COMPOSITIO MATHEMATICA

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Compositio Math. **159** (2023), 1607–1672.

doi:10.1112/S0010437X23007261

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

https://doi.org/10.1112/S0010437X23007261 Published online by Cambridge University Press

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Abstract

We investigate the maximal finite length submodule of the Breuil–Kisin prismatic cohomology of a smooth proper formal scheme over a p-adic ring of integers. This submodule governs pathology phenomena in integral p-adic cohomology theories. Geometric applications include a control, in low degrees and mild ramifications, of (1) the discrepancy between two naturally associated Albanese varieties in characteristic p, and (2) the kernel of the specialization map in p-adic étale cohomology. As an arithmetic application, we study the boundary case of the theory due to Fontaine and Laffaille, Fontaine and Messing, and Kato. Also included is an interesting example, generalized from a construction in Bhatt, Morrow and Scholze's work, which illustrates some of our theoretical results being sharp, and negates a question of Breuil.

Contents

Received 27 June 2022, accepted in final form 9 March 2023, published online 30 June 2023.
2020 Mathematics Subject Classification 14F30, 11F80 (primary).
Keywords: prismatic cohomology, crystalline cohomology, Fontaine–Messing theory.

1. Introduction

Let $\mathcal{O}_K$ be a mixed characteristic discrete valuation ring (DVR) with perfect residue field k and fraction field K. Let $\mathcal{X}$ be a smooth proper (formal) scheme over $\mathcal{O}_K$. It is natural to ask how the geometry of $\mathcal{X}_k$ and $\mathcal{X}_K$ are related. Recall that the proper base change theorem [Sta21, Tag 0GJ2] says that, for any prime ℓ, there is a specialization map

$$\mathrm{Sp}\colon \mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}_\ell) \to \mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell).$$

The smooth base change theorem says [Sta21, Tag0GKD] that the above map is an isomorphism for any $\ell \neq p$.

The lack of a smooth base change theorem for when $\ell = p$ is related to many interesting 'pathology' phenomena in p-adic cohomology theories. In this paper, we try to investigate these pathologies using the recent advances of prismatic cohomology theory.

The driving philosophy in this paper is as follows. Recall that in [BMS18], [BMS19], and [BS22], the authors attached a natural cohomology theory, known as the prismatic cohomology, to the mixed characteristic family $\mathcal{X}/\mathcal{O}_K$. This cohomology can be thought of as 'the universal p-adic cohomology theory', therefore we expect a certain well-defined piece inside prismatic cohomology to be 'the universal source of pathology' in all p-adic cohomology theories. Before explicating the above, let us first say that the comparison between étale torsion and crystalline torsion as in [BMS18, Theorem 1.1(ii)] serves as the initial inspiration. Now let us showcase two more such pathologies and state what our main theorem specializes to in these two cases.

Albanese and reduction

Let us assume, in addition to the above, that $\mathcal{X}$ possesses an $\mathcal{O}_K$-point x. Associated with the pair $(\mathcal{X}, x)$ is a functorial map of abelian varieties $f\colon \mathrm{Alb}(\mathcal{X}_k) \to \mathcal{A}_k$ over k, where $\mathcal{A}$ is the Néron model of the Albanese of $(\mathcal{X}_K, x_K)$. The smooth and proper base change theorem tells us that f is a p-power isogeny. What can we say about $\ker(f)$?

Theorem 1.1 (Corollary 4.6). *Let e be the ramification index of $\mathcal{O}_K$.*

(1) *If $e < p - 1$ then the map $f\colon \mathrm{Alb}(\mathcal{X}_k) \to \mathrm{Alb}(\mathcal{X}_K)_k$ is an isomorphism.*
(2) *If $e < 2(p - 1)$ then $\ker(f)$ is p-torsion.*
(3) *If $e = p - 1$ then $\ker(f)$ is p-torsion and of multiplicative type, hence must be a form of several copies of μ_p. Moreover, there is a canonical injection of $\mathcal{O}_K$-modules,*

$$\mathbb{D}(\ker(f)) \otimes_k (\mathcal{O}_K/p) \hookrightarrow \mathrm{H}^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}).$$

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Here $\mathbb{D}(-)$ denotes the Dieudonné module of the said finite flat group scheme. If one translates this result into a statement concerning maps between Picard schemes, then our result slightly refines an old result by Raynaud [Ray79, Thèoréme 4.1.3] in the setting of smooth central fibers (see Remark 4.8).

Kernel of specialization

The p-adic specialization map is not an isomorphism, as it is almost never surjective, for the rank of the source is at most half of the rank of the target. One can still ask whether the p-adic specialization map is injective or not.

THEOREM 1.2 (Corollary 4.15). *Let e be the ramification index of $\mathcal{O}_K$ and let $i \in \mathbb{N}$. Consider the specialization map $\mathrm{Sp}^i \colon \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}_p) \to \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p)$.*

(1) *If $e \cdot (i-1) < p-1$, then Sp^i is injective.*
(2) *If $e \cdot (i-1) < 2(p-1)$, then $\ker(\mathrm{Sp}^i)$ is annihilated by p^{i-1}.*
(3) *If $e \cdot (i-1) = p-1$, then $\ker(\mathrm{Sp}^i)$ is p-torsion, and there is a $\mathrm{Gal}(\overline{k}/k)$-equivariant injection,*

$$\ker(\mathrm{Sp}^i) \otimes_{\mathbb{F}_p} \big(\mathcal{O}_K \otimes_W W(\overline{k})\big)/p \hookrightarrow \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}) \otimes_W W(\overline{k}).$$

The above two theorems are of similar shape, and that is because they are shadows of the same result concerning prismatic cohomology, which we explain next.

Prismatic input

Choose a uniformizer $\pi \in \mathcal{O}_K$. There is a canonical surjection $\mathfrak{S} := W(k)[\![u]\!] \twoheadrightarrow \mathcal{O}_K$ with kernel generated by the Eisenstein polynomial of π, which has degree given by the ramification index e. Let $\varphi_{\mathfrak{S}}$ be the endomorphism on $\mathfrak{S}$ which restricts to usual Frobenius on $W(k)$ and sends u to u^p. The triple $(\mathfrak{S}, (E), \varphi_{\mathfrak{S}})$ is known as the Breuil–Kisin prism associated with $(\mathcal{O}_K, \pi)$ [BS22, Example 1.3(3)].

