q^n.
\end{aligned}$$

In T we will sometimes use additive notation, sometimes multiplicative notation. Thus we have $x \otimes q + x \otimes q' = x \otimes qq'$ and $x \otimes q + x' \otimes q = (x + x') \otimes q$ in additive notation, $(x \otimes q)(x \otimes q') = x \otimes qq'$ and $(x \otimes q)(x' \otimes q) = (x + x') \otimes q$ in multiplicative notation. The exponential map $\exp\colon \mathbf{h} \longrightarrow T$, $h \longmapsto \exp\big(\pi(h)\big)$ is a surjective group homomorphism; it can also be written as $\sum h_i \otimes z_i \longmapsto \sum h_i \otimes e^{2\pi z_i}$. The kernel of exp is iX. The following subgroups of T will be important later:

$$\begin{aligned}
T_i &:= \{h_i \otimes q \mid q \in \mathbb{C}^\times\} \cong \mathbb{C}^\times \quad (i = 0, \ldots, r),\\
T_d &:= \{d \otimes q \mid q \in \mathbb{C}^\times\} \cong \mathbb{C}^\times,\\
T_+ &:= \exp(\mathbf{h}_{\mathbb{R}}) = \Big\{\sum_{k=0}^{r} h_k \otimes q_k + d \otimes q \;\Big|\; q_k > 0 \text{ for } k = 0, \ldots, r \text{ and } q > 0\Big\},\\
T_c &:= \exp(i\mathbf{h}_{\mathbb{R}}) = \Big\{\sum_{k=0}^{r} h_k \otimes q_k + d \otimes q \;\Big|\; |q_k| = |q| = 1 \quad \text{for } k = 0, \ldots, r\Big\}.
\end{aligned}$$

These subgroups can be used to write down two decompositions of T which we will use frequently. The first one is the *polar decomposition*, $T = T_+ \times T_c$. Corresponding to this decomposition we write $t = t_+t_c$ for elements of T. The second one is the decomposition $T = (T_0 \times \cdots \times T_r) \times T_d \cong (T_0 \times \cdots \times T_r) \times \mathbb{C}^\times$, here, we write $t = \sum h_i \otimes q_i + d \otimes q = (\sum h_i \otimes q_i, q)$.

One-parameter subgroups in T and characters of T. Each one-parameter subgroup $\mu \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, X)$ (of these, there are only the maps $\mu_a\colon z \longmapsto za$, $a \in X$) extends to a one-parameter subgroup $\overline{\mu} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}, \mathbf{h})$, which itself lifts to a one-parameter subgroup

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$\bar{\bar{\mu}} \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{C}^\times, T)$:

$$\begin{tikzcd}
\mathbb{C}^\times \arrow[r] & T = X \otimes \mathbb{C}^\times & \bar{\bar{\mu}}\colon q \longmapsto a \otimes q \\
\mathbb{C} \arrow[u, "\exp"] \arrow[r] \arrow[ur, phantom, "\circ"] & \mathbf{h} = X \otimes \mathbb{C} \arrow[u, "\exp"'] & \bar{\mu}\colon z \longmapsto a \otimes z \\
\mathbb{Z} \arrow[u] \arrow[r] \arrow[ur, phantom, "\circ"] & X = X \otimes \mathbb{Z} \arrow[u] & \mu\colon z \longmapsto az
\end{tikzcd}$$

where $\exp\colon \mathbb{C} \longrightarrow \mathbb{C}^\times$, $z \longmapsto e^{2\pi z}$. Each character $\mu \in \operatorname{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) = X^*$ extends to a map $\bar{\mu} \in \operatorname{Hom}_{\mathbb{Z}}(\mathbf{h}, \mathbb{C})$, which in turn lifts to a map $\bar{\bar{\mu}} \in \operatorname{Hom}_{\mathbb{Z}}(T, \mathbb{C}^\times)$:

$$\begin{tikzcd}
T = X \otimes \mathbb{C}^\times \arrow[r] & \mathbb{C}^\times & \bar{\bar{\mu}}\colon x \otimes q \longmapsto q^{\mu(x)} \\
\mathbf{h} = X \otimes \mathbb{C} \arrow[u, "\exp"] \arrow[r] \arrow[ur, phantom, "\circ"] & \mathbb{C} \arrow[u, "\exp"'] & \bar{\mu}\colon x \otimes z \longmapsto z\mu(x) \\
X = X \otimes \mathbb{Z} \arrow[u] \arrow[r] \arrow[ur, phantom, "\circ"] & \mathbb{Z} \arrow[u] & \mu\colon x \longmapsto \mu(x)
\end{tikzcd}$$

If there is no danger of confusion, we will drop the bar and the double bar.

The one-parameter subgroups U_i and V_i. For each simple root α_i $(i = 0, \ldots, r)$ we define one-parameter subgroups

$$\begin{aligned}
x_i &:= x_{\alpha_i}\colon \mathbb{C} \longrightarrow G, \quad x_{\alpha_i}(c) := \exp\bigl(\pi(ce_i)\bigr), \quad U_i := U_{\alpha_i} := x_{\alpha_i}(\mathbb{C}),\\
y_i &:= x_{-\alpha_i}\colon \mathbb{C} \longrightarrow G, \quad x_{-\alpha_i}(c) := \exp\bigl(\pi(cf_i)\bigr), \quad V_i := U_{-\alpha_i} := x_{-\alpha_i}(\mathbb{C}).
\end{aligned}$$

The exponential maps $\exp\colon \mathbb{C}e_i \longrightarrow U_i$, $ce_i \longmapsto x_i(c)$ and $\exp\colon \mathbb{C}f_i \longrightarrow V_i$, $cf_i \longmapsto y_i(c)$ are group isomorphisms. Since $[e_i, f_j] = 0$, we know that $x_i(c)$ and $y_j(c')$ commute for $i \neq j$. Using [K, (3.8.1)], one proves that the one-parameter subgroups are normalized by T:

$$\exp\bigl(\pi(h)\bigr) x_i(c) \exp\bigl(\pi(h)\bigr)^{-1} = x_i(e^{\alpha_i(h)}c), \tag{12}$$

$$\exp\bigl(\pi(h)\bigr) y_i(c) \exp\bigl(\pi(h)\bigr)^{-1} = y_i(e^{-\alpha_i(h)}c). \tag{13}$$

But (12), (13) and the commutativity of T imply $G' = [G, G]$, which justifies calling G' the "derived Kac-Moody group". Furthermore, $G = G' \rtimes T_d$. In this sense, we will use the notation (x, q) for elements of G.

The building blocks G_i. Let $\mathbf{g}_i := \langle e_i, f_i, h_i \rangle$ and $G_i := \langle T_i, U_i, V_i \rangle$. Since all elements of $\mathbf{g}_i$ act locally finite on V [K, Section 1.2], the exponential map $\exp\colon \mathbf{g}_i \longrightarrow G_i$, $x \longmapsto \exp\bigl(\pi(x)\bigr)$ is well-defined. Just as we have an isomorphism of Lie algebras $\varphi_i\colon \mathrm{sl}(2, \mathbb{C}) \xrightarrow{\sim} \mathbf{g}_i$,

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$\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) \longmapsto e_i$, $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) \longmapsto f_i$, $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right) \longmapsto h_i$, we have a group isomorphism $\phi_i \colon \mathrm{SL}(2,\mathbb{C}) \xrightarrow{\sim} G_i$ given by

$$\phi_i\colon \qquad \begin{pmatrix}1&c\\0&1\end{pmatrix} \longmapsto x_i(c), \quad \begin{pmatrix}1&0\\c&1\end{pmatrix} \longmapsto y_i(c), \quad \begin{pmatrix}q&0\\0&q^{-1}\end{pmatrix} \longmapsto h_i \otimes q.$$

From (12), (13) and the formulae given in the proof of [C, Lemma 6.1.1], we know that the G_i are normalized by T. Furthermore, $K_i := \phi_i\big(\mathrm{SU}(2)\big) \cong \mathrm{SU}(2)$ is a compact form of G_i and we denote by n_i the element

$$n_i := \phi_i\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right) = \phi_i\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\phi_i\left(\begin{pmatrix}1&0\\-1&1\end{pmatrix}\right)\phi_i\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right) = x_i(1)y_i(-1)x_i(1).$$

The normalizer of T and the Weyl group $\mathcal{W}$. Let $N := \langle T, n_0, \ldots, n_r\rangle$. Then $T \lhd N$ and N is the normalizer of T in G. $\mathcal{W} := N/T$ is called the Weyl group of $G = G(A)$; the $s_i := [\![n_i]\!]$ generate $\mathcal{W}$, $(\mathcal{W}, S)$ (where $S := \{s_0, \ldots, s_r\}$) is a Coxeter system, and $\mathcal{W}$ is isomorphic to the Weyl group of the corresponding Lie algebra $\mathbf{g} = \mathbf{g}(A)$.

Borel subgroups. Above, we associated a one-parameter subgroup to each simple root. This can be done for any real root. Let α be a real root, say $\alpha = w \cdot \alpha_i$ for some simple root α_i and some $w = [\![n]\!] \in \mathcal{W}$. Then

$$x_\alpha\colon \mathbb{C} \longrightarrow G, \quad x_\alpha(c) := n x_{\alpha_i}(c) n^{-1}, \quad U_\alpha := x_\alpha(\mathbb{C}) = nU_i n^{-1}$$

is the one-parameter subgroup associated to α. Now we can define the following subgroups of G:

$$U_+ := \langle U_\alpha \mid \alpha > 0,\ \alpha \text{ real}\rangle, \quad U_- := \langle U_\alpha \mid \alpha < 0,\ \alpha \text{ real}\rangle,$$
$$B_+ := \langle T, U_+\rangle, \quad B_- := \langle T, U_-\rangle.$$

The involution ω_0 and the compact form K. The compact involution $\omega_0 \colon \mathbf{g} \longrightarrow \mathbf{g}$ lifts to an involution $\hat\omega_0 \colon G \longrightarrow G$. For elements of T and the one-parameter subgroups one gets

$$\hat\omega_0(t) = t^{-1} \quad \text{for } t \in T_+, \tag{14}$$
$$\hat\omega_0(t) = t \quad \text{for } t \in T_c, \tag{15}$$
$$\hat\omega_0\big(x_i(c)\big) = y_i(-\bar c), \tag{16}$$
$$\hat\omega_0\big(y_i(c)\big) = x_i(-\bar c). \tag{17}$$

The fixed point set $K := \{g \in G \mid \hat\omega_0(g) = g\}$ is called the *compact form* of G. From [PK, Corollary 4(b)] we know that there is an Iwasawa decomposition for G, *i.e.*,

$$G = KT_+U_+.$$

REMARK. Whether there is a Cartan decomposition $G = KT_+K$ for Kac-Moody groups, is an open problem. If there were, this would simplify some of the proofs in chapter 2.

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Lifting the representations of $\mathbf{g}(A)$. Denote the set of all locally finite elements of $\mathrm{End}\big(L(\Lambda)\big)$ by F_{id}, and let $\Lambda \in D$. From $F_{\mathrm{ad}} \subset F_{\pi_\Lambda}$ we get $\pi_\Lambda(F_{\mathrm{ad}}) \subset F_{\mathrm{id}}$, whence $\hat{\pi}_\Lambda \colon \exp(F_{\mathrm{ad}}) \longrightarrow \mathrm{GL}\big(L(\Lambda)\big)$, $\exp(x) \longmapsto \exp\big(\pi_\Lambda(x)\big)$ is well-defined. If this map extends to a group homomorphism $\hat{\pi}_\Lambda \colon G(A) \longrightarrow \mathrm{GL}\big(L(\Lambda)\big)$ we say that π_Λ *lifts to* $\hat{\pi}_\Lambda$. In this case, one has a commuting diagram

$$\begin{tikzcd} G(A) \arrow[r, "\hat{\pi}_\Lambda"] & \mathrm{GL}\big(L(\Lambda)\big) \\ F_{\mathrm{ad}} \arrow[u, "\exp"] \arrow[r, "\pi_\Lambda"'] \arrow[ur, phantom, "\circ"] & F_{\mathrm{id}} \arrow[u, "\exp"'] \end{tikzcd}$$

By [MP, Proposition 6.1.12] all the representations π_Λ, $\Lambda \in D$ can be lifted to a representation of $G(A)$. In particular, $\hat{\pi}_{\lambda_0}, \ldots, \hat{\pi}_{\lambda_r}, \hat{\pi}_\delta$ are called the *fundamental representations of* G.

Action of the torus on $L(\Lambda)$. Each element $t \in T$ acts as a $\bar{\bar{\mu}}(t)$-multiple of the identity on $L(\Lambda)_\mu$.

Action of the one-parameter subgroups on $L(\Lambda)$. Let $v \in L(\Lambda)$. Since $\pi_\Lambda(e_i)$ acts locally nilpotent, the sum

$$\hat{\pi}_\Lambda\big(x_{\alpha_i}(c)\big)v = \exp\big(\pi_\Lambda(ce_i)\big)(v) = \sum_n \frac{1}{n!}\pi_\Lambda(ce_i)^n v = \sum_n \frac{c^n}{n!}\pi_\Lambda(e_i)^n v$$

is finite, say terminates at $n = N$ (N depends on v, but not on c). If $v \in L(\Lambda)_\mu$, then

$$\hat{\pi}_\Lambda\big(x_{\alpha_i}(c)\big)v = v + cv_1 + c^2v_2 + \cdots + c^N v_N \quad \text{where} \quad v_k \in L(\Lambda)_{\mu+k\alpha_i}. \tag{18}$$

In the same vein,

$$\hat{\pi}_\Lambda\big(x_\alpha(c)\big)v = v + cv_1 + c^2v_2 + \cdots + c^N v_N \quad \text{where} \quad v_k \in L(\Lambda)_{\mu+k\alpha}$$

for any real root α.

Action of the Weyl group. Elements of the Weyl group (or, more precisely, their representatives) permute weight spaces. If $w = [\![n]\!] \in \mathcal{W}$, then [S3, p. 5-48]

$$n \cdot V_\mu = V_{w(\mu)}. \tag{19}$$

Action of the compact form K. From the contravariance of the hermitian form $\langle\ |\ \rangle\colon L(\Lambda) \times L(\Lambda) \longrightarrow \mathbb{C}$ with respect to $\mathbf{g}(A)$ we easily deduce, starting with the exponential generators of $G(A)$, the global contravariance of $\langle\ |\ \rangle$ with respect to $G(A)$, *i.e.*,

$$\Big\langle \hat{\pi}_\Lambda(g)v \mid \hat{\pi}_\Lambda\big(\hat{\omega}_0(g)\big)w \Big\rangle = \langle v \mid w \rangle \tag{20}$$

for all $g \in G(A)$, $v, w \in L(\Lambda)$. In particular, we get that the compact form $K = \{g \in G \mid \hat{\omega}_0(g) = g\}$ acts by unitary operators on $L(\Lambda)$ and thus on $H(\Lambda)$.

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Other useful formulas with respect to $\hat{\omega}_0$ are given by the explicit adjoints with respect to $\langle \mid \rangle$:

$$\hat{\pi}_\Lambda(t)^* = \hat{\pi}_\Lambda\big(\hat{\omega}_0(t^{-1})\big) \quad \text{for all } t \in T, \tag{21}$$

$$\hat{\pi}_\Lambda\big(x_i(c)\big)^* = \hat{\pi}_\Lambda\big(y_i(\bar{c})\big) \quad \text{and} \quad \hat{\pi}_\Lambda\big(y_i(c)\big)^* = \hat{\pi}_\Lambda\big(x_i(\bar{c})\big) \tag{22}$$

for all $i = 1, \ldots, r$, $c \in \mathbb{C}$.

2**. Trace class elements in Kac-Moody groups.** In this chapter we investigate the functional analytic properties of the elements of G when they are viewed as operators acting on $L(\Lambda)$, $V_\circ = \bigoplus_{i=0}^{r} L(\lambda_i)$ or $V = \bigoplus_{\Lambda \in D} L(\Lambda)$. We start by looking at suitable subgroups of G, and ask: Which of the elements of the subgroup under consideration are

(F1)	selfadjoint	(F4)	bounded
(F2)	unitary	(F5)	bounded invertible
(F3)	normal	(F6)	algebraic trace class operators

on $L(\Lambda)$, resp. $V_\circ$, resp. V? And, if the completions of these spaces are denoted by $H(\Lambda)$, $H_\circ$, H, respectively: Which elements are

(F7) trace class operators on $H(\Lambda)$, resp. $H_\circ$, resp. H?

For convenience, we recall the definitions (for more, *cf*. [Con] or [Go]). If V is a unitary space, then a linear map $\phi: V \longrightarrow V$ is called *selfadjoint* if $\langle \phi v \mid w\rangle = \langle v \mid \phi w\rangle$ for all v, w, *unitary* if $\langle \phi v \mid \phi w\rangle = \langle v \mid w\rangle$ for all v, w, *normal* if $\phi\phi^* = \phi^*\phi$, *bounded* if $\|\phi\| < \infty$, *bounded invertible* if $\|\phi\|, \|\phi^{-1}\| < \infty$, and *compact* if the closure of $\phi(\{v \mid \|v\| < 1\})$ is a compact subset of V.

A linear map $\phi: V \longrightarrow V$ is called an *algebraic trace class operator* with respect to a given orthonormal basis $B = \{e_i\}_{i\in\mathbb{N}}$, if $\sum_i |\langle e_i \mid \phi e_i\rangle| < \infty$. In this case we call $\mathrm{trace}^B(\phi) := \sum_i \langle e_i \mid \phi e_i\rangle$ the *algebraic trace of* ϕ. Please note that it depends on the chosen orthonormal basis, whether a map $\phi: V \longrightarrow V$ is an algebraic trace class operator or not. The set of all algebraic trace class operators with respect to a basis B is denoted by $\mathrm{End}_1^B(V)$.

Let H be a Hilbert space. An operator $\Phi \in B(H)$ is called *trace class operator* if $\sum_i \langle e_i \mid |\Phi| e_i\rangle < \infty$, where $|\Phi| = (\Phi^*\Phi)^{\frac{1}{2}}$ is the absolute value of Φ. The set of all trace class operators is denoted by $B_1(H)$. The (Hilbert space) *trace of* Φ, $\mathrm{trace}(\Phi) := \sum_i \langle e_i \mid \Phi e_i\rangle$, is independent of the choice of basis [Con, p. 274]. We have $B_1(H) \lhd B(H)$ and $B_1(H) \subset \mathrm{End}_1^B(H)$ [Con, p. 274].

For arbitrary subsets $U \subset G$ we introduce the following notation:

$$\begin{aligned}
U_\Lambda^{\mathrm{b}} &:= \big\{ g \in U \mid \hat{\pi}_\Lambda(g): L(\Lambda) \longrightarrow L(\Lambda) \text{ is bounded} \big\},\\
U_\Lambda^{\mathrm{bi}} &:= \big\{ g \in U \mid \hat{\pi}_\Lambda(g), \hat{\pi}_\Lambda(g^{-1}): L(\Lambda) \longrightarrow L(\Lambda) \text{ are bounded} \big\},\\
U_\Lambda^{\mathrm{atr}} &:= \big\{ g \in U \mid \hat{\pi}_\Lambda(g): L(\Lambda) \longrightarrow L(\Lambda) \text{ is an algebraic trace class operator} \big\},\\
U_\Lambda^{\mathrm{tr}} &:= \big\{ g \in U \mid \hat{\pi}_\Lambda(g) \text{ extends to a trace class operator on } H(\Lambda) \big\};
\end{aligned}$$

similarly, we use the symbols $U_\circ^{\rm b}$, $U_\circ^{\rm bi}$, $U_\circ^{\rm atr}$, $U_\circ^{\rm tr}$ and $U^{\rm b}$, $U^{\rm bi}$, $U^{\rm atr}$, $U^{\rm tr}$ (substitute $\hat\pi_\Lambda$ by $\hat\pi_\circ$ resp. $\hat\pi$). Furthermore, let

$$U^{>1} := \big\{ g = (x, q) \in U \,\big|\, |q| > 1 \big\},$$
$$U^{=1} := \big\{ g = (x, q) \in U \,\big|\, |q| = 1 \big\},$$
$$U^{<1} := \big\{ g = (x, q) \in U \,\big|\, |q| < 1 \big\}.$$

The main results of this chapter are: We have $G^{>1} \subset G_\circ^{\rm tr} \subset G^{>1} \cup G^{=1}$ (Theorem 1), and $G^{>1}$ is invariant under conjugation by arbitrary group elements (Theorem 2). We proceed in three steps: First, we take a thorough look at the torus T; then we deal with the Levi subgroups $L_\gamma = \langle T, U_\gamma, U_{-\gamma}\rangle$; finally we come to the group G itself.

2.1. *Torus elements as operators on $L(\Lambda)$, $V_\circ$ and V.* We recall the definition of the *fundamental chamber* F and the *Tits cone* X:

$$F := \big\{ h \in \mathbf{h}_\mathbb{R} \mid \alpha_i(h) \geq 0 \text{ for all } i \big\}, \quad X := \bigcup_{w\in W} w \cdot F. \tag{23}$$

LEMMA 3 (ACTION OF T ON $L(\Lambda)$). *Let G be an affine Kac-Moody group, T the torus of G, $\Lambda \in \langle \lambda_0, \dots, \lambda_r\rangle_{\mathbb{N}_0}$ and $t = \exp(h' + ih'')$, $h', h'' \in \mathbf{h}_\mathbb{R}$. Then:*
(F1) $\hat\pi_\Lambda(t)$ *is selfadjoint* $\Leftrightarrow$ $\mu(h'') \in \frac{1}{2}\mathbb{Z}$ *for all* $\mu \in P(\Lambda)$*;*
(F2) $\hat\pi_\Lambda(t)$ *is unitary* $\Leftrightarrow$ $h' = 0$ $\Leftrightarrow$ $t \in T_c$*;*
(F3) $\hat\pi_\Lambda(t)$ *is normal;*
(F4) $T_\Lambda^{\rm b} = \exp(X) \times T_c$*;*
(F5) $T_\Lambda^{\rm bi} = \exp(\mathbb{R}c) \times T_c$*;*
(F6) $T_\Lambda^{\rm atr} = T^{>1}$*;*
(F7) $T_\Lambda^{\rm tr} = T^{>1}$.

PROOF. Because $\hat\pi_\Lambda(t)$ acts diagonally on the weight spaces $L(\Lambda)_\mu$ as multiplication by $\overline{\overline{\mu}}(t)$, it is natural to look at the set of eigenvalues of $\hat\pi_\Lambda(t)$, called the *spectrum of* $\hat\pi_\Lambda(t)$, $\operatorname{spec}\hat\pi_\Lambda(t) := \{\overline{\overline{\mu}}(t) \mid \mu \in P(\Lambda)\}$. If $t = \exp(h)$, it is also convenient to look at the *spectrum of* $\pi_\Lambda(h)$, $\operatorname{spec}\pi_\Lambda(h) := \{\mu(h) \mid \mu \in P(\Lambda)\}$, instead.

(F1): $\hat\pi_\Lambda(t)$ is selfadjoint iff $\overline{\overline{\mu}}(t) \in \mathbb{R}$ for all $\mu \in P(\Lambda)$. Since $\overline{\overline{\mu}}(t) = \overline{\overline{\mu}}(t_+)\overline{\overline{\mu}}(t_c)$, where $\overline{\overline{\mu}}(t_+) \in \mathbb{R}^+$ and $\overline{\overline{\mu}}(t_c) \in S^1$ (here, we implicitly use $\Lambda \in \langle\lambda_0, \dots, \lambda_r\rangle_{\mathbb{N}_0} \Rightarrow \Lambda \in X^* \Rightarrow P(\Lambda) \subset X^*$, *cf.* Lemma 1(1)), this holds iff $\overline{\overline{\mu}}(t_c) = e^{2\pi i\mu(h'')} = \pm 1$ iff $\mu(h'') \in \frac{1}{2}\mathbb{Z}$ for all $\mu \in P(\Lambda)$. $\diamondsuit$

(F2): Similar to (F1), $\hat\pi_\Lambda(t)$ is unitary iff $\overline{\overline{\mu}}(t) \in S^1$ iff $\overline{\overline{\mu}}(t_+) = e^{2\pi\mu(h')} = 1$ iff $\mu(h') = 0$ for all $\mu \in P(\Lambda)$. By Lemma 1(2) this implies $\alpha_i(h') = 0$ for all i, whence $h' \in \mathbb{R}c$. But we also have $\Lambda(h') = 0$, which finally forces $h' = 0$. $\diamondsuit$

(F3): Since T is commutative, any $\hat\pi_\Lambda(t)$ is normal. $\diamondsuit$

(F4): $\hat\pi_\Lambda(t)$ is bounded iff $\operatorname{spec}\hat\pi_\Lambda(t)$ is bounded; in this case $\|\hat\pi_\Lambda(t)\| = \sup\{|\overline{\overline{\mu}}(t)| \mid \mu \in P(\Lambda)\} < \infty$. We discuss a) the case $t \in T_c$, b) the case $t \in T_+$, then c) the general case.

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a) If $t \in T_c$, then $\overline{\overline{\mu}}(t) \in S^1$ for all $\mu \in P(\Lambda)$, whence $\|\hat{\pi}_\Lambda(t)\| = \sup\{|\overline{\overline{\mu}}(t)| \mid \mu \in P(\Lambda)\} = 1$.

b) $t \in T_+$. This is where the real work is. Recall the definition of the fundamental chamber and the Tits cone (23). One has $X = \mathbb{R}c \cup \{h \in \mathbf{h}_\mathbb{R} \mid \delta(h) > 0\}$ [K, Proposition 5.8.b], $X \cap (-X) = \mathbb{R}c$, and $\mathbf{h}_\mathbb{R}$ decomposes disjointly into the four sets

$$\mathbb{R}c$$
$$X_1 := X \setminus \mathbb{R}c = \left\{h \in \mathbf{h}_\mathbb{R} \mid \delta(h) > 0\right\} = \bigcup_{w \in W} w \cdot (F \setminus \mathbb{R}c),$$
$$X_2 := -X \setminus \mathbb{R}c = \left\{h \in \mathbf{h}_\mathbb{R} \mid \delta(h) < 0\right\},$$
$$X_3 := \langle \nabla \rangle_\mathbb{R} \setminus \mathbb{R}c = \left\{h \in \mathbf{h}_\mathbb{R} \mid \delta(h) = 0\right\} \setminus \mathbb{R}c.$$

Correspondingly, we discuss spec $\hat{\pi}_\Lambda(t)$, $t = \exp(h)$, for each of these four cases.

- $t = \exp(h)$, $h \in \mathbb{R}c$. Since $\hat{\pi}_\Lambda(t) = \Lambda(t) \cdot \mathrm{id}_{L(\Lambda)}$, we have spec $\hat{\pi}_\Lambda(t) = \{\Lambda(t)\}$ and $\|\hat{\pi}_\Lambda(t)\| = |\Lambda(t)|$.
- $t = \exp(h)$, $h \in X_1$. Let $t = \exp(h)$, where $h \in F \setminus \mathbb{R}c$. In this case

$$\sup\left\{\mu(h) \mid \mu \in P(\Lambda)\right\} = \Lambda(h), \tag{24}$$
$$\inf\left\{\mu(h) \mid \mu \in P(\Lambda)\right\} = -\infty, \tag{25}$$
$$\operatorname{spec} \pi_\Lambda(h) = \left\{\mu(h) \mid \mu \in P(\Lambda)\right\} \text{ is a discrete subset of } \mathbb{R}, \tag{26}$$
$$\text{each eigenvalue has finite multiplicity.} \tag{27}$$

Ad (24): Since $\Lambda \in P(\Lambda)$, we have $\Lambda(h) \in \{\mu(h) \mid \mu \in P(\Lambda)\}$; since $\mu \in P(\Lambda) \Rightarrow \mu \leq \Lambda \Rightarrow \mu = \Lambda - \sum_{i=0}^r n_i\alpha_i$ ($n_i \geq 0$), we get $\mu(h) \leq \Lambda(h)$.