In [BMS19, BS22], the authors attached an $\mathfrak{S}$-perfect complex $\mathrm{R\Gamma}_{\Delta}(\mathcal{X}/\mathfrak{S})$ with a Frobenius operator. Similar to the classical crystalline story, the Frobenius operator is also an isogeny. A concrete consequence of having an isogenous Frobenius map is that the torsion submodule in $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$ is p-power torsion [BMS18, Proposition 4.3(i)]. Hence, the torsion must be supported on $\mathrm{Spec}(\mathfrak{S}/p)$. Note that $\mathfrak{S}/p \cong k[\![u]\!]$ is a DVR. An upshot of the above discussion is that we have three descriptions of a submodule in $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$:

(1) the u^∞-torsion submodule in $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$, henceforth denoted by $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty]$;
(2) the maximal finite length submodule in $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$; and
(3) the submodule in $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$ supported at the closed point in $\mathrm{Spec}(\mathfrak{S})$.

To make the point that the above *is* the universal source of pathology in p-adic cohomology theory, let us exhibit the connection between u^∞-torsion and our previous results.

THEOREM 1.3.

(1) *(Theorem 4.2) Concerning the natural map $f \colon \mathrm{Alb}(\mathcal{X}_k) \to \mathrm{Alb}(\mathcal{X}_K)_k$, we have a natural isomorphism of Dieudonné modules,*

$$\mathbb{D}(\ker(f))^{(-1)} \cong \mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u],$$

where $(-)^{(-1)}$ denotes the Frobenius untwist *and $(-)[u]$ denotes the u-torsion submodule.*
(2) *(Theorem 4.14) As for the kernel of the p-adic specialization map, we have a natural isomorphism of $\mathrm{Gal}(\overline{k}/k)$-representations,*

$$\ker(\mathrm{Sp}^i) \cong \big(\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty]/u \otimes_{W(k)} W(\overline{k})\big)^{\varphi=1}.$$

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In view of the aforementioned statements, the reader can probably guess what our main result, concerning the structure of u^∞-torsion in prismatic cohomology, should look like.

Theorem 1.4 (Theorem 3.6 and Corollary 3.22). *Let us write $\mathfrak{M}^i := \mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})[u^\infty]$, and write* $\mathrm{Ann}(-)$ *for the annihilator ideal of an $\mathfrak{S}$-module.*

(1) *If $e \cdot (i-1) < p-1$, then $\mathfrak{M}^i = 0$.*
(2) *If $e \cdot (i-1) < 2(p-1)$, then $\mathrm{Ann}(\mathfrak{M}^i) + (u) \supset (p^{i-1}, u)$.*
(3) *If $e \cdot (i-1) = p-1$, then $\mathrm{Ann}(\mathfrak{M}^i) \supset (p, u)$. Moreover, the semi-linear Frobenius on $\mathfrak{M}^i$ is bijective, and there is a natural injection $\mathfrak{M}^i \otimes_k (\mathcal{O}_K/p) \hookrightarrow \mathrm{H}^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.*

Remark 1.5. (1) We also prove the mod p^n analogs. As a consequence we obtain the following result (Corollary 3.8) concerning the shape of prismatic cohomology. Let i be an integer satisfying $e \cdot (i-1) < p-1$. Then there exists a (non-canonical) isomorphism of $\mathfrak{S}$-modules,

$$\mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S}) \simeq \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathfrak{S}.$$

(2) Min [Min21, Corollary 5.4] has previously obtained Theorem 1.4(1), and his method will show the mod p^n analog when $e \cdot i < p-1$. Let us briefly explain the appearance of $i-1$ in our result, which might seem odd at first glance. It is due to the fact that the prismatic Verschiebung operator V_i becomes canonically divisible by E when restricted to the p^∞-torsion submodule or the u^∞-torsion submodule, and these submodules with the usual prismatic Frobenius and the 'divided Verschiebung' are canonically (generalized) Kisin modules of height $i-1$ instead of i. For more details, see Corollary 3.16.

(3) One may ask if there can be a better trick/argument showing better bounds on the vanishing of u-torsion. Later on we shall explain a generalization of a construction in [BMS18, § 2.1] with u-torsion in cohomological degree 2 and ramification index $p-1$. Hence, our result is actually sharp in terms of the largest $e \cdot (i-1)$ allowed.

To illustrate Remark 1.5(2), let us quickly sketch a proof of Theorem 1.4(1). The following proof was provided by the referee; the same strategy has essentially appeared in the proof of [LL20, Corollary 7.25]. However, we would like to point out that this argument does not immediately extend to the derived mod p^n-prismatic cohomology.

Proof sketch of Theorem 1.4(1). We have a short exact sequence of $k[\![u]\!]$-modules,

$$0 \to \mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})/p \to \mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S}, \mathcal{O}/p) \to \mathrm{H}^{i+1}_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})[p] \to 0.$$

There are Frobenius φ and Verschiebung V_i acting on the first two items in a compatible way, such that the composition of these two operators in either way is equal to multiplication by $u^{e \cdot i}$, hence these two operators descend to the third item with their compositions satisfying the same relation. This forces the third item to be u-torsion-free whenever $e \cdot i < p-1$ (see, for example, [FKW21, Lemma 2.2.1] or the proof of Corollary 3.4). Since any u-power torsion is necessarily also p-power torsion, we win. □

Special fiber telling Hodge numbers of the generic fiber

As a third geometric application of our result, we revisit the question discussed in [Li22]: what mild condition on $\mathcal{X}$ guarantees that the Hodge numbers of the generic fiber X can be read off from the special fiber $\mathcal{X}_0$? In [Li22] the first named author obtained a result along these lines, with technical input of prismatic cohomology and the structural result in [Min21]. However, it was noted that the results in [Li22] are not optimal in the unramified case, when compared with what one got from results by Fontaine–Messing, Kato, and Wintenberger. We analyzed the situation and concluded that it is because we lack knowledge of the shape of u^∞-torsion in prismatic cohomology in the boundary degree. This paper is partially motivated by the hope of improving results in [Li22], and our improvement is the following theorem.