Ad (25): For all $n \in \mathbb{N}_0$ we have $\Lambda - n\delta \in P(\Lambda)$; now $\delta(h) > 0$ implies (25).

Ad (26): Let $J := \{i \mid \alpha_i(h) > 0\}$, which is nonempty because we assumed that $h \in F \setminus \mathbb{R}c$. For each real number R the set $\{\mu(h) \mid \mu \in P(\Lambda)\} \cap [R, \infty[$ is finite, because

$$\mu(h) = \Lambda(h) - \sum_{i=0}^r n_i\alpha_i(h) = \Lambda(h) - \sum_{i \in J} n_i\alpha_i(h) \geq R \Leftrightarrow \sum_{i \in J} n_i\alpha_i(h) \leq \Lambda(h) - R$$

and the latter can be realized only by finitely many tupels $(n_i)_{i \in J}$, $n_i \in \mathbb{N}_0$.

Ad (27): Follows from Lemma 1(4).

If $t = \exp(h)$, where $h \in X_1$, say $h = w.\tilde{h}$ for some $w \in W$ and $\tilde{h} \in F \setminus \mathbb{R}c$, then by the invariance of $P(\Lambda)$ under the action of W,

$$\left\{\mu(h) \mid \mu \in P(\Lambda)\right\} = \left\{\mu(\tilde{h}) \mid \mu \in P(\Lambda)\right\}.$$

Hence

$$\sup\left\{\mu(t) \mid \mu \in P(\Lambda)\right\} < \infty,$$
$$\inf\left\{\mu(t) \mid \mu \in P(\Lambda)\right\} = 0,$$
$$\operatorname{spec} \hat{\pi}_\Lambda(t) = \left\{\mu(t) \mid \mu \in P(\Lambda)\right\} \text{ is a discrete subset of } \mathbb{R}^+,$$
$$\text{each eigenvalue has finite multiplicity,}$$

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i.e., [Con, Proposition II.4.6]:

$$\text{If } t = \exp(h),\ h \in X_1, \text{ then } \hat{\pi}_\Lambda(t) \text{ is a compact operator on } L(\Lambda). \tag{28}$$

• Let $t = \exp(h)$, $h \in X_2$. These are the inverses of the elements we were talking about a moment ago, thus

$$\begin{gathered}\text{If } t = \exp(h),\ h \in X_2, \text{ then } \hat{\pi}_\Lambda(t) \text{ is an unbounded operator on } L(\Lambda);\\ \operatorname{spec} \hat{\pi}_\Lambda(t) \text{ has no limit points.}\end{gathered} \tag{29}$$

• Now for the last case, $t = \exp(h)$, $h \in X_3$. Here, $\hat{\pi}_\Lambda(t)$ is unbounded in a very unpleasant way. First, any eigenvalue has infinite multiplicity, since $\mu(h) = (\mu - n\delta)(h)$ for all $n \in \mathbb{N}_0$. On the other hand, $\{\mu(h) \mid \mu \in P(\Lambda)\}$ is bounded neither below nor above: Since $h \notin \mathbf{c}$, there is an $i \in \{0, \ldots, r\}$ such that $\alpha_i(h) \neq 0$. If $\alpha := \alpha_i + n\delta$ then $\alpha^\vee = h_i + n\frac{a_i}{a_i^\vee}c$ [K, Proposition 5.1.d.ii]; furthermore because of $\delta(h) = 0$

$$(w_\alpha \cdot \Lambda)(h) = \big(\Lambda - \Lambda(\alpha^\vee)\alpha\big)(h) = \Lambda(h) - \Lambda(h_i)\alpha_i(h) - n\underbrace{\frac{a_i}{a_i^\vee}\Lambda(c)\alpha_i(h)}_{\neq 0},$$

which, for suitable n, gets arbitrarily large or small. Thus:

$$\begin{gathered}\text{If } t = \exp(h),\ h \in X_3, \text{ then } \hat{\pi}_\Lambda(t) \text{ is an unbounded operator on } L(\Lambda);\\ \operatorname{spec} \hat{\pi}_\Lambda(t) \text{ has 0 as a limit point.}\end{gathered} \tag{30}$$

To summarize b): If $t = \exp(h)$, $h \in \mathbf{h}_\mathbb{R}$, then $\hat{\pi}_\Lambda(t)$ is bounded iff $h \in X$.

c) $t \in T$ arbitrary. Since $\hat{\pi}_\Lambda(t) = \hat{\pi}_\Lambda(t_+ t_c) = \hat{\pi}_\Lambda(t_+)\hat{\pi}_\Lambda(t_c)$ and $\|\hat{\pi}_\Lambda(t_c)\| = 1$, the map $\hat{\pi}_\Lambda(t)$ is bounded iff $\hat{\pi}_\Lambda(t_+)$ is bounded, *i.e.*, iff $t_+ \in \exp(X)$. $\diamondsuit$

(F5): Using $X \cap (-X) = \mathbb{R}c$ and the statements in (F4), we obtain (F5). $\diamondsuit$

(F6), (F7): Since $\hat{\pi}_\Lambda(t)$ acts diagonally, it already extends to a trace class operator on $H(\Lambda)$ if it is just an algebraic trace class operator, *i.e.*, if and only if

$$\sum_{\mu \in P(\Lambda)} \dim L(\Lambda)_\mu |\mu(t)| < \infty.$$

In [K], the *character* ch_V of a $\mathbf{g}$-module $V = \oplus_\mu V_\mu$ is defined by $\mathrm{ch}_V := \sum_\mu \dim V_\mu e^\mu$, where $e^\mu \colon \mathbf{h} \longrightarrow \mathbb{C}^\times$, $e^\mu(h) := e^{\mu(h)}$. The character ch_V is thus a function defined on (a subset of) $\mathbf{h}$. The set where ch_V converges absolutely, is denoted by $Y(V)$. The character ch_V can be interpreted as the trace of $\hat{\pi}_\Lambda(t)$, for

$$\operatorname{trace}\big(\hat{\pi}_\Lambda(t)\big) = \sum_\mu \dim L(\Lambda)_\mu \mu(t) = \sum_\mu \dim L(\Lambda)_\mu e^{2\pi\mu(h)} = \mathrm{ch}_{L(\Lambda)}(2\pi h).$$

By [K, Section 11.10], for affine algebras and $\Lambda \in D$,

$$Y\big(L(\Lambda)\big) = \Big\{h \in \mathbf{h} \;\Big|\; \sum_\mu \dim L(\Lambda)_\mu |e^{\mu(h)}| < \infty\Big\} = \Big\{h \in \mathbf{h} \;\Big|\; \mathrm{Re}\big(\delta(h)\big) > 0\Big\}.$$

Of course $\mathrm{Re}\big(\delta(h)\big) > 0 \Leftrightarrow \mathrm{Re}\big(\delta(2\pi h)\big) > 0$, thus $\hat{\pi}_\Lambda(t)$ is a trace class operator if and only if $\mathrm{Re}\big(\delta(h)\big) > 0$; in more detail, the following statements are equivalent:

(a) $\hat{\pi}_\Lambda(t)$ extends to a trace class operator on $H(\Lambda)$,

(b) $\mathrm{Re}\big(\delta(h)\big) > 0$,

(c) $|\delta(t)| > 1$,

(d) $t = (x, q)$ where $|q| > 1$.

This finishes the proof of Lemma 3. ■

PROPOSITION 4 (ACTION OF T ON $V_\circ$). *Let G be an affine Kac-Moody group, T the torus of G and $t = \exp(h' + ih'')$, $h', h'' \in \mathbf{h}_\mathbb{R}$. Then*

(F1) *$\hat{\pi}_\circ(t)$ is selfadjoint $\Leftrightarrow h'' \in \frac{1}{2}X \Leftrightarrow t \in T_+ \times \exp\left(\frac{1}{2}iX\right)$;*

(F2) *$\hat{\pi}_\circ(t)$ is unitary $\Leftrightarrow h' = 0 \Leftrightarrow t \in T_c$;*

(F3) *$\hat{\pi}_\circ(t)$ is normal;*

(F4) *$T_\circ^{\mathrm{b}} = \exp(X) \times T_c$;*

(F5) *$T_\circ^{\mathrm{bi}} = \exp(\mathbb{R}c) \times T_c$;*

(F6) *$T_\circ^{\mathrm{atr}} = T^{>1}$;*

(F7) *$T_\circ^{\mathrm{tr}} = T^{>1}$.*

PROOF. First recall that $\hat{\pi}_\circ(t) = \bigoplus_{i=0}^r \hat{\pi}_{\lambda_i}(t)$, whence $\hat{\pi}_\circ(t)$ is selfadjoint, unitary, …, trace class if and only if all $\hat{\pi}_{\lambda_i}(t)$ are so.

(F1): $\Rightarrow$: Let $h'' = \sum m_j h_j + md$. By Lemma 3 (F1) we have $\mu(h'') \in \frac{1}{2}\mathbb{Z}$ for all $\mu \in P(\lambda_i)$, all $i = 0, \ldots, r$. In particular, $\lambda_i(h'') \in \frac{1}{2}\mathbb{Z}$ forces $m_i \in \frac{1}{2}\mathbb{Z}$ and $(\lambda_i - \delta)(h'') \in \frac{1}{2}\mathbb{Z}$ forces $m \in \frac{1}{2a_0}\mathbb{Z}$. But there is a stronger argument which even forces $m \in \frac{1}{2}\mathbb{Z}$: By Lemma 1 (3), $\lambda_0 - \alpha_0 - \delta \in P(\lambda_0)$. Now

$$(\lambda_0 - \alpha_0 - \delta)(h'') = \underbrace{(\lambda_0 - \delta)(h'')}_{\in \frac{1}{2}\mathbb{Z}} - \underbrace{\sum m_j \alpha_0(h_j)}_{\in \frac{1}{2}\mathbb{Z}} - m \underbrace{\alpha_0(d)}_{=1} \in \frac{1}{2}\mathbb{Z}$$

whence $m \in \frac{1}{2}\mathbb{Z}$. Thus $h'' \in \frac{1}{2}X$. $\Leftarrow$: Let $h'' \in \frac{1}{2}X$, *i.e.*, $2h'' \in X$. Since $\lambda_i \in X^*$, we know that $P(\lambda_i) \subset X^*$. Thus, for all $\mu \in P(\lambda_i)$ and all i, $\mu(2h'') \in \mathbb{Z}$. ◇

(F2)–(F7): Follows immediately from Lemma 3. ■

PROPOSITION 5 (ACTION OF T ON V). *Let G be an affine Kac-Moody group, T the torus of G and $t = \exp(h' + ih'')$, $h', h'' \in \mathbf{h}_\mathbb{R}$. Then*

(F1) *$\hat{\pi}(t)$ is selfadjoint $\Leftrightarrow h'' \in \frac{1}{2}\langle\nabla\rangle_\mathbb{Z} \Leftrightarrow t \in T_+ \times \exp\left(\frac{1}{2}i\langle\nabla\rangle_\mathbb{Z}\right)$;*

(F2) *$\hat{\pi}(t)$ is unitary $\Leftrightarrow h' = 0 \Leftrightarrow t \in T_c$;*

(F3) *$\hat{\pi}(t)$ is normal;*

(F4) *$T^{\mathrm{b}} = \{\mathbf{1}\}$;*

(F5) *$T^{\mathrm{bi}} = \{\mathbf{1}\}$;*

(F6) *$T^{\mathrm{atr}} = \emptyset$;*

(F7) *$T^{\mathrm{tr}} = \emptyset$.*

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PROOF. (F1): $\hat{\pi}(t)$ is selfadjoint iff all $\hat{\pi}_\Lambda(t)$, $\Lambda \in D$ are. Let $h'' = \sum m_j h_j + md$. As above, $m_j \in \frac{1}{2}\mathbb{Z}$. In addition, $n\delta(h'') \in \frac{1}{2}\mathbb{Z}$ (for all $n \in \mathbb{C}$!) forces $m = 0$. Conversely, let $h'' \in \frac{1}{2}\langle\nabla\rangle_{\mathbb{Z}}$, and $\mu \in P(\Lambda)$, $\Lambda \in D$. Since $\mu \leq \Lambda$, we know that $\mu(h'') \in \frac{1}{2}\mathbb{Z} \Leftrightarrow \Lambda(h'') \in \frac{1}{2}\mathbb{Z}$. Now, for $\Lambda = \sum n_j\lambda_j + n\delta$:

$$\Lambda(h'') = \sum n_j \underbrace{\lambda_j(h'')}_{\in \frac{1}{2}\mathbb{Z}} - n \underbrace{\delta(h'')}_{=0} \in \frac{1}{2}\mathbb{Z}. \qquad \diamondsuit$$

(F2), (F3): Follows immediately from Lemma 3 (F2), (F3). $\diamondsuit$

(F4), (F5): Since T acts diagonally on the weight spaces, $\hat{\pi}(t)$ is bounded iff $\sup\{|\mu(h)| \mid \mu \in P(\Lambda), \Lambda \in D\} < \infty$. We claim that this only holds for $h = 0$. Suppose $h = \sum m_j h_j + md \neq 0$, then $m_j \neq 0$ for some j or $m \neq 0$. In the first case, we consider the integral dominant weights $N\lambda_j$, $N \in \mathbb{N}$, and get

$$\sup\big\{|\mu(h)| \bigm| \mu \in P(\Lambda), \Lambda \in D\big\} \geq \sup_{N\in\mathbb{N}}\big\{|N\lambda_j(h)|\big\} = \sup_{N\in\mathbb{N}}\{N|m_j|\} = +\infty,$$

in the second case, we consider the integral dominant weights $N\delta$, $N \in \mathbb{N}$, and get

$$\sup\big\{|\mu(h)| \bigm| \mu \in P(\Lambda), \Lambda \in D\big\} \geq \sup_{N\in\mathbb{N}}\big\{|N\delta(h)|\big\} = \sup_{N\in\mathbb{N}}\{N|m|\} = +\infty.$$

Now (F5) is obvious. $\diamondsuit$

(F6), (F7): Since $\hat{\pi}(t)$ acts diagonally, it is of algebraic trace class iff it is of trace class. But $T^{\mathrm{tr}} \subset T^{\mathrm{b}}$, whence $T^{\mathrm{tr}} = T^{\mathrm{atr}} = \emptyset$. ∎

REMARK. In view of the things to come (definition of an adjoint quotient) we see that V is "too large".