Theorem 1.6 (Theorem 4.17, improvement of [Li22, Theorem 1.1]). *Let $\mathcal{X}$ be a smooth proper p-adic formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$ of ramification index e. Let T be the largest integer such that $e \cdot (T-1) \leq p-1$.*

(1) *Assume there is a lift of $\mathcal{X}$ to $\mathfrak{S}/(E^2)$. Then for all i, j satisfying $i+j<T$, we have equalities*
$$h^{i,j}(X) = \mathfrak{h}^{i,j}(\mathcal{X}_k),$$
where the latter denotes virtual Hodge numbers of $\mathcal{X}_k$, defined as in [Li22, Definition 3.1].

(2) *Assume, furthermore, that $e \cdot (\dim \mathcal{X}_k - 1) \leq p-1$. Then the special fiber $\mathcal{X}_k$ knows the Hodge numbers of the rigid generic fiber X.*

Along the way, we also improve the results in [Li22] concerning the integral Hodge–de Rham spectral sequence (see Theorem 4.18), and obtain a curious degeneration statement of the 'Nygaard–Prism' spectral sequence (see Theorem 4.22) in the unramified case.

Application to integral p-adic Hodge theory

It is a central theme in integral p-adic Hodge theory to understand Galois representations such as $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p)$ in terms of linear algebraic data such as a certain crystalline cohomology of $\mathcal{X}$ together with natural structures. The first result along such lines is that of Fontaine and Messing [FM87] and Kato [Kato87], which treats the case of $e=1$ (namely the unramified base) and $i<p-1$.[1] Later on Breuil [Bre98] generalized the above to semistable $\mathcal{X}$, whereas Faltings [Fal99] studied the analog for p-divisible groups allowing arbitrary ramification index e. A few years later, Caruso [Car08] made progress allowing $e>1$ as long as $e \cdot (i+1) < p-1$.[2]

Of interest to us is Breuil's question [Bre02, Question 4.1].

Question 1.7. Assume $i<p-1$ and let S be the p-adic divided power envelope of $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$. Then the (torsion-free) crystalline cohomology $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}/S)/\mathrm{tors}$ together with its natural structure (e.g., divided Frobenius operator, filtration and connection) should be a 'strongly divisible lattice' and 'corresponds' to the Galois representation $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p)$.

All works mentioned above can be thought of as solving various special cases of the above question. In [LL20, Theorem 7.22], a connection with u-torsion in prismatic cohomology is observed. We showed that, fixing an $i<p-1$ and a smooth proper formal scheme $\mathcal{X}/\mathcal{O}_K$, the mod p^n analog of the above question has a positive answer in degree i if and only if both of ith and $(i+1)$th mod p^n prismatic cohomology of $\mathcal{X}/\mathfrak{S}$ are u-torsion-free. We then used Caruso's result on the mod p analog as a starting point to do an induction to show the vanishing as in Theorem 1.4(1) and Remark 1.5(1), which in turn implies the mod p^n analog of Breuil's question for all n and $e \cdot i < p-1$ (see [LL20, Corollary 7.25]). In particular, this gives an affirmative answer to Breuil's original question when $e \cdot i < p-1$. In this paper, the aforementioned vanishing of u-torsion is easily deduced, hence gives a 'shortcut' to the above result bypassing Caruso's work.

In private communications with Breuil, we were encouraged to study his question beyond the above bound. To our surprise, we discovered that the construction in [BMS18, § 2.1] can be

[1] See also [AMMN22] for an approach of a different flavor.
[2] For the mod p analog, Caruso's work even allows $e \cdot i < p-1$.

generalized to a counterexample with $e = p-1$ and $i = 1$ to Breuil's question (see Example 1.10). Note that in this example, we have $e \cdot i = p-1$, hence our previous result was actually sharp.

The other extreme of (e, i) with $e \cdot i = p-1$ is $e = 1, i = p-1$. In this case, Fontaine and Messing [FM87] and Kato [Kato87] showed that the crystalline cohomology $\mathrm{H}^{p-1}_{\mathrm{crys}}(\mathcal{X}_n/W_n)$ together with its natural structure is still a Fontaine–Laffaille module. According to [FL82] this Fontaine–Laffaille module is associated with a Galois representation $\rho^{p-1}_{n,\mathrm{FL}}$. It is only natural to ask the following question.

Question 1.8. What is the relation between $\rho^{p-1}_{n,\mathrm{FL}}$ and $\mathrm{H}^{p-1}_{\mathrm{\acute{e}t}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}/p^n)$?

Although we have not found any discussion on this question, there seems to be consensus among experts that these two Galois representations are different. We are not aware of any particular past expectation. Our entire § 5 is more or less devoted to this question, and we arrive at the following statement.

Theorem 1.9 (Theorem 5.28). *There exists a natural map* $\eta\colon \mathrm{H}^{p-1}_{\mathrm{\acute{e}t}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z})(p-1) \to \rho^{p-1}_{n,\mathrm{FL}}$ *of G_K-representations such that* $\ker(\eta)$ *is an unramified representation of G_K killed by p, and* $\operatorname{coker}(\eta)$ *sits in a natural exact sequence* $0 \to \ker(\eta) \to \operatorname{coker}(\eta) \to \ker(\mathrm{Sp}^{p-1}_n)$.

Here Sp^{p-1}_n denotes the specialization map of mod p^n étale cohomology in degree $p-1$, which is also known to be an unramified G_K-representation killed by p (see Corollary 4.15(3)). The appearance of $\ker(\eta)$ is due to the defect of a key functor in integral p-adic Hodge theory, which is well known to experts; whereas the potential u-torsion in degree p of mod p^n prismatic cohomology of $\mathcal{X}$ is solely responsible for the appearance of $\ker(\mathrm{Sp}^{p-1}_n)$.

Example and open questions

Now let us discuss an interesting example, generalized from [BMS18, § 2.1].

Example 1.10. Let $\mathcal{E}/W(\overline{k})$ be a lift of an ordinary elliptic curve over an algebraically closed field $\overline{k}$ of characteristic p. Fix an $n \in \mathbb{N}$ and let $\mathcal{O}_K := W(\overline{k})[\zeta_{p^n}]$. Over $\mathcal{O}_K$ we have a tautological map of group schemes $\chi\colon \mathbb{Z}/p^n \to \mu_{p^n}$ sending 1 to ζ_{p^n}.