2.2. *The Levi subgroups.* Consider the Levi subgroups $L_\gamma := \langle T, U_\gamma, U_{-\gamma}\rangle$. Let K_γ be the compact form of L_γ, *i.e.*, the fixed point set of L_γ with respect to $\hat{\omega}_0$. L_γ has a Cartan decomposition $L_\gamma = K_\gamma T K_\gamma$. We now use the fact that δ extends to a group homomorphism on G [S2]. Since $K_\gamma \subset \ker|\delta|$, we have $L_\gamma^{>1} = K_\gamma T^{>1} K_\gamma$. On the other hand, since the elements of $K_\gamma \subset K$ are bounded invertible, we also have $L_\gamma^{\mathrm{b}} = K_\gamma T^{\mathrm{b}} K_\gamma$ and $L_\gamma^{\mathrm{tr}} = K_\gamma T^{\mathrm{tr}} K_\gamma$. Now, by Proposition 4(F7),

$$L_\gamma^{\mathrm{tr}} = K_\gamma T^{\mathrm{tr}} K_\gamma = K_\gamma T^{>1} K_\gamma = L_\gamma^{>1}. \tag{31}$$

2.3. *Trace class elements.*

THEOREM 1. *Let $G = G(A)$ be a (minimal) Kac-Moody group of affine type and $\Lambda \in D \setminus \mathbb{C}\delta$. Then the set G_Λ^{tr} of all elements g for which $\hat{\pi}_\Lambda(g): L(\Lambda) \longrightarrow L(\Lambda)$ extends to a trace class operator on $H(\Lambda)$ satisfies*

$$G^{>1} \subset G_\Lambda^{\mathrm{tr}} \subset G^{>1} \cup G^{=1}.$$

From this, we immediately get $G^{>1} \subset G_\circ^{\mathrm{tr}} \subset G^{>1} \cup G^{=1}$.

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PROOF. We can now use all our propositions and previous work to prove $G^{>1} \subset G^{\mathrm{tr}}_{\Lambda}$. Let $x \in G^{>1}$. Since G is generated by the torus and by the one-parameter subgroups U_γ (γ real root), x can be written as

$$x = t_1 u_1 \cdot \cdots \cdot t_N u_N \quad (t_i \in T, u_i \in U_{\gamma_i}).$$

Each one-parameter subgroup is normalized by the torus, whence this can be rewritten as

$$x = t u'_1 \cdot \cdots \cdot u'_N \quad (t \in T^{>1}, u'_i \in U_{\gamma_i})$$

(where $t \in T^{>1}$ because $U_{\gamma_i} \subset \ker \delta$). Artificially write t as a product $t = t'_1 \cdots t'_N$, where $t'_i \in T^{>1}$. This is always possible, for example one could take $t = (\tau, q) = (\tau, q^{\frac{1}{N}})(\mathbf{1}, q^{\frac{1}{N}}) \cdots (\mathbf{1}, q^{\frac{1}{N}})$. Once again, we reorder the factors and get

$$x = t'_1 u''_1 \cdot \cdots \cdot t'_N u''_N \quad (t'_i \in T^{>1}, u''_i \in U_{\gamma_i}).$$

Each of the factors $t'_i u''_i$ is $\in L^{>1}_{\gamma_i,\Lambda} = L^{\mathrm{tr}}_{\gamma_i}$ (*cf.* (31)), thus $x \in G^{\mathrm{tr}}_{\Lambda}$. Now let $x \in G^{<1}$. Then $x^{-1} \in G^{>1}$ (because $1 = \delta(\mathbf{1}) = \delta(x)\delta(x^{-1})$). If $\hat{\pi}_\Lambda(x)$ were bounded, then because of $B_1\big(H(\Lambda)\big) \lhd B\big(H(\Lambda)\big)$ we would have $\mathrm{id} = \hat{\pi}_\Lambda(x)\hat{\pi}_\Lambda(x)^{-1} \in B_1\big(H(\Lambda)\big)$. Thus $G^{<1} \cap G^{\mathrm{b}}_{\Lambda} = \emptyset$; in particular elements of $G^{<1}$ are never trace class operators. ∎

THEOREM 2. *Let $G = G(A)$ be a (minimal) Kac-Moody group of affine type. Then $G^{>1}$ is invariant under conjugation by arbitrary elements of G.*

PROOF. If $x \in G^{>1}$, $y \in G$, then $\delta(yxy^{-1}) = \delta(x)$. ∎

2.4. *Trace Invariance.* We will now prove that, in a certain sense, we get an adjoint quotient on a subset of G. This subset is, after our elaborations on trace class elements (Theorems 1 and 2), chosen to be $G^{>1}$. In fact we have

THEOREM 3. *a) Let G be an affine Kac-Moody group and $\Lambda \in D$. Then $\chi_\Lambda :=$ trace $\circ \hat{\pi}_\Lambda \colon G^{>1} \longrightarrow \mathbb{C}$ is invariant under conjugation by arbitrary group elements, i.e., a class function. b) The trace function $\chi = (\chi_0, \ldots, \chi_r, \chi_\delta) \colon G^{>1} \longrightarrow \mathbb{C}^{r+1} \times \mathbb{C}^\times$ is a class function and factors to $\bar{\chi}$:*

$$\begin{array}{ccc} G^{>1} & \xrightarrow{\ \chi\ } & \mathbb{C}^{r+1} \times \mathbb{C}^\times \\ \downarrow & \circ & \nearrow_{\bar{\chi}} \\ G^{>1}/\mathrm{Ad} & & \end{array}$$

PROOF. b) follows from a). Using the Iwasawa decomposition $G = KT_+U_+$ [PK] reduces the problem of invariance of χ_Λ under conjugation by arbitrary group elements to the problem of invariance under conjugation by elements of the subgroups K, T_+, U_+.

Invariance of χ_Λ under K. Let $x \in G^{>1}$, $k \in K$. Since elements of K act as unitary operators, we have $\chi_i(x) = \chi_i(kxk^{-1})$ [Con, p. 274].

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Invariance of χ_Λ *under* T_+. Let $t \in T_+$. Then $\hat{\pi}_\Lambda(t)$ is selfadjoint and acts on $L(\Lambda)_\mu$ as multiplication by $\overline{\overline{\mu}}(t)$. Thus, in $L(\Lambda)_\mu$,

$$\begin{aligned}\left\langle e_i \mid \hat{\pi}_\Lambda(t)\hat{\pi}_\Lambda(x)\hat{\pi}_\Lambda(t^{-1})e_i\right\rangle &= \left\langle \hat{\pi}_\Lambda(t)e_i \mid \hat{\pi}_\Lambda(x)\hat{\pi}_\Lambda(t^{-1})e_i\right\rangle \\ &= \overline{\overline{\mu}}(t)\overline{\overline{\mu}}(t^{-1})\left\langle e_i \mid \hat{\pi}_\Lambda(x)e_i\right\rangle = \left\langle e_i \mid \hat{\pi}_\Lambda(x)e_i\right\rangle\end{aligned}$$

whence $\operatorname{trace}\bigl(\hat{\pi}_\Lambda(t)\hat{\pi}_\Lambda(x)\hat{\pi}_\Lambda(t^{-1})\bigr) = \operatorname{trace}\bigl(\hat{\pi}_\Lambda(x)\bigr)$.

Invariance of χ_Λ *under* U_+. It is sufficient to prove invariance under conjugation by elements $x_i(c)$ of the one-parameter subgroups U_i. To prove this, we use the decomposition of $L(\Lambda)$ into α_i-strings:

$$L(\Lambda) = \bigoplus_{\mu \in P(\Lambda)} L(\Lambda)_\mu = \bigoplus(\alpha_i\text{-strings of subspaces}).$$

Since in the latter identity we only put together finitely many weight spaces to give an α_i-string, each basis B of $L(\Lambda)$ adapted to the weight space decomposition is automatically also adapted to the decomposition of $L(\Lambda)$ into α_i-strings. For each α_i-string $s \subset L(\Lambda)$ let p_s resp. i_s denote the canonical projection resp. injection. Since the α_i-strings are invariant under $x_i(c)$, we get

$$\begin{aligned}p_s \circ x_i(c) \circ \hat{\pi}_\Lambda(x) \circ x_i(c)^{-1} \circ i_s &= \bigl(p_s \circ x_i(c) \circ i_s\bigr)\bigl(p_s \circ \hat{\pi}_\Lambda(x) \circ i_s\bigr)\bigl(p_s \circ x_i(c)^{-1} \circ i_s\bigr) \\ &= \bigl(p_s \circ x_i(c) \circ i_s\bigr)\bigl(p_s \circ \hat{\pi}_\Lambda(x) \circ i_s\bigr)\bigl(p_s \circ x_i(c) \circ i_s\bigr)^{-1}.\end{aligned}$$

Since $\dim s < \infty$, we know

$$\begin{aligned}&\operatorname{trace}\bigl(p_s \circ x_i(c) \circ \hat{\pi}_\Lambda(x) \circ x_i(c)^{-1} \circ i_s\bigr) \\ &\quad = \operatorname{trace}\bigl(p_s \circ x_i(c) \circ i_s\bigr)\bigl(p_s \circ \hat{\pi}_\Lambda(x) \circ i_s\bigr)\bigl(p_s \circ x_i(c) \circ i_s\bigr)^{-1} \\ &\quad = \operatorname{trace}\bigl(p_s \circ \hat{\pi}_\Lambda(x) \circ i_s\bigr).\end{aligned}$$

The trace formula (32), applied to $L(\Lambda) = \bigoplus s$, now gives

$$\begin{aligned}\chi_i\bigl(x_i(c) \circ \hat{\pi}_\Lambda(x) \circ x_i(c)^{-1}\bigr) &= \operatorname{trace}\bigl(x_i(c) \circ \hat{\pi}_\Lambda(x) \circ x_i(c)^{-1}\bigr) \\ &= \sum_s \operatorname{trace}\bigl(p_s \circ x_i(c) \circ \hat{\pi}_\Lambda(x) \circ x_i(c)^{-1} \circ i_s\bigr) \\ &= \sum_s \operatorname{trace}\bigl(p_s \circ \hat{\pi}_\Lambda(x) \circ i_s\bigr) \\ &= \operatorname{trace}\bigl(\hat{\pi}_\Lambda(x)\bigr) = \chi_i(x)\end{aligned}$$

and that's what we wanted. ∎

It is now time to mention some open problems. 1) What exactly is G^{b}? Or, since we already know $G^{>1} \subset G^{\mathrm{b}} \subset G^{>1} \cup G^{=1}$: What are the elements of $G^{\mathrm{b}} \cap G^{=1}$? One conjecture is, that $G^{\mathrm{b}} \cap G^{=1}$ is equal to the subgroup generated by the center of G and the compact form K. 2) What exactly is G^{tr}? Or, as we already know $G^{>1} \subset G^{\mathrm{tr}} \subset G^{>1} \cup G^{=1}$, what is $G^{\mathrm{tr}} \cap G^{=1}$? Here the conjecture is that $G^{\mathrm{tr}} \cap G^{=1} = \emptyset$. (Which would follow from the conjecture about G^{b} above.) 3) If $G^{=1} \cap G^{\mathrm{tr}} \neq \emptyset$, is G^{tr} invariant under conjugation?

3. Steinberg cross-sections in Kac-Moody groups.

In chapter 2 we proved that the trace function χ is well-defined on $G^{>1}$ and factors to a map $\overline{\chi}\colon G^{>1}/\operatorname{Ad} \longrightarrow \mathbb{C}^{r+1} \times \mathbb{C}^{>1}$. In order to understand the nature of $\overline{\chi}$, we transfer the notion of a cross-section, as introduced by Steinberg in his investigations about linear algebraic groups [St1, St2], to affine Kac-Moody groups: Define a map ω and a set C, both called a *Steinberg cross-section,* by

$$\omega\colon \mathbb{C}^{r+1} \times \mathbb{C}^{\times} \longrightarrow G, \quad (c_0, \ldots, c_r, q) \longmapsto \Big(\prod_{i=0}^{r} x_i(c) n_i, q\Big),$$

$$C := \operatorname{im}(\omega) = \prod_{i=0}^{r} U_i n_i \times \mathbb{C}^{\times}.$$

It is possible to prove that $\chi \circ \omega$ (*i.e.*, the trace function restricted to C) still has some of the nice properties of the finite dimensional case (Theorems 4 and 5). An interesting new feature is the existence of a $\mathbb{C}^{\times}$-action on the cross-section (Proposition 15), which leads to a functional identity (Theorem 6).

3.1. *A trace formula.* If $V = \oplus_\omega V_\omega$ then an orthonormal basis B of V is *adapted to the decomposition* $V = \oplus_\omega V_\omega$ if B is the union of orthonormal bases of the V_ω's. In our case, bases of $L(\Lambda)$ will always be chosen adapted to the weight space decomposition $L(\Lambda) = \oplus_\mu L(\Lambda)_\mu$. Later we will need the following

LEMMA 6 (TRACE FORMULA). *Let* $V = \oplus_\omega V_\omega$, *where* $\dim V_\omega < \infty$, B *be an adapted basis of* V, *let* $i_\omega\colon V_\omega \longrightarrow V$ *and* $p_\omega\colon V \longrightarrow V_\omega$ *be the canonical injection resp. projection and* $\phi \in \operatorname{End}_1^B(V)$ *an algebraic trace class operator, then*

$$\operatorname{trace}^B(\phi) = \sum_\omega \operatorname{trace}(p_\omega \circ \phi \circ i_\omega). \tag{32}$$

■

3.2. $\chi \circ \omega$ *for Kac-Moody groups.* To find out how the trace function $\chi \circ \omega$ behaves, we will first look at the components $\chi_i \circ \omega$. To simplify the notation we write V instead of $L(\lambda_i)$, V_μ instead of $L(\lambda_i)_\mu$ and (x, q) instead of $\hat{\pi}_{\lambda_i}(x, q)$. Let B be an orthonormal basis of V adapted to $V = \oplus_\mu V_\mu$ and suppose $(x, q) \in \operatorname{End}_1^B(V)$. By the trace formula (32) we have

$$\chi_i(x, q) = \operatorname{trace}(x, q) = \sum_\mu \operatorname{trace}\big(p_\mu \circ (x, q) \circ i_\mu\big). \tag{33}$$

Since $(\mathbf{1}, q) = d \otimes q$ acts diagonally on V_μ as multiplication by $\mu(d \otimes q) = q^{\mu(d)}$, we get

$$\begin{aligned} \operatorname{trace}\big(p_\mu \circ (x, q) \circ i_\mu\big) &= \operatorname{trace}\big(p_\mu \circ (x, 1) \circ (\mathbf{1}, q) \circ i_\mu\big) \\ &= q^{\mu(d)} \operatorname{trace}\big(p_\mu \circ (x, 1) \circ i_\mu\big), \end{aligned} \tag{34}$$

which reduces the problem of determining $\chi_i(x, q)$ to the problem of determining $\operatorname{trace}\big(p_\mu \circ (x, 1) \circ i_\mu\big)$.

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Determining trace$(p_\mu \circ x \circ i_\mu)$. We will proceed along the lines of [St2, p. 69], but give the details missing there. Let $y_j := x_j(c_j)n_j$; then $x = x_0(c_0)n_0 \cdots x_r(c_r)n_r = y_0 \cdots y_r$ and one has

$$(p_\mu \circ x \circ i_\mu) = (p_\mu \circ y_0 \circ i_\mu) \circ \cdots \circ (p_\mu \circ y_r \circ i_\mu). \tag{35}$$

PROOF OF (35) BY INDUCTION ON r. Let $v \in V_\mu$ where $\mu = \sum m_j\lambda_j + n\delta$, and $k \in \{0, \ldots, r\}$. Then $w_k(\mu) = \mu - \mu(h_k)\alpha_k = \mu - m_k\alpha_k$ and by (19), $n_k \cdot v \in V_{\mu - m_k\alpha_k}$. Furthermore

$$y_k \cdot v = n_k \cdot v + c_k v_1 + c_k^2 v_2 + \cdots, \text{ (where } v_l \in V_{\mu - m_k\alpha_k + l\alpha_k} \text{ by (18))}, \tag{36}$$

$$(p_\mu \circ y_k \circ i_\mu)(v) = \begin{cases} c_k^{m_k} v_{m_k}, & m_k \geq 0 \\ 0, & m_k < 0. \end{cases} \tag{37}$$

It now follows that, if we abbreviate $w_0 := n_r \cdot v$, $w_1 := c_r v_1$, $w_2 := c_r^2 v_2$, *etc.*,

$$\begin{aligned} (p_\mu \circ x \circ i_\mu)(v) &= p_\mu\big(y_0 \cdots y_r(v)\big) \\ &= p_\mu\big(y_0 \cdots y_{r-1}(w_0 + w_1 + \cdots + w_{m_r} + \cdots)\big) \\ &= p_\mu\big(y_0 \cdots y_{r-1}(w_{m_r})\big) + \sum_{k \neq m_r} p_\mu\big(y_0 \cdots y_{r-1}(w_k)\big) \\ &= (p_\mu \circ y_0 \circ i_\mu) \cdots (p_\mu \circ y_{r-1} \circ i_\mu)(p_\mu \circ y_r \circ i_\mu)(v) \\ &\qquad + \sum_{k \neq m_r} p_\mu\big(y_0 \cdots y_{r-1}(w_k)\big). \end{aligned}$$

Each of the summands $p_\mu\big(y_0 \cdots y_{r-1}(w_k)\big)$, $k \neq m_r$, is zero: Repeated application of (19) and (18) gives

$$\begin{aligned} w_k &\in V_{\mu + *\alpha_r} \quad (* \neq 0 \text{ because } k \neq m_r) \\ n_{r-1} w_k &\in V_{\mu + *\alpha_r + \star\alpha_{r-1}} \quad (\text{maybe } \star = 0) \\ y_{r-1} w_k &\in \bigoplus_{k_{r-1}} V_{\mu + *\alpha_r + k_{r-1}\alpha_{r-1}} \\ &\vdots \\ y_0 \cdots y_{r-1} w_k &\in \bigoplus_{k_0} \cdots \bigoplus_{k_{r-1}} V_{\mu + *\alpha_r + k_{r-1}\alpha_{r-1} + \cdots + k_0\alpha_0}. \end{aligned}$$

From the linear independence of $\{\alpha_0, \ldots, \alpha_r\}$ it is immediate that none of the latter vector spaces equals V_μ, whence (35). ∎

Which μ contribute? Which μ contribute to the sum (33)? The μ with $\mu \not\leq \lambda_i$ do not contribute, since $V_\mu = \{0\}$ for such μ. Non-dominant μ also do not contribute: Let $\mu = \sum_{j=0}^r m_j\lambda_j + n\delta$ with $m_k < 0$; we first get (by (37)) $p_\mu \circ y_k \circ i_\mu = 0$ and then (by (35)) $p_\mu \circ x \circ i_\mu = 0$. This motivates taking a closer look at the sets

$$D(\lambda_i) := \{\mu \mid \mu \leq \lambda_i \quad \wedge \quad \mu \in D\}.$$

By [K, Proposition 12.5.a], $D(\lambda_i)$ is the set of all dominant weights and $D(\lambda_i) = P(\lambda_i) \cap D$. A simple but important observation is the following: If $\mu = \sum m_j\lambda_j + n\delta \in D(\lambda_i)$, then

$$\lambda_i - \left(\sum m_j\lambda_j + n\delta\right) = \lambda_i - \mu = \sum n_j\alpha_j \tag{38}$$

for some $n_0, \ldots, n_r \in \mathbb{N}_0$. If both sides of (38) are evaluated at the central element c, one gets a necessary condition for the tupels $(m_0, \ldots, m_r)$:

$$\sum_{j=0}^{r} m_j a_j^\vee = a_i^\vee. \tag{39}$$

LEMMA 7. *Let $i \in \{0, \ldots, r\}$ and $D(\lambda_i) \bmod \delta := \{\sum m_j\lambda_j \mid \sum m_j\lambda_j + n\delta \in D(\lambda_i)$ for some $n\}$. Then $D(\lambda_i) \bmod \delta$ is finite and decomposes into three types of elements:*

I: λ_i *itself*,
II: λ_j *with* $j \neq i$ *and* $a_j^\vee = a_i^\vee$,
III: $\sum m_j\lambda_j$ *with* $\sum m_j \geq 2$ *and* $m_j = 0$ *if* $a_j^\vee \geq a_i^\vee$.

Correspondingly, each element $\mu \in D(\lambda_i)$ is exactly of one of the following types:

I: $\mu = \lambda_i + n\delta, \quad n \leq n_i(\lambda_i) = 0$,
II: $\mu = \lambda_j + n\delta$, *where* $j \neq i$ *and* $a_j^\vee = a_i^\vee$, $\quad n \leq n_i(\lambda_j) \leq 0$,
III: $\mu = \sum m_j\lambda_j + n\delta$, *where* $\sum m_j \geq 2$ *and* $m_j = 0$ *if* $a_j^\vee \geq a_i^\vee$.

PROOF. The finiteness of $D(\lambda_i) \bmod \delta$ and decomposition into the three types I, II, III is immediate from (39). If both sides of (38) are evaluated at d, we get $n \leq 0$; thus the number $n_i(\nu) := \max\{n \in \frac{1}{a_0}\mathbb{N} \mid \nu + n\delta \in D(\lambda_i)\}$ exists for all $\nu \in D(\lambda_i) \bmod \delta$ and is ≤ 0. ∎

It is possible to determine $D(\lambda_i) \bmod \delta$ and $D(\lambda_i)$ explicitly for all affine Kac-Moody algebras and all $i = 0, \ldots, r$; this was done in Appendix B of the German version of the present work[2] [B].

EXAMPLE. $E_6^{(1)}$, $i = 2$. Equation (39) becomes

$$m_0 + m_1 + 2m_2 + 3m_3 + 2m_4 + m_5 + 2m_6 = 2$$

and $D(\lambda_2) \bmod \delta$ contains at most the elements I: λ_2, II: λ_4, λ_6, III: $2\lambda_0, 2\lambda_1, 2\lambda_5, \lambda_0 + \lambda_1, \lambda_0 + \lambda_5, \lambda_1 + \lambda_5$. (In the appendix B mentioned above it is proved that actually, $D(\lambda_2) \bmod \delta$ consists of I: λ_2, II: –, III: $2\lambda_1$, $\lambda_0 + \lambda_5$.) ∎

[2] ...which has been deleted in this version, since the actual knowledge of these sets turned out to be redundant for the proofs of my theorems. Of course, for doing examples, this is still quite useful.

$(\chi\circ\omega)(c_0,\ldots,c_r,q)$ *as a formal power series.* By (35) and (37) we know what the action of $x=\prod_{i=0}^r x_i(c_i)n_i=\prod_{i=0}^r y_i$ on V_μ is: There is a linear map $\Upsilon_\mu\colon V_\mu\longrightarrow V_\mu$, independent of $c_0,\ldots,c_r$, such that $p_\mu\circ x\circ i_\mu$ has the form

$$p_\mu\circ x\circ i_\mu\colon V_\mu\longrightarrow V_\mu,\quad v\longmapsto c_0^{m_0}\cdots c_r^{m_r}\Upsilon_\mu(v).$$

If a_μ denotes the trace of $\Upsilon_\mu\colon V_\mu\longrightarrow V_\mu$, then

$$\operatorname{trace}(p_\mu\circ x\circ i_\mu)=a_\mu\cdot c_0^{m_0}\cdots c_r^{m_r}. \tag{40}$$

THEOREM 4. *Let G be an affine Kac-Moody group and let ω, χ be defined as above. Then, for the components $\chi_i\circ\omega$ of $\chi\circ\omega$, we have: $(\chi_i\circ\omega)(c_0,\ldots,c_r,q)$ is a formal power series in q^{-1} with coefficients in $\mathbb{C}[c_0,\ldots,c_r]$:*

$$\begin{aligned}(\chi_i\circ\omega)(c_0,\ldots,c_r,q)&=\sum_{\substack{\mu\in D(\lambda_i)\text{ and}\\ \mu=\Sigma m_j\lambda_j+n\delta}} a_\mu q^{\mu(d)}c_0^{m_r}\cdots c_r^{m_r}\\ &=\sum_n\Big(\underbrace{\sum_{m_0}\cdots\sum_{m_r}}_{\textit{finite sum}}a_\mu c_0^{m_0}\cdots c_r^{m_r}\Big)q^n.\end{aligned} \tag{41}$$

PROOF. (33), (34), (40) and Lemma 7. ∎

REMARK. It would be nice to know something about the a_μ. A glance at the formulas of the preceding paragraphs gives $a_\mu=\operatorname{trace}\big(p_\mu\circ x_0(1)n_0\cdots x_r(1)n_r\circ i_\mu\big)$. Is there a connection between $x_0(1)n_0\cdots x_r(1)n_r$ and the Coxeter element $\operatorname{cox}=n_0\cdots n_r=x_0(0)n_0\cdots x_r(0)n_r$?

Regularity of $\chi\circ\omega$. Before discussing the regularity of $\chi\circ\omega$ we have a look at the sum (41) and consider $(\chi_i\circ\omega)(c_0,\ldots,c_r,q)$ as polynomials in $c_0,\ldots,c_r$ with coefficients in $\mathbb{C}[[q^{-1}]]$:

$$(\chi_i\circ\omega)(c_0,\ldots,c_r,q)=\sum_{m_0}\cdots\sum_{m_r}\Big(\sum_n a_\mu q^n\Big)c_0^{m_0}\cdots c_r^{m_r}. \tag{42}$$

Using the decomposition of the set of dominant weights into types I, II, III we get

$$(\chi_i\circ\omega)(c_0,\ldots,c_r,q)=(\cdots)c_i+\underbrace{\sum(\cdots)c_j}_{\substack{\text{summands,}\\ \text{containing } c_j\text{'s}\\ \text{with } a_j^\vee=a_i^\vee}}+\underbrace{\text{other summands}}_{\substack{\text{summands of higher}\\ \text{order, not containing}\\ c_j\text{'s with } a_j^\vee\geq a_i^\vee.}}. \tag{43}$$

If we define $b_n:=a_{\lambda_i+n\delta}$, then the coefficient of c_i in (43) is

$$b_0+b_{-1}q^{-1}+b_{-2}q^{-2}+\cdots. \tag{44}$$

LEMMA 8. $b_0\neq 0$.