With the above notation, we consider the smooth proper Deligne–Mumford stack $\mathcal{X} := [\mathcal{E}_{\mathcal{O}_K}/(\mathbb{Z}/p^n)]$ where the action of $\mathbb{Z}/p^n$ is via the character χ and the embedding $\mu_{p^n} \subset \mathcal{E}[p^n]$ (by the theory of canonical subgroup; see [Katz73, § 3.4]). Note that its special fiber is $\mathcal{E}_{\overline{k}} \times B(\mathbb{Z}/p^n)$ and its generic fiber is an elliptic curve $\mathcal{E}'_K := \left(\mathcal{E}_{\mathcal{O}_K}/\mu_{p^n}\right)_K$. In view of the pathologies discussed at the beginning of this introduction, let us record some facts concerning this example.

- The Albanese map has Néron model given by the 'further quotient' map: $\mathcal{X} \to \mathcal{E}' := \mathcal{E}_{\mathcal{O}_K}/\mu_{p^n}$, and the special fiber of this map factors as $\mathcal{E}_{\overline{k}} \times B(\mathbb{Z}/p^n) \to \mathcal{E}_{\overline{k}} \xrightarrow{f} \mathcal{E}_{\overline{k}}/\mu_{p^n}$. Note that $\ker(f) = \mu_{p^n}$.
- The fundamental group of $\mathcal{X}_{\overline{k}}$ is abelian, with torsion given by $\mathbb{Z}/p^n$ due to the factor of $B(\mathbb{Z}/p^n)$. By the universal coefficient theorem, we have $\mathrm{H}^2_{\mathrm{\acute{e}t}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}_p)_{\mathrm{tors}} \cong \mathbb{Z}/p^n$ whereas $\mathrm{H}^2_{\mathrm{\acute{e}t}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p)$ is torsion-free. Hence, we have $\ker(\mathrm{Sp}^2) = \mathbb{Z}/p^n$.
- One can go through the Leray spectral sequence for the cover $\mathcal{E} \to \mathcal{X}$ to compute the prismatic cohomology of $\mathcal{X}/\mathfrak{S}$. The most relevant computation is

$$\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty] \cong \mathfrak{S}/((u+1)^{p^{n-1}} - 1, p^n).$$

- Finally, we compute the crystalline cohomology of $\mathcal{X}/S$ and to our surprise we have $\mathrm{H}^1_{\mathrm{crys}}(\mathcal{X}/S) \cong S \oplus J$, where J is the ideal

$$\{x \in S \mid p^n \text{ divides } x \cdot ((u+1)^{p^n} - 1)\}.$$

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In particular, it is torsion-free of rank 2 yet not free. This gives a counterexample to Question 1.7.

By standard approximation techniques, one can cook up schematic examples having all the above features. When $n = 1$, we have $e = p - 1$, therefore our results above (stated and proved only for formal schemes) are sharp. For more details (see § 6).

Combining a generalized version of the above construction with our Theorem 1.1, we get a geometric proof of Raynaud's theorem [Ray74, Théorème 3.3.3] on prolongations of finite flat commutative group schemes over mixed characteristic DVR (see § 6.1).

Finally, let us end this introduction with two natural questions awaiting exploration. We consider them to be the next step in understanding pathological torsion in p-adic cohomology theory.

Question 1.11. Is there a smooth proper (formal) scheme $\mathcal{X}$ over an unramified base W which has u-torsion in its pth prismatic cohomology? Note that p is the smallest possible cohomological degree according to our result, and when $p = 2$ this is achieved by the above example.

Question 1.12 (see Question 3.10). Recall $\mathfrak{M}^i \coloneqq \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta})[u^\infty]$.

(1) Let β be the smallest exponent such that $p^\beta \in \mathrm{Ann}(\mathfrak{M}^i)$, and let γ be the exponent such that $\mathrm{Ann}(\mathfrak{M}^i) + (u) = (u, p^\gamma)$. Is there a bound on β and γ in terms of e and i?
(2) In light of the above example, we guess that β and/or γ are bounded above by $\log_p((e \cdot (i-1))/(p-1)) + 1$ when p is odd.

Remark 1.13. Confirming the above guess will give us results along the following lines. If $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W)$ has torsion *not* annihilated by p^N, then $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p)$ has torsion *not* annihilated by $p^{N-c(e,i)}$, with $c(e,i)$ being some constants depending only on e and i. Note that this would be a relation between torsion in étale and crystalline cohomology 'converse' to that established in [BMS18, Theorem 1.1(ii)]. When $e \cdot i < 2(p-1)$, our Theorem 1.4(2) can be translated into such a statement. Since our Theorem 1.4(2) does not seem to be optimal, we do not pursue that direction in this paper.

The links between each section are as follows.

$$\begin{tikzcd}
 & \boxed{\text{Section } 2} \arrow[d] \\
\boxed{\text{Section } 3} \arrow[r] \arrow[d] & \boxed{\text{Section } 5} \arrow[d, dashed, no head] \\
\boxed{\text{Section } 4} \arrow[r, dashed, no head] & \boxed{\text{Section } 6}
\end{tikzcd}$$

Notation and conventions

Let k be a perfect field of characteristic $p > 0$ with $W = W(k)$ its Witt ring. Let K be a totally ramified finite extension of $W(k)[1/p]$ of degree e, and let $\mathcal{O}_K$ be its ring of integers. Choose a uniformizer $\pi \in \mathcal{O}_K$ whose Eisenstein polynomial we denote by E with $E(0) = a_0 p$; we get a surjection $\mathfrak{S} \coloneqq W[\![u]\!] \twoheadrightarrow \mathcal{O}_K$ sending u to π. We equip $\mathfrak{S}$ with the δ-structure with $\varphi_{\mathfrak{S}}(u) = u^p$.

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The pair $(\mathfrak{S},(E))$ is the so-called Breuil–Kisin prism (see [BS22, Example 1.3.(3)]). Denote the p-adic divided power envelope of $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$ by S.