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PROOF. As above, let $y_j := x_j(c)n_j$; we will prove

$$p_{\lambda_i} \circ y_j \circ i_{\lambda_i} \colon V_{\lambda_i} \longrightarrow V_{\lambda_i} \quad \text{is the identity if } j \neq i \tag{45}$$

$$p_{\lambda_i} \circ y_i \circ i_{\lambda_i} \colon V_{\lambda_i} \longrightarrow V_{\lambda_i} \quad \text{is not the zero map.} \tag{46}$$

Ad (45): V_{λ_i} is invariant under n_j as well as $x_j(c)$, because on the one hand $n_j \cdot V_{\lambda_i} = V_{w_j(\lambda_i)} = V_{\lambda_i}$, and on the other hand $x_j(c) \cdot V_{\lambda_i} = V_{\lambda_i}$ by (18). But G_j is generated by n_j and the $x_j(c)$, thus V_{λ_i} is invariant under G_j. Using $\dim V_{\lambda_i} = 1$ and $G_j = [G_j, G_j]$ it follows that all elements of G_j act as the identity on V_{λ_i}—in particular, y_j does. $\diamondsuit$

Ad (46): Let $V_{\lambda_i} = \mathbb{C}v$ and $V_{\lambda_i - \alpha_i} = \mathbb{C}w$ where $w := n_i \cdot v$ (note that $\lambda_i - \alpha_i = w_i(\lambda_i)$ implies $\dim V_{\lambda_i-\alpha_i} = \dim V_{\lambda_i} = 1$). By (18), $y_i \cdot v = x_i(c) \cdot w = w + w'$, where $w' \in V_{\lambda_i}$. If $p_{\lambda_i} \circ y_i \circ i_{\lambda_i}$ were the zero map, then also $w' = (p_{\lambda_i} \circ y_i \circ i_{\lambda_i})(v) = 0$ and

$$x_i(c) \cdot w = w; \quad \Rightarrow U_i \cdot V_{\lambda_i-\alpha_i} = V_{\lambda_i-\alpha_i}$$
$$x_{-\alpha_i}(c) \cdot w \overset{\ddagger}{=} w; \quad \Rightarrow V_i \cdot V_{\lambda_i-\alpha_i} = V_{\lambda_i-\alpha_i}$$

($\ddagger$ because $\lambda_i - \alpha_i - k\alpha_i = w_i(\lambda_i + k\alpha_i)$ and $\lambda_i + k\alpha_i \not\leq \lambda_i$). Since n_i is in the subgroup generated by U_i and V_i, this would imply $n_i \cdot V_{\lambda_i-\alpha_i} = V_{\lambda_i-\alpha_i}$, contradicting $n_i \cdot V_{\lambda_i-\alpha_i} = V_{\lambda_i}$. $\diamondsuit$

b_0 is the trace of the 1×1-matrix of the linear map

$$p_{\lambda_i} \circ x \circ i_{\lambda_i} \colon V_{\lambda_i} \longrightarrow V_{\lambda_i} \tag{47}$$

from (40). Equations (35), (45), (46) now imply that $p_{\lambda_i} \circ x \circ i_{\lambda_i} = p_{\lambda_i} \circ y_i \circ i_{\lambda_i}$ is not the zero map, *i.e.*, $b_0 \neq 0$. ∎

The coefficient infront of c_j on the right hand side of (43) has the form

$$b_{j,0} + b_{j,-1}q^{-1} + b_{j,-2}q^{-2} + \cdots \tag{48}$$

where $b_{j,k} := a_{\lambda_j + k\delta}$ and $b_{j,0}$ is possibly $= 0$.

EXAMPLE. If $n_i(\lambda_j)$ denotes the first of the numbers k for which $b_{j,k}$ of (48) is $\neq 0$, then in the case of the algebra of type $B_r^{(1)}$ and the module $L(\lambda_0)$, we have $n_0(\lambda_1) = -1$.

In the finite dimensional case, $\chi \circ \omega = \chi|_C$ induces an isomorphism of algebraic varieties $C \longrightarrow \mathbb{C}^r$. In our situation, we shall obtain a slightly weaker result (Theorem 7). As a first step, we shall see in Theorem 5 that the Jacobian determinant of $\chi \circ \omega$ is a function of the "modular" coordinate q alone, non-zero for sufficiently large q.

Taking a first look at the functional determinant of $\chi \circ \omega$, it seems that because of (42) it is polynomial in $c_0, \ldots, c_r$ and a formal power series in q^{-1}. A second look simplifies the situation considerably: We have

$$\det(\chi \circ \omega)'(c_0, \ldots, c_r, q) = \begin{vmatrix} \frac{\partial(\chi_0 \circ \omega)}{\partial c_0} & \cdots & \frac{\partial(\chi_0 \circ \omega)}{\partial c_r} & \frac{\partial(\chi_0 \circ \omega)}{\partial q} \\ \vdots & & \vdots & \vdots \\ \frac{\partial(\chi_r \circ \omega)}{\partial c_0} & \cdots & \frac{\partial(\chi_r \circ \omega)}{\partial c_r} & \frac{\partial(\chi_r \circ \omega)}{\partial q} \\ \frac{\partial(\chi_\delta \circ \omega)}{\partial c_0} & \cdots & \frac{\partial(\chi_\delta \circ \omega)}{\partial c_r} & \frac{\partial(\chi_\delta \circ \omega)}{\partial q} \end{vmatrix}$$

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$$
= \begin{vmatrix}
\frac{\partial(\chi_0 \circ \omega)}{\partial c_0} & \cdots & \frac{\partial(\chi_0 \circ \omega)}{\partial c_r} & \frac{\partial(\chi_0 \circ \omega)}{\partial q} \\
\vdots & & \vdots & \vdots \\
\frac{\partial(\chi_r \circ \omega)}{\partial c_0} & \cdots & \frac{\partial(\chi_r \circ \omega)}{\partial c_r} & \frac{\partial(\chi_r \circ \omega)}{\partial q} \\
0 & \cdots & 0 & 1
\end{vmatrix}
$$

$$
= \begin{vmatrix}
\frac{\partial(\chi_0 \circ \omega)}{\partial c_0} & \cdots & \frac{\partial(\chi_0 \circ \omega)}{\partial c_r} \\
\vdots & & \vdots \\
\frac{\partial(\chi_r \circ \omega)}{\partial c_0} & \cdots & \frac{\partial(\chi_r \circ \omega)}{\partial c_r}
\end{vmatrix} =: \Delta
$$

so the partial derivatives by q have vanished. We can now formulate one of the main results:

THEOREM 5. *The functional determinant Δ, after suitably reordering indices, has the form*

$$
\begin{vmatrix}
\boxed{} & * & * & * \\
0 & \boxed{} & * & * \\
 & & \ddots & \\
0 & 0 & 0 & \boxed{}
\end{vmatrix}
$$

where the blocks are determined by the equivalence relation $i \sim j \Leftrightarrow a_i^\vee = a_j^\vee$ and the blocks on the diagonal do not contain any c_j's, whence only depends on q: $\Delta \in \mathbb{C}[[q^{-1}]]$. Δ is invertible in $\mathbb{C}[[q^{-1}]]$, since all $\det \square$ are.

PROOF. We order indices by descending dual labels and decompose the matrix into blocks determined by the equivalence relation $i \sim j \Leftrightarrow a_i^\vee = a_j^\vee$. The way $\chi_i \circ \omega$ looks like in (43), it is obvious that the blocks on the diagonal do not contain c_j's and that the blocks below the diagonal are zero. Hence $\Delta \in \mathbb{C}[[q^{-1}]]$.

EXAMPLE. In the case of $E_6^{(1)}$, the block structure is given by $a_3^\vee = 3$, $a_2^\vee = a_4^\vee = a_6^\vee = 2$, $a_0^\vee = a_1^\vee = a_5^\vee = 1$. Thus

$$
\Delta = \begin{vmatrix}
* & & & & & & \\
 & * & 0 & 0 & & \star & \\
 & 0 & * & 0 & & & \\
 & 0 & 0 & * & & & \\
 & & & & * & 0 & 0 \\
 & 0 & & & 0 & * & 0 \\
 & & & & 0 & 0 & *
\end{vmatrix}
$$

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(the zeros *inside* the blocks on the diagonal are obtained from additional information from the appendix B in [B]), where $* \in \mathbb{C}[[q^{-1}]]$. In $\star$ there may be c_j's. $\diamondsuit$

We still have to prove that the determinants of the blocks on the diagonal are invertible. To this end, we have to prove that the constant term of the formal power series does not vanish. Let $\square = (m_{ij})$ be one of these blocks. The m_{ii} look like (44) and the m_{ij} ($i \neq j$) look like (48). We now use an old acquaintance, the Leibniz formula

$$\det \square = \sum_{\sigma} \operatorname{sgn} \sigma m_{1\sigma(1)} \cdots m_{N\sigma(N)} = m_{11}m_{22} \cdots m_{NN} + \sum_{\sigma \neq \mathrm{id}} \operatorname{sgn} \sigma m_{1\sigma(1)} \cdots m_{N\sigma(N)}$$

and will prove

$$m_{11}m_{22} \cdots m_{NN} = A_0 + A_1 q^{-1} + A_2 q^{-2} + \cdots \quad \text{where } A_0 \neq 0, \tag{49}$$

$$\sum_{\sigma \neq \mathrm{id}} \operatorname{sgn} \sigma m_{1\sigma(1)} \cdots m_{N\sigma(N)} = B_1 q^{-1} + B_2 q^{-2} + \cdots. \tag{50}$$

(49) is immediate, since all m_{ii} look like (44). As for (50), we make the

CLAIM. *If* $\lambda_{j_1}, \ldots, \lambda_{j_s}$ ($s > 1$) *have the same level, then not only are all the numbers* $n_{j_1}(\lambda_{j_2}), n_{j_2}(\lambda_{j_3}), \ldots, n_{j_s}(\lambda_{j_1}) \leq 0$, *but at least one of them is* < 0.

For: Suppose $n_{j_1}(\lambda_{j_2}) = n_{j_2}(\lambda_{j_3}) = \cdots = n_{j_s}(\lambda_{j_1}) = 0$. Then

$$\begin{aligned} \nu_1 &:= \lambda_{j_1} - \lambda_{j_2} \geq 0 \\ \nu_2 &:= \lambda_{j_2} - \lambda_{j_3} \geq 0 \\ &\ \vdots \\ \nu_s &:= \lambda_{j_s} - \lambda_{j_1} \geq 0. \end{aligned}$$

Adding these up yields $\nu_1 + \cdots + \nu_s = 0$. Since $\nu_i \geq 0$, we get $\nu_1 = \cdots = \nu_s = 0$ and in particular $\lambda_{j_1} = \lambda_{j_2}$, which is a contradiction. $\diamondsuit$

Since $\sigma \neq \mathrm{id}$, there is a "cycle" $m_{i_1 i_2} m_{i_2 i_3} \cdots m_{i_s i_1}$ ($s \geq 2$) in each of the summands of the sum $\sum_{\sigma \neq \mathrm{id}}$. But

$$m_{ij} = \frac{\partial (\chi_i \circ \omega)}{\partial c_j} = \sum_{\substack{\mu = \lambda_j + n\delta \\ n \leq n_i(\lambda_j)}} a_\mu q^n,$$

and at least one of the numbers $n_{i_j}(\lambda_{i_{j+1}})$ is not just ≤ 0 but even < 0—which proves (50). By (49) and (50), the determinant $\det \square \in \mathbb{C}[[q^{-1}]]$ is invertible. $\blacksquare$

3.3. *A $\mathbb{C}^\times$-action on the cross-section.* Preview. We will first determine the set H of all $h \in \mathbf{h}$ such that $(\operatorname{cox} - \mathrm{id})(h) \in \mathbf{c}$. This set turns out to be $H = \mathbb{C}b \oplus \mathbb{C}c$, for some element $b \in X$. Then, we investigate the set $\Phi_{\mathrm{cox}} := \{\alpha > 0 \mid \operatorname{cox}^{-1} \alpha < 0\}$ and, for all $\beta \in \Phi_{\mathrm{cox}}$, determine the value $\beta(b)$. After these preparations, we define a $\mathbb{C}^\times$-action on the Steinberg cross-section C, then get a functional identity for $\chi \circ \omega$.

The subspace H. If W is a Coxeter group with generators $s_1, \dots, s_n$, then $\mathrm{cox} := s_1 \cdots s_n$ is called *the Coxeter element* of W. For each permutation $\sigma \in S_n$, the elements $\mathrm{cox}_\sigma := s_{\sigma(1)} \cdots s_{\sigma(n)}$ are also called Coxeter elements. If A is a GCM, not of type $A_1^{(1)}$, $r \geq 2$, then all Coxeter elements $\mathrm{cox}_\sigma \in \mathcal{W}(A)$ are conjugate in $\mathcal{W}$ [H, pp. 74, 174]. Many statements about cox thus also hold for all cox_σ.

PROPOSITION 9. *Let A be an affine GCM, $\mathbf{g} = \mathbf{g}(A)$ the associated Kac-Moody algebra and $\mathrm{cox}_\sigma = s_{\sigma(0)} \cdots s_{\sigma(r)}$ a Coxeter element of the Weyl group $\mathcal{W} = \mathcal{W}(A)$. Let*

$$H := \left\{h \in \mathbf{h} \mid (\mathrm{cox}_\sigma - \mathrm{id})(h) \in \mathbf{c}\right\}.$$

Then $\dim H = 2$ *and* $H \subset \mathbf{h}'$.

PROOF. The subspace $\mathbf{h}' \subset \mathbf{h}$ is invariant under $\mathcal{W}$; if $s \in \mathcal{W}$, denote by s' the restriction $\mathbf{h}' \longrightarrow \mathbf{h}'$. Consider the sets

$$H' := \left\{h \in \mathbf{h}' \mid (\mathrm{cox}'_\sigma - \mathrm{id})(h) \in \mathbf{c}\right\},$$
$$H := \left\{h \in \mathbf{h} \mid (\mathrm{cox}_\sigma - \mathrm{id})(h) \in \mathbf{c}\right\}.$$

About H': In [Co] one finds the characteristic polynomials of the Coxeter transformations cox'_σ; in the table where they are listed [Co, p. 474] you can read off that the algebraic multiplicity of the eigenvalue 1 of cox'_σ is two for all affine GCM. From [Co, Theorem 3.1] and the identity $\bigcap_{i=0}^r \ker \alpha_i = \mathbf{c}$ [K, Proposition 1.6] we now get

$$\mathrm{cox}'_\sigma(h) = h \iff h \in \mathbf{c} \tag{51}$$

whence the geometric multiplicity of the eigenvalue 1 of cox'_σ is equal to one. The Jordan normal form contains the Jordan block $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ (and no other Jordan blocks with eigenvalue 1). Once again applying (51), we get

$$h \in H' \iff (\mathrm{cox}'_\sigma - \mathrm{id})(h) \in \mathbf{c} \iff (\mathrm{cox}'_\sigma - \mathrm{id})^2(h) = 0.$$

Together with the statement above about the Jordan normal form of cox_σ, it follows that $\dim H' = 2$.

About H: The relation between the matrix of s' with respect to the basis $\{h_0, \dots, h_r\}$ and the matrix of s with respect to the basis $\{h_0, \dots, h_r, d\}$ is:

$$s = \left(\begin{array}{c|c} s' & * \\ \hline 0 & 1 \end{array}\right);$$

hence the algebraic multiplicity of the eigenvalue 1 of cox_σ is 3 for all affine GCM. The argument using [Co, Theorem 3.1] and [K, Proposition 1.6] still holds in $\mathbf{h}$, thus

$$\mathrm{cox}_\sigma(h) = h \iff h \in \mathbf{c} \tag{52}$$

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and the geometric multiplicity of the eigenvalue 1 of cox_σ is again equal to one. The Jordan normal form of cox_σ contains the Jordan block

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(and no other Jordan blocks with eigenvalue 1). From (52) we get

$$h \in H \quad \Longleftrightarrow \quad (\mathrm{cox}_\sigma - \mathrm{id})^2(h) = 0 \tag{53}$$

whence $\dim H = 2$. Since $H' \subset H$, obviously, and H' and H have the same dimension, it follows that $H = H'$. In particular, $H \subset \mathbf{h}'$. ∎

With respect to the basis $\{h_0, \ldots, h_r, d\}$ we have $\mathrm{cox}_\sigma \in M_n(\mathbb{Z})$. By (53), this implies that H is spanned by two vectors $\in X$. One of them we already know: $c \in H$. A second vector which is linearly independent of c, is obtained by solving the equation $(\mathrm{cox}_\sigma - \mathrm{id})x = c$.

PROPOSITION 10. *Let A be an affine GCM, $\mathbf{g} = \mathbf{g}(A)$ the Kac-Moody algebra associated with A and $\mathrm{cox} = s_0 \cdots s_r$ the Coxeter element of the Weyl group $W = W(A)$. Then $H = \mathbb{C}b \oplus \mathbb{C}c$, where c is the central element and b is determined by*

$$(\mathrm{cox} - \mathrm{id})b = kc \ (k \in \mathbb{N} \text{ minimal}), \quad b \in \langle \nabla \rangle_{\mathbb{N}_0}, \quad b - c \notin \langle \nabla \rangle_{\mathbb{N}_0}.$$

The number k as well as the components of c and b with respect to the basis $\nabla = \{h_0, \ldots, h_r\}$ turn out to be as listed in the table below[3]*:*

type	k	c	b
$A_r^{(1)}$	$r+1$	(1,1,...,1,1)	$(r, r-1, \ldots, 1, 0)$
$B_r^{(1)}$, r even	2	(1,1,2,...,2,2,1)	$(r-1, r-1, 2r-4, 2r-6, \ldots, 2, 0)$
$B_r^{(1)}$, r odd	1	(1,1,2,...,2,2,1)	$(\frac{r-1}{2}, \frac{r-1}{2}, r-2, r-3, \ldots, 1, 0)$
$C_r^{(1)}$	2	(1,1,...,1,1)	$(r, r-1, \ldots, 1, 0)$
$D_r^{(1)}$, r even	1	(1,1,2,...,2,1,1)	$(\frac{r}{2}-1, \frac{r}{2}-1, r-3, r-4, \ldots, 1, 0, 0)$
$D_r^{(1)}$, r odd	2	(1,1,2,...,2,1,1)	$(r-2, r-2, 2r-6, 2r-8, \ldots, 2, 0, 0)$
$E_6^{(1)}$	1	(1,1,2,3,2,1,2)	$(1, 2, 3, 3, 1, 0, 1)$
$E_7^{(1)}$	1	(1,2,3,4,3,2,1,2)	$(3, 5, 6, 6, 3, 1, 0, 2)$
$E_8^{(1)}$	1	(1,2,3,4,5,6,4,2,3)	$(4, 7, 9, 10, 10, 9, 4, 1, 3)$
$F_4^{(1)}$	1	(1,2,3,2,1)	$(2, 3, 3, 1, 0)$
$G_2^{(1)}$	1	(1,2,1)	$(1, 1, 0)$

[3] The components of c, which are of course the dual labels, have been listed for completeness' sake.

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$A_2^{(2)}$	1	(1,2)	(1,1)
$A_{2r}^{(2)}$, r even	1	(1,2,...,2,2)	$(\frac{r}{2}, r-1, r-2, \ldots, 1, 0)$
$A_{2r}^{(2)}$, r odd	1	(1,2,...,2,2)	$(\frac{r+1}{2}, r, r-1, \ldots, 2, 1)$
$A_{2r-1}^{(2)}$, r even	1	(1,1,2,...,2,2)	$(\frac{r}{2}, \frac{r}{2}, r-1, r-2, \ldots, 2, 1)$
$A_{2r-1}^{(2)}$, r odd	1	(1,1,2,...,2,2)	$(\frac{r-1}{2}, \frac{r-1}{2}, r-2, r-3, \ldots, 1, 0)$
$D_r^{(1)}$, r even	1	(1,2,...,2,1)	$(\frac{r}{2}, r-1, r-2, \ldots, 1, 0)$
$D_r^{(1)}$, r odd	2	(1,2,...,2,1)	$(r, 2r-2, 2r-4, \ldots, 2, 0)$
$E_6^{(2)}$	1	(1,2,3,4,2)	(2,3,3,2,0)
$D_4^{(3)}$	1	(1,2,3)	(1,1,0)

PROOF. Explicit calculation of all cases. Write the s_i as matrices with respect to the basis $\{h_0, \ldots, h_r, d\}$ of $\mathbf{h}$, determine the matrix of cox, and solve the linear system of equations $(\mathrm{cox} - \mathrm{id})x = c$. ∎

REMARK. There is a more conceptual approach to Proposition 10, at least in the case of bipartite graphs $X_r^{(k)}$, due to Steinberg [St3] and further elaborated in [BLM]. In particular, H can be written in the form

$$H = \mathbb{C}c \oplus \mathbb{C}\tilde{c},$$

where $\tilde{c}$ is obtained from c by changing every "second" sign (*cf.* [BLM, Proposition 8]). Moreover, the image of H in the "finite" Cartan subalgebra $\mathbf{h}'/\mathbb{C}c$ is conjugate under the finite Weyl group into the space $\mathbb{C}w_{\beta^\vee}$, spanned by the simple coweight $w_{\beta^\vee}$ corresponding to the fork node β of the diagram $X_r^{(k)}$ ([BLM, Prop. 12,13], [St3, Part 2]).

EXAMPLE. In the case $A_1^{(1)}$, with Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, the matrix of cox $= s_0 s_1$ turns out to be

$$\mathrm{cox} = \begin{pmatrix} -1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and for $b := h_0$ we have $(\mathrm{cox} - \mathrm{id})b = 2c$. ♢

The set Φ_{cox}. To determine $\Phi_{\mathrm{cox}} := \{\alpha > 0 \mid \mathrm{cox}^{-1}\alpha < 0\}$ we use a corrected version of [K, Exercise 3.12]:

LEMMA 11. *Let* $w = w_{i_1} \cdots w_{i_t}$ *be an expression of minimal length and* $\Phi_w := \{\alpha > 0 \mid w^{-1}\alpha < 0\}$. *Then:*

(a) $\Phi_w = \{\alpha_{i_1}, w_{i_1}\alpha_{i_2}, \ldots, w_{i_1} \cdots w_{i_{t-1}}\alpha_{i_t}\}$;

(b) *For each* $\beta \in \Phi_w$, *the sequence* $\beta, w_{i_1}\beta, \ldots, w_{i_t} \cdots w_{i_1}\beta$ *contains exactly one change of sign* $+ \longrightarrow -$; *At the place where this occurs, there is a simple root;*

(c) *If* $\alpha_{j(\beta)}$ *denotes the simple root associated with* $\beta \in \Phi_w$ *in (b), then for all* $\lambda \in \mathbf{h}^*$ *we have the identity*

$$\lambda - w\lambda = \sum_{\beta \in \Phi_w} \lambda(h_{j(\beta)})\beta. \tag{54}$$

PROOF OF (a) (*cf.* [H, EXERCISE 5.6.1]). Let $\beta \in \Phi_w$. The sequence

$$\beta, w_{i_1}\beta, \ldots, w_{i_t} \cdots w_{i_1}\beta \tag{55}$$

contains at least one change of sign $+ \longrightarrow -$, say

$$w_{i_{N-1}} \cdots w_{i_1}\beta > 0 \quad \text{and} \quad w_{i_N} w_{i_{N-1}} \cdots w_{i_1}\beta < 0.$$

Since w_{i_N} permutes the set $R_+ \setminus \{\alpha_{i_N}\}$, it follows that $w_{i_{N-1}} \cdots w_{i_1}\beta = \alpha_{i_N}$ resp. $\beta = w_{i_1} \cdots w_{i_{N-1}}\alpha_{i_N} =: \gamma_N$; thus $\Phi_w \subset \{\gamma_1, \ldots, \gamma_t\}$. Since the expression for w was supposed to be of minimal length we get, by [H, Proposition 5.6], equality of these sets.

PROOF OF (b). Suppose there were two changes of sign occuring in sequence (55), say

$$\beta \cdots \quad \underbrace{w_{i_{N-1}} \cdots w_{i_1}\beta}_{=\alpha_{i_N} > 0} \quad \underbrace{w_{i_N} \cdots w_{i_1}\beta}_{=w_{i_N}\alpha_{i_N} < 0} \quad \cdots \quad \underbrace{w_{i_{M-1}} \cdots w_{i_1}\beta}_{=\alpha_{i_M} > 0} \quad \underbrace{w_{i_M} \cdots w_{i_1}\beta}_{=w_{i_M}\alpha_{i_M} < 0} \quad \cdots w^{-1}\beta$$

(*cf.* the proof of (a)). Then $\alpha_{i_M} = w_{i_{M-1}} \cdots w_{i_N}\alpha_{i_N} < 0$, which is $\in R_-$ by [K, Lemma 3.11 b], contradiction.

PROOF OF (c) (INDUCTION BY t). The case $t = 1$ is trivial. Let $v := w_{i_1} \cdots w_{i_{t-1}}$, then $\Phi_v = \{\alpha_{i_1}, w_{i_1}\alpha_{i_2}, \ldots, w_{i_1} \cdots w_{i_{t-2}}\alpha_{i_{t-1}}\}$ by (a) and furthermore

$$\begin{aligned}
\lambda - w\lambda &= \lambda - v\lambda + v\lambda - w\lambda \\
&= \sum_{\beta \in \Phi_v} \lambda(h_{j(\beta)})\beta + w_{i_1} \cdots w_{i_{t-1}}(\lambda - w_{i_t}\lambda) \\
&= \sum_{\beta \in \Phi_v} \lambda(h_{j(\beta)})\beta + \lambda(h_{i_t}) w_{i_1} \cdots w_{i_{t-1}}\alpha_{i_t} \\
&= \sum_{\beta \in \Phi_w} \lambda(h_{j(\beta)})\beta,
\end{aligned}$$

the latter because $\Phi_w = \Phi_v \cup \{w_{i_1} \cdots w_{i_{t-1}}\alpha_{i_t}\}$ and since the sequence (55) for $\beta = w_{i_1} \cdots w_{i_{t-1}}\alpha_{i_t}$ has a since change $+ \longrightarrow -$ at i_t:

$$\begin{aligned}
& w_{i_{t-1}} \cdots w_{i_1}(w_{i_1} \cdots w_{i_{t-1}}\alpha_{i_t}) = \alpha_{i_t} > 0 \\
\text{and} \quad & w_{i_t} w_{i_{t-1}} \cdots w_{i_1}(w_{i_1} \cdots w_{i_{t-1}}\alpha_{i_t}) = w_{i_t}\alpha_{i_t} < 0.
\end{aligned}$$

∎

We would like to apply Lemma 11 to the Coxeter element cox, so we need:

LEMMA 12. $\mathrm{cox} = w_0 w_1 \cdots w_r$ *is an expression of minimal length.*

PROOF. Our strategy is: If $\mathrm{cox} = w_0 w_1 \cdots w_r$ were of minimal length, then $\Phi_{\mathrm{cox}} = \{\alpha_0, w_0\alpha_1, \ldots, w_0 \cdots w_{r-1}\alpha_r\}$ by Lemma 11. We will now, reversely, prove that the elements of the latter set are different from each other and are $\in \Phi_{\mathrm{cox}}$; by [H, Proposition 5.6.b] we get what we want.