We always use C and its cousins like $C^\flat$ or A_{inf} to denote the usual construction associated with the completion of an algebraic closure $\overline{K}$ of K in p-adic Hodge theory. We use $G_K := \mathrm{Gal}(\overline{K}/K)$ denote the absolute Galois group. Similarly, $G_k := \mathrm{Gal}(\overline{k}/k)$.

We use $\mathcal{X}$ to denote a smooth proper p-adic formal scheme on $\mathrm{Spf}(\mathcal{O}_K)$, $\mathcal{X}_0$ to denote its reduction mod π and X to denote its rigid generic fiber.

On $(\mathcal{O}_K)_{\mathrm{qSyn}}$ we have the sheaf Δ given by (left Kan extended) prismatic cohomology relative to $\mathfrak{S}$. We use $\Delta^{(1)}$ to denote its $\varphi_{\mathfrak{S}}$ twist. This sheaf of Frobenius-twisted prismatic cohomology admits a decreasing filtration called the Nygaard filtration (see [BS22, § 15]), which will be denoted by $\mathrm{Fil}^{\bullet}_{\mathrm{N}}$. Let us note that $\mathrm{R\Gamma}_{\Delta}(\mathcal{X}/\mathfrak{S}) \cong \mathrm{R\Gamma}_{\mathrm{qSyn}}(\mathcal{X},\Delta)$ and $\varphi_{\mathfrak{S}}^{*}\mathrm{R\Gamma}_{\Delta}(\mathcal{X}/\mathfrak{S}) \cong \mathrm{R\Gamma}_{\mathrm{qSyn}}(\mathcal{X},\Delta^{(1)})$.

For any $n \in \mathbb{N} \cup \{\infty\}$, we use subscript $(-)_n$ to denote the derived mod p^n of a quasi-syntomic sheaf; for example, $\mathrm{R\Gamma}_{\mathrm{qSyn}}(\mathcal{X},\Delta_n^{(1)}) := \mathrm{R\Gamma}_{\mathrm{qSyn}}(\mathcal{X},\Delta^{(1)}/p^n)$.

In this paper we only consider relative prismatic cohomology, and hopefully readers will not confuse our notation with the absolute prismatic cohomology developed in [BL22].

2. Various modules and their Galois representations

In this section, we discuss three types of Frobenius modules – Kisin modules, Breuil modules and Fontaine–Laffaille modules – and their associated Galois representations. Roughly speaking, various cohomologies discussed in this paper will have these structures and functors to Galois representations just model comparison to étale cohomology. The major difference between the current work and [LL20] is that we now focus on the boundary case $eh = p-1$. So it is necessary to discuss *nilpotent* objects for Fontaine–Laffaille modules and Breuil modules when $e = 1$ and $h = p-1$.

2.1 Kisin modules

We review (generalized) Kisin modules from [LL20, § 6.1]. Let $(\mathfrak{S}, E(u))$ be the Breuil–Kisin prism over $\mathcal{O}_K$, with $d = E(u) = E$ the Eisenstein polynomial of a fixed uniformizer $\pi \in \mathcal{O}_K$. A φ-module $\mathfrak{M}$ over $\mathfrak{S}$ is an $\mathfrak{S}$-module $\mathfrak{M}$ together with a $\varphi_{\mathfrak{S}}$-semi-linear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$. Write $\varphi^*\mathfrak{M} = \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Note that $1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is an $\mathfrak{S}$-linear map. A *(generalized) Kisin module* $\mathfrak{M}$ *of height* h is a φ-module $\mathfrak{M}$ of finite $\mathfrak{S}$-type together with an $\mathfrak{S}$-linear map $\psi : \mathfrak{M} \to \varphi^*\mathfrak{M}$ such that $\psi \circ (1 \otimes \varphi_{\mathfrak{M}}) = E^h \,\mathrm{id}_{\varphi^*\mathfrak{M}}$ and $(1 \otimes \varphi_{\mathfrak{M}}) \circ \psi = E^h \,\mathrm{id}_{\mathfrak{M}}$. The map between generalized Kisin modules is a $\mathfrak{S}$-linear map that is compatible with φ and ψ. We denote by $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,h}$ the category of (generalized) Kisin modules of height h. As explained in [LL20], the main difference between generalized Kisin modules and classical theory Kisin modules is that the classical theory only discusses the situation where $\mathfrak{M}$ has no u-torsion, while Kisin modules from prismatic cohomology could have u-torsion in general. In the following, when we need to restrict to the classical theory, we will call $\mathfrak{M}$ either *classical* or *u-torsion-free*. Let $\mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,\mathrm{c}}$ denote the full subcategory of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,h}$ consists of classical Kisin modules of height h and killed by p^n for some $n \in \mathbb{N}$.

We now review some technologies to deal with classical Kisin modules on the boundary case and extend them to generalized Kisin modules. Following [Kis09, (1.2.10)] and [Gao17, § 2.1], we call a φ-module $\mathfrak{M}$ *multiplicative* (respectively, *nilpotent*) if $(1 \otimes \varphi) : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is surjective (respectively, if $\lim_{n\to\infty} \varphi^n(x) = 0,\ \forall x \in \mathfrak{M}$).

Remark 2.1. In [Kis09, (1.2.10)] and [Gao17, § 2.1], the authors define *multiplicative* to mean $(1 \otimes \varphi) : \varphi^* \mathfrak{M} \to \mathfrak{M}$ is *bijective*. For a classical Kisin module $\mathfrak{M}$ these two concepts are the same as $1 \otimes \varphi$ is always injective. But for generalized Kisin modules, as u-torsion exists, bijection of $1 \otimes \varphi$ is too restrictive. For example, $\mathfrak{S}/(p, u)\mathfrak{S}$ with the usual Frobenius is multiplicative but $1 \otimes \varphi$ is not injective.

Let $\mathfrak{M}$ be a φ-module over $\mathfrak{S}$ of finite $\mathfrak{S}$-type. Set $M := \mathfrak{M}/u\mathfrak{M}$ and write $q \colon \mathfrak{M} \to M = \mathfrak{M}/u\mathfrak{M}$. By the Fitting lemma, we have $M = M^{\mathrm{m}} \oplus M^{\mathrm{n}}$ where φ is bijective on M^{m} and nilpotent on M^{n}.