Since w_i permutes the set $R_+ \setminus \{\alpha_i\}$, we get for $k = 0, \ldots, r$

$$\begin{aligned} \alpha_k &\in R_+ \\ w_{k-1}\alpha_k &\in R_+ \quad \text{and moreover} \quad \in \langle \alpha_{k-1}, \alpha_k \rangle_{\mathbb{N}_0} \\ w_{k-2}w_{k-1}\alpha_k &\in R_+ \quad \text{and moreover} \quad \in \langle \alpha_{k-2}, \alpha_{k-1}, \alpha_k \rangle_{\mathbb{N}_0} \\ &\cdots \\ w_0 \cdots w_{k-1}\alpha_k &\in R_+; \end{aligned}$$

furthermore, for the same reason,

$$\begin{aligned} \alpha_k &\in R_+ \\ w_k\alpha_k = -\alpha_k &\in R_- \\ w_{k+1}w_k\alpha_k &\in R_- \quad \text{even} \quad \in -\langle \alpha_k, \alpha_{k+1} \rangle_{\mathbb{N}_0} \\ &\cdots \\ w_r \cdots w_k\alpha_k &\in R_- \end{aligned}$$

i.e., $\mathrm{cox}^{-1}(w_0 \cdots w_{k-1}\alpha_k) = w_r \cdots w_k \alpha_k \in R_-$. Thus, $w_0 \cdots w_{k-1}\alpha_k \in \Phi_{\mathrm{cox}}$ for $k = 0, \ldots, r$. These elements are different from each other, since

$$\begin{aligned} \alpha_0 &= \alpha_0, \\ w_0\alpha_1 &= \alpha_1 + \text{ linear combination of } \alpha_0, \\ w_0 w_1 \alpha_2 &= \alpha_2 + \text{ linear combination of } \alpha_0, \alpha_1, \end{aligned}$$

etc. ∎

COROLLARY 13 (AND DEFINITION).

$$\Phi_{\mathrm{cox}} = \{\alpha_0, w_0\alpha_1, \ldots, w_0 \cdots w_{r-1}\alpha_r\} =: \{\beta_0, \beta_1, \ldots, \beta_r\}.$$

In Section 3.4 we will get a functional identity for $\chi \circ \omega$, but to this end we need the values $\beta_i(b)$. These can be determined:

LEMMA 14. *If $b \in \mathbf{h}$ is the element of $\mathbf{h}$ satisfying $(\mathrm{cox} - \mathrm{id})b = kc$, cf. Proposition 10, then*

$$\big(\beta_0(b), \ldots, \beta_r(b)\big) = k(a_0^\vee, \ldots, a_r^\vee).$$

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PROOF. We will first prove that $\big(\beta_0(b), \ldots, \beta_r(b)\big)$ is proportional to $(a_0^\vee, \ldots, a_r^\vee)$. For each $\alpha_j \in \mathbf{h}^*$, (54) implies

$$\alpha_j - \operatorname{cox} \alpha_j = \sum_i \alpha_j(h_i)\beta_i = \sum_i a_{ij}\beta_i.$$

Evaluated at b this gives

$$\alpha_j(b) - (\operatorname{cox} \alpha_j)(b) = \sum_i a_{ij}\beta_i(b). \tag{56}$$

But we also have

$$\operatorname{cox} b - b = kc \quad \Rightarrow \quad \operatorname{cox}^{-1} b = b - kc$$

and, if $\langle\ ,\ \rangle$ denotes the dual pairing $\mathbf{h} \times \mathbf{h}^* \longrightarrow \mathbb{C}$,

$$(\operatorname{cox} \alpha_j)(b) = \langle b, w_0 \cdots w_r \alpha_j\rangle = \langle s_r \cdots s_0 b, \alpha_j\rangle = \langle b - kc, \alpha_j\rangle = \langle b, \alpha_j\rangle = \alpha_j(b);$$

whence the lefthand side of (56) is zero. Since the dual Kac labels describe the linear dependence of the rows of the Cartan matrix, we end up with the proportionality we claimed, say

$$\big(\beta_0(b), \ldots, \beta_r(b)\big) = \kappa(a_0^\vee, \ldots, a_r^\vee),$$

for some $\kappa \in \mathbb{C}$. The constant κ can be determined from $\beta_0(b) = \alpha_0(b) = \kappa a_0^\vee = \kappa$ by using the data from the table in Proposition 10; it turns out that in all cases $\kappa = k$. ∎

EXAMPLE. For $B_3^{(1)}$, and with respect to the basis $\{h_0, \ldots, h_3\}$, we get

$$\operatorname{cox} = \begin{pmatrix} 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & -2 \\ 1 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \kappa = -\frac{\alpha_3(b)}{a_3^\vee} = -\frac{-1}{1} = 1.$$

A $\mathbb{C}^\times$-action on C. For the time being, we will use the notation of chapter 1. The center $\mathbf{c}$ is parametrized by $\gamma\colon \mathbb{C} \longrightarrow \mathbf{h}$, $\zeta \longmapsto kc \otimes \zeta$, ($k$ is the number from Proposition 10), and the center of G is parametrized by $\bar{\bar{\gamma}}\colon \mathbb{C}^\times \longrightarrow T$, $z \longmapsto kc \otimes z$ (*cf.* [K2, p. 190]). For each $z \in \mathbb{C}^\times$ we want to find an element $\bar{\bar{\mu}}(z) \in T$ such that

$$\bar{\bar{\mu}}(z)^{-1} C \bar{\bar{\mu}}(z) = C\bar{\bar{\gamma}}(z) \quad \text{bzw.} \quad \bar{\bar{\mu}}(z) C \bar{\bar{\gamma}}(z) \bar{\bar{\mu}}(z)^{-1} = C;$$

in this sense, the translation by the central element $\bar{\bar{\gamma}}(z)$ can be reversed by conjugation with $\bar{\bar{\mu}}(z)$ and we get an action of $\mathbb{C}^\times$ on C:

$$\mathbb{C}^\times \times C \longrightarrow C, \quad z.(x, q) := \bar{\bar{\mu}}(z)(x, q)\bar{\bar{\gamma}}(z)\bar{\bar{\mu}}(z)^{-1}.$$

The one-parameter subgroup $\bar{\bar{\mu}}\colon \mathbb{C}^\times \longrightarrow T$, $z \longmapsto b \otimes z$, where $b \in \langle\nabla\rangle_{\mathbb{N}_0}$ as in Proposition 10, satisfies our needs: First, we have

$$\begin{aligned}
(\operatorname{cox} - \mathrm{id})b = kc &\iff \operatorname{cox}(b \otimes \zeta) - b \otimes \zeta = kc \otimes \zeta && \forall\, \zeta \in \mathbb{C} \\
&\overset{1}{\Longrightarrow} \exp(-b \otimes \zeta) \exp\big(\operatorname{cox}(b \otimes \zeta)\big) = \exp(kc \otimes \zeta) && \forall\, \zeta \in \mathbb{C} \\
&\overset{2}{\iff} \exp(-b \otimes \zeta) \widehat{\operatorname{cox}} \exp(b \otimes \zeta) \widehat{\operatorname{cox}}^{-1} = \exp(kc \otimes \zeta) && \forall\, \zeta \in \mathbb{C} \\
&\overset{3}{\iff} \bar{\bar{\mu}}(z)^{-1} \widehat{\operatorname{cox}}\, \bar{\bar{\mu}}(z)\, \widehat{\operatorname{cox}}^{-1} = \bar{\bar{\gamma}}(z) && \forall\, z \in \mathbb{C}^\times \\
&\overset{4}{\iff} \bar{\bar{\mu}}(z)^{-1} \widehat{\operatorname{cox}}\, \bar{\bar{\mu}}(z) = \widehat{\operatorname{cox}}\, \bar{\bar{\gamma}}(z) && \forall\, z \in \mathbb{C}^\times
\end{aligned}$$

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(1: $\exp\colon \mathbf{h} \longrightarrow T$ is a group homomorphism; 2: $s_i\colon \mathbf{h} \longrightarrow \mathbf{h}$ lifts to an action on T, $\mathrm{c\hat{o}x} = n_0 n_1 \cdots n_r$; 3: definition of $\overline{\overline{\mu}}$, $\overline{\overline{\gamma}}$ and substituting $e^{\zeta} \leftrightarrow z$; 4: $\gamma(z) \in \mathrm{center}(G)$). We now write γ, μ instead of $\overline{\overline{\gamma}}$, $\overline{\overline{\mu}}$.

Let

$$\beta_0 := \alpha_0, \beta_1 := s_0(\alpha_1), \ldots, \beta_r := s_0 \cdots s_{r-1}(\alpha_r), \tag{57}$$

using $n_i x_\alpha(c) n_i^{-1} = x_{s_i(\alpha)}(c)$ we get

$$\begin{aligned} &x_0(c_0) n_0 x_1(c_1) n_1 x_2(c_2) n_2 \cdots x_r(c_r) n_r \\ &\quad = x_0(c_0)\big[n_0 x_1(c_1) n_0^{-1}\big] n_0 n_1 x_2(c_2) n_2 \cdots x_r(c) n_r \\ &\quad = x_0(c_0) x_{s_0(\alpha_1)}(c_1)\big[n_0 n_1 x_2(c_2) n_1^{-1} n_0^{-1}\big] n_0 n_1 n_2 \cdots x_r(c_r) n_r \\ &\quad = \cdots = x_0(c_0) x_{s_0(\alpha_1)}(c_1) x_{s_0 s_1(\alpha_2)}(c_2) \cdots x_{s_0 \cdots s_{r-1}(\alpha_r)}(c_r) n_0 \cdots n_r \\ &\quad = x_{\beta_0}(c_0) x_{\beta_1}(c_1) \cdots x_{\beta_r}(c_r) \mathrm{c\hat{o}x}. \end{aligned}$$

It follows that

$$\omega(c_0, \ldots, c_r, q) = \Big[\prod_{i=0}^{r} x_i(c_i) n_i, q\Big] = \Big[\Big(\prod_{i=0}^{r} x_{\beta_i}(c_i)\Big)\mathrm{c\hat{o}x}, q\Big].$$

Right translation of elements of the cross-section by $\gamma(z)$ and subsequent conjugation by $\mu(z)$ gives

$$\begin{aligned} \mu(z)\Big[\Big(\prod_{i=0}^{r} x_{\beta_i}(c_i)\Big)\mathrm{c\hat{o}x}, q\Big]\gamma(z)\mu(z)^{-1} &= \Big[\mu(z)\Big(\prod_{i=0}^{r} x_{\beta_i}(c_i)\Big)\mathrm{c\hat{o}x}\gamma(z)\mu(z)^{-1}, q\Big] \\ &= \Big[\prod_{i=0}^{r}\Big(\mu(z) x_{\beta_i}(c_i)\mu(z)^{-1}\Big)\mu(z)\mathrm{c\hat{o}x}\gamma(z)\mu(z)^{-1}, q\Big] \\ &= \bigg[\bigg(\prod_{i=0}^{r} x_{\beta_i}\Big(\beta_i\big(\mu(z)\big)c_i\Big)\bigg)\mathrm{c\hat{o}x}, q\bigg] \\ &= \Big[\Big(\prod_{i=0}^{r} x_{\beta_i}(z^{\beta_i(b)}c_i)\Big)\mathrm{c\hat{o}x}, q\Big] \\ &= \Big[\Big(\prod_{i=0}^{r} x_{\beta_i}(z^{ka_i^\vee}c_i)\Big)\mathrm{c\hat{o}x}, q\Big] \in C. \end{aligned} \tag{58}$$

In the last equality, we used Lemma 14. Now we just proved :

PROPOSITION 15. *Let $b \in \mathbf{h}$ be the element satisfying* $(\mathrm{cox} - \mathrm{id})b = kc$, *see Proposition 10. Then right translation by $\gamma(z) := kc \otimes z$ and subsequent conjugation by $\mu(z) := b \otimes z$ defines a $\mathbb{C}^\times$-action on C, and this $\mathbb{C}^\times$-action is given by*

$$z \cdot \omega(c_0, \ldots, c_r, q) = \omega(z^{ka_0^\vee}c_0, \ldots, z^{ka_r^\vee}c_r, q). \tag{59}$$

A functional identity for $\chi \circ \omega$.

THEOREM 6. *The components of the trace function satisfy the following functional identity (with k as in Proposition 10):*

$$(\chi_i \circ \omega)(z^{ka_0^\vee}c_0, \ldots, z^{ka_r^\vee}c_r, q) = z^{ka_i^\vee}(\chi_i \circ \omega)(c_0, \ldots, c_r, q) \tag{60}$$

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PROOF. Since the χ_i are invariant under conjugation by elements of T (*cf.* chapter 2), and central elements $\gamma(z) = kc \otimes z$ act as a $\lambda_i(kc \otimes z)$-multiple of the identity on $L(\lambda_i)$, we arrive at

$$\begin{aligned}(\chi_i \circ \omega)(z^{ka_0^\vee} c_0, \dots, z^{ka_r^\vee} c_r, q) &= \chi\big(z \cdot \omega(c_0, \dots, c_r, q)\big) \\ &= \chi_i\big(\mu(z)\omega(c_0, \dots, c_r, q)\gamma(z)\mu(z)^{-1}\big) \\ &= \chi_i\big(\omega(c_0, \dots, c_r, q)\gamma(z)\big) \\ &= z^{\lambda_i(kc)}\chi_i\big(\omega(c_0, \dots, c_r, q)\big) \\ &= z^{ka_i^\vee}(\chi_i \circ \omega)(c_0, \dots, c_r, q).\end{aligned}$$ ■

For each $q \in \mathbb{C}^\times$, let $p_q \colon G \longrightarrow G'$, $(x, q) \longmapsto x$ and $C_q := p_q(C)$; furthermore, let

$$\chi_q := \chi|_{C_q \times \{q\}} \colon C_q \times \{q\} \longrightarrow \mathbb{C}^{r+1} \times \{q\}$$

denote the restriction of χ. The following is the analogue of Steinberg's result on the regularity of $\chi \circ \omega$.

THEOREM 7. *For sufficiently large q, the restriction χ_q induces a $\mathbb{C}^\times$-equivariant isomorphism of algebraic varieties*

$$C_q \longrightarrow \mathbb{C}^{r+1}.$$

PROOF. By Theorem 6, χ_q is a $\mathbb{C}^\times$-equivariant morphism between $(r+1)$-dimensional affine spaces with respect to the same set of weights. Since, for sufficiently large q, its Jacobian is non-zero (Theorem 5), we get that it is an isomorphism (*cf.* [S1, Section 8.1]). ■

REMARK. One may as well employ the statement of Theorem 7 to deduce the equality of the $\mathbb{C}^\times$-weights on C_q and $\mathbb{C}^{r+1}$ to deduce the fact that $k = \kappa$ in Lemma 14.

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