Lemma 2.2. *With notation as above, there exists a unique $W(k)$-linear section $[\cdot] : M^{\mathrm{m}} \to \mathfrak{M}$ such that $[\cdot]$ is φ-equivariant and $q \circ [\cdot] = \mathrm{id}|_{M^{\mathrm{m}}}$.*

Proof. Pick any $\overline{x} \in M^{\mathrm{m}}$. Since φ on M^{m} is bijective, there exists a unique $\overline{x}_n \in M$ such that $\varphi^n(\overline{x}_n) = \overline{x}$. Select $x_n \in \mathfrak{M}$ a lift of $\overline{x}_n$ and define $[\overline{x}] := \lim_{n \to \infty} \varphi^n(x_n)$. We first check that $\varphi^n(x_n)$ converges to an $x \in \mathfrak{M}$ such that $q(x) = \overline{x}$. Indeed, since $\varphi(\overline{x}_{n+1}) = \overline{x}_n$, $\varphi(x_{n+1}) - x_n = uy_n$ with $y_n \in \mathfrak{M}$. So $\varphi^{n+1}(x_{n+1}) - \varphi^n(x_n) = \varphi^n(u)\varphi^n(y)$ and hence $\varphi^n(x_n)$ converges to an $x \in \mathfrak{M}$ and clearly $q(x) = \overline{x}$. Suppose that $x'_n \in \mathfrak{M}$ is another lift of $\overline{x}_n$. Then $x'_n - x_n = uz_n$ with $z_n \in \mathfrak{M}_n$. Then $\varphi^n(x'_n) - \varphi^n(x_n) = u^{p^n}\varphi^n(z_n)$. So $\{\varphi^n(x'_n)\}$ also converges to x. This implies that $x = [\overline{x}]$ does not depend on the choice of lift x_n of $\overline{x}_n = \varphi^{-n}(\overline{x})$. Hence, the section $[\cdot] : M^{\mathrm{m}} \to \mathfrak{M}$ is well defined and satisfies $q \circ [\cdot] = \mathrm{id}_{M^{\mathrm{m}}}$. For any $a \in W(k)$, it is clear that $a[\overline{x}] = [a\overline{x}]$ from construction of $[\overline{x}]$. So $[\cdot]$ is $W(k)$-linear. If $\overline{y} = \varphi(\overline{x})$ then $\varphi(x_n)$ is a lift of $\varphi^{-n}(y)$ and $[\overline{y}] = \lim_{n \to \infty} \varphi^{n+1}(x_n) = \varphi(x) = \varphi([\overline{x}])$. So $[\cdot]$ is φ-equivariant. Finally, suppose there is another section $[\cdot]' : M^{\mathrm{m}} \to \mathfrak{M}$. Then $[\overline{x}] - [\overline{x}]' \in u\mathfrak{M}$ for any $\overline{x} \in M^{\mathrm{m}}$. Then $[\overline{x}] - [\overline{x}]' = \varphi^n([\overline{x}_n] - [\overline{x}_n]') \in u^{p^n}\mathfrak{M}$. This forces $[\overline{x}] = [\overline{x}]'$. □

Lemma 2.3. *Let $\mathfrak{M}$ be a φ-module with finite $\mathfrak{S}$-type. Then there exists an exact sequence of φ-modules*

$$0 \longrightarrow \mathfrak{M}^{\mathrm{m}} \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{M}^{\mathrm{n}} \longrightarrow 0 \tag{2.4}$$

such that $\mathfrak{M}^{\mathrm{m}}$ is multiplicative and $\mathfrak{M}^{\mathrm{n}}$ has no non-trivial multiplicative submodule. Furthermore, the above exact sequence is functorial for $\mathfrak{M}$, and if $\mathfrak{M}$ is in $\mathrm{Mod}^{\varphi,h,\mathrm{c}}_{\mathfrak{S},\mathrm{tor}}$ then so are $\mathfrak{M}^{\mathrm{m}}$ and $\mathfrak{M}^{\mathrm{n}}$.

Proof. Note that [Kis09, Prop. (1.2.11)] has treated the situation that $\mathfrak{M}$ has no u-torsion, but our idea here is slightly different. By the above lemma, we can set $\mathfrak{M}^{\mathrm{m}}$ to be the $\mathfrak{S}$-submodule of $\mathfrak{M}$ generated by $[M^{\mathrm{m}}]$ and $\mathfrak{M}^{\mathrm{n}} := \mathfrak{M}/\mathfrak{M}^{\mathrm{m}}$. Clearly, $1 \otimes \varphi : \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^{\mathrm{m}} \to \mathfrak{M}^{\mathrm{m}}$ is surjective. Consider the right exact sequence $\mathfrak{S} \otimes_{W(k)} [M^{\mathrm{m}}] \to \mathfrak{M} \to \mathfrak{M}^{\mathrm{n}} \to 0$. Modulo u, we have the right exact (indeed, exact) sequence $M^{\mathrm{m}} \to M \to \mathfrak{M}^{\mathrm{n}}/u\mathfrak{M}^{\mathrm{n}} \to 0$. So $\mathfrak{M}^{\mathrm{n}}/u\mathfrak{M}^{\mathrm{n}} \simeq M^{\mathrm{n}}$ and also forces $\mathfrak{M}^{\mathrm{m}}/u\mathfrak{M}^{\mathrm{m}} = M^{\mathrm{m}}$. Hence, φ on $\mathfrak{M}^{\mathrm{n}}$ is topologically nilpotent as φ on M^{n} is nilpotent, thus $\mathfrak{M}^{\mathrm{n}}$ cannot have non-trivial multiplicative submodule. So we obtain exact sequence (2.4) which is functorial for $\mathfrak{M}$ because $[\cdot]$ is clearly functorial for $\mathfrak{M}$ by the above lemma.

If $\mathfrak{M} \in \mathrm{Mod}^{\varphi,h,\mathrm{c}}_{\mathfrak{S},\mathrm{tor}}$ then $\mathfrak{M}^{\mathrm{m}}$ has no u-torsion. Note that the exact sequence (2.4) modulo u becomes the exact sequence $0 \to M^{\mathrm{m}} \to M \to M^{\mathrm{n}} \to 0$. Then $\mathfrak{M}^{\mathrm{n}}$ cannot have u-torsion as $\mathfrak{M}$ has no u-torsion. Hence, both $\mathfrak{M}^{\mathrm{m}}$ and $\mathfrak{M}^{\mathrm{n}}$ have no u-torsion. Then both $\mathfrak{M}^{\mathrm{m}}$ and $\mathfrak{M}^{\mathrm{n}}$ have E-height h by [Fon90, Prop. B 1.3.5] as required. □

But for a generalized Kisin module $\mathfrak{M}$ with height h, it is unclear whether we can define $\psi : \mathfrak{M}^{\mathrm{m}} \to \varphi^* \mathfrak{M}^{\mathrm{m}}$ such that $\mathfrak{M}^{\mathrm{m}}$ has height h. Luckily, we will not need such a statement.

Let $M[p^n]$ denote the p^n-torsion in M. For later application, we need the following two statements.

Lemma 2.5. *Let M be a finitely generated $\mathfrak{S}$-module. Assume M/p^nM are u-torsion-free for all $n>0$. Then $M/(M[p^n]+pM)$ are also u-torsion-free for all $n>0$.*

Proof. Suppose $x \in M$ is a lift of a u-torsion in $M/(M[p^n]+pM)$, hence satisfies $u \cdot x = y + p \cdot z$ for some $y \in M[p^n]$ and $z \in M$. Multiplying the equation by p^n, we get $u \cdot p^n \cdot x = p^{n+1} \cdot z$. As $M/p^{n+1}M$ also has no u-torsion by assumption, we see that $p^n \cdot x = p^{n+1} \cdot \widetilde{z}$ for some $\widetilde{z} \in M$. Writing $x = (x - p \cdot \widetilde{z}) + p \cdot \widetilde{z}$ shows that in fact $x \in M[p^n] + pM$, as required. □

Proposition 2.6. *Let M be a finitely generated generalized Breuil–Kisin module. Assume M/p^nM are u-torsion free for all $n>0$. Then there exists a $\mathbb{Z}_p$-module N and an isomorphism of $\mathfrak{S}$-modules $M \simeq N \otimes_{\mathbb{Z}_p} \mathfrak{S}$.*

Proof. First let us treat the case when M is torsion. In this case M is killed by a power of p (see [BMS18, Proposition 4.3(i)]). Denote $\mathrm{Im}(M \xrightarrow{p} M) = pM =: M_1$. We claim M_1/p^nM_1 are also u-torsion-free for all $n>0$. Granting this claim, by induction on the exponent of p annihilating M, we know M_1 satisfies the conclusion. Here, for the starting point of induction, we used the fact that a finitely generated $\mathfrak{S}/p$-module is u-torsion-free if and only if it is free. Then by [Min21, Lemma 5.9], we get the conclusion for M.

We now verify the claim. Applying the snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\cdot p^n"] & M \arrow[r] \arrow[d, "\cdot p^n"] & M/M_1 \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M \arrow[r] & M/M_1 \arrow[r] & 0 \end{tikzcd}$$

yields an exact sequence

$$0 \to M/(M[p^n]+pM) \to M_1/p^nM_1 \to M/p^nM.$$

Here $M[p^n]$ denotes the p^n-torsion in M. Since M/p^nM has no u-torsion by assumption, it suffices to show the same for $M/(M[p^n]+pM)$. Applying Lemma 2.5 gives the claim.

Next we turn to the general case. By [BMS18, Proposition 4.3], we have two short exact sequences of generalized Breuil–Kisin modules

$$0 \to M_{\mathrm{tor}} \to M \to M_{\mathrm{tf}} \to 0$$

and

$$0 \to M_{\mathrm{tf}} \to M_{\mathrm{fr}} \to M_0 \to 0.$$

Here M_{tor} is the torsion submodule, M_{tf} is the torsion-free quotient, M_{fr} is the reflexive hull of M (which is free as $\mathfrak{S}$ is a two-dimensional regular Noetherian domain), and M_0 has finite length. The first sequence implies that M_{tor}/p^n injects into M/p^n, therefore M_{tor} satisfies the assumption. Since we have treated the torsion case, we see that M_{tor} satisfies the conclusion. Now we claim M_0 vanishes. This immediately implies that $M_{\mathrm{tf}} = M_{\mathrm{fr}}$ is free, hence the first sequence splits, and $M = M_{\mathrm{tor}} \oplus M_{\mathrm{tf}}$ has the shape of a $\mathbb{Z}_p$-module.

Finally, let us justify the claim that $M_0 = 0$. Taking the second sequence above, derived mod p, gives an inclusion $M_0[p] \subset M_{\mathrm{tf}}/p$. Since M_0 has finite length, we see that $M_0[p]$ must be u^∞-torsion. If we can show that M_{tf}/p is u-torsion-free, then we get $M_0[p] = 0$ which implies $M_0 = 0$ as it must be p^∞-torsion. We are now reduced to showing $M/(M_{\mathrm{tor}} + p \cdot M)$ is

https://doi.org/10.1112/S0010437X23007261 Published online by Cambridge University Press

u-torsion-free. Since $M_{\mathrm{tor}} = M[p^n]$ for sufficiently large n, we finish the proof by appealing to Lemma 2.5. $\square$

2.2 Breuil modules

Fix $0 \le h \le p-1$. Let S be the p-adically completed PD envelope of $\theta : \mathfrak{S} \twoheadrightarrow \mathcal{O}_K, u \mapsto \pi$, and for $i \ge 1$ write $\mathrm{Fil}^i S \subseteq S$ for the (closure of the) ideal generated by $\{\gamma_n(E) = E^n/n!\}_{n \ge i}$. For $i \le p-1$, one has $\varphi(\mathrm{Fil}^i S) \subseteq p^i S$, so we may define $\varphi_i : \mathrm{Fil}^i S \to S$ where $\varphi_i := p^{-i}\varphi$. We have $c_1 := \varphi(E(u))/p \in S^\times$. Note that $S \subset K_0[\![u]\!]$. Define $I_+ := S \cap uK_0[\![u]\!]$. Clearly, $S/I_+ = W(k)$. Let $S_n := S/p^n S$. Let ${}^{\sim}\mathrm{Mod}_S^{\varphi,h}$ denote the category whose objects are triples $(\mathcal{M}, \mathrm{Fil}^h \mathcal{M}, \varphi_h)$, consisting of:

(1) two S-modules $\mathcal{M}$ and $\mathrm{Fil}^h \mathcal{M}$;
(2) an S-module map $\iota : \mathrm{Fil}^h \mathcal{M} \to \mathcal{M}$ whose image contains $\mathrm{Fil}^h S \cdot \mathcal{M}$; and
(3) a φ-semi-linear map $\varphi_h : \mathrm{Fil}^h \mathcal{M} \to \mathcal{M}$ such that for all $s \in \mathrm{Fil}^h S$ and $x \in \mathcal{M}$ we have

$$\varphi_h(sx) = (c_1)^{-h}\varphi_h(s)\varphi_h(E(u)^h x).$$

Morphisms are given by S-linear maps compatible with ιs and commuting with φ_h. Let ${}'\mathrm{Mod}_S^{\varphi,h}$ denote the full subcategory of ${}^{\sim}\mathrm{Mod}_S^{\varphi,h}$ whose objects $(\mathcal{M}, \mathrm{Fil}^h \mathcal{M}, \varphi_h)$ satisfy the following conditions.

(1) ι is injective such that $\mathrm{Fil}^h \mathcal{M}$ is regarded as a submodule of $\mathcal{M}$.
(2) $\varphi_h(\mathrm{Fil}^h \mathcal{M})$ generates $\mathcal{M}$ as S-modules.

A sequence is defined to be *short exact* if it is short exact as a sequence of S-module, and induces a short exact sequence on Fil^hs. Let $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ denote the full subcategory of ${}'\mathrm{Mod}_S^{\varphi,h}$ with the underlying module $\mathcal{M}$ killed by a p-power, and the triple $\mathcal{M}$ can be a written as successive extensions of triples $\mathcal{M}_i$ in ${}'\mathrm{Mod}_S^{\varphi,h}$ with each underlying module $\mathcal{M}_i \simeq \bigoplus_{\mathrm{finite}} S_1$.

Let $\nabla : S \to S$ be $W(k)$-linear continuous derivation such that $\nabla(u) = 1$. Let $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,\nabla}$ denote the category of the object $(\mathcal{M}, \mathrm{Fil}^h \mathcal{M}, \varphi_h, \nabla)$ where $(\mathcal{M}, \mathrm{Fil}^h \mathcal{M}, \varphi_h)$ is an object in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ and ∇ is $W(k)$-linear morphism $\nabla : \mathcal{M} \to \mathcal{M}$ such that the following assertions hold.

(1) For all $s \in S$ and $x \in \mathcal{M}$, $\nabla(sx) = \nabla(s)x + s\nabla(x)$.
(2) $E\nabla(\mathrm{Fil}^h \mathcal{M}) \subset \mathrm{Fil}^h \mathcal{M}$.
(3) The following diagram commutes:

$$\begin{array}{ccc} \mathrm{Fil}^h \mathcal{M} & \xrightarrow{\varphi_h} & \mathcal{M} \\ {\scriptstyle E(u)\nabla}\downarrow & & \downarrow{\scriptstyle c_1\nabla} \\ \mathrm{Fil}^h \mathcal{M} & \xrightarrow{u^{p-1}\varphi_h} & \mathcal{M} \end{array} \tag{2.7}$$

An object $\mathcal{M}$ in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ is called a (torsion) *Breuil module*.

Now let us recall the relation of classical torsion Kisin modules and objects in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. For each such $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,c}$, we construct an object $\mathcal{M} := \underline{\mathcal{M}}(\mathfrak{M}) \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ as follows: $\mathcal{M} := S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and

$$\mathrm{Fil}^h \mathcal{M} := \{x \in \mathcal{M} | (1 \otimes \varphi_{\mathfrak{M}})(x) \in \mathrm{Fil}^h S \otimes_{\mathfrak{S}} \mathfrak{M}\};$$

https://doi.org/10.1112/S0010437X23007261 Published online by Cambridge University Press

and $\varphi_h : \mathrm{Fil}^h \mathcal{M} \to \mathcal{M}$ is defined as the composite of the map

$$\mathrm{Fil}^h \mathcal{M} \xrightarrow{1\otimes\varphi_{\mathfrak{M}}} \mathrm{Fil}^h S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_h\otimes 1} S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = \mathcal{M}.$$

For any $\mathcal{M} \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$, define a semi-linear $\varphi : \mathcal{M} \to \mathcal{M}$ by $\varphi(x) = (c_1)^{-h}\varphi_h(E^h x)$. Similar to the situation of Kisin module, we say $\mathcal{M}$ is *multiplicative* (respectively, *nilpotent*) if $1 \otimes \varphi : S \otimes_{\varphi,S} \mathcal{M} \to \mathcal{M}$ is surjective (respectively, $\lim_{n\to\infty} \varphi^n(x) = 0$, $\forall x \in \mathcal{M}$). Clearly, if $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,\mathrm{c}}$ is multiplicative (respectively, nilpotent) then so is $\underline{\mathcal{M}}(\mathfrak{M})$.

Remark 2.8. Here our definition of multiplicative is different from that in [Gao17, Def. 2.2.2] where $\mathcal{M}$ is called multiplicative if $\mathrm{Fil}^h \mathcal{M} = \mathrm{Fil}^h S\mathcal{M}$. Indeed, these two definitions are equivalent. Suppose that $\mathrm{Fil}^h \mathcal{M} = \mathrm{Fil}^h S\mathcal{M}$. Since $\varphi_h(ax) = \varphi_h(a)\varphi(x)$ for any $a \in \mathrm{Fil}^h S$ and $x \in \mathcal{M}$, $\{\varphi(x) = c_1^{-h}\varphi_h(E^h x)\}$ and $\{\varphi_h(\mathrm{Fil}^h S\mathcal{M})\}$ generate the same subsets in $\mathcal{M}$. This implies that $\varphi(\mathcal{M})$ generates $\mathcal{M}$. Conversely, suppose that $\varphi(\mathcal{M})$ generates $\mathcal{M}$. To show that $\mathrm{Fil}^h \mathcal{M} = \mathrm{Fil}^h S\mathcal{M}$, we can reduce to the case that $\mathcal{M}$ is finite S_1-free by dévissage. See the last part of the proof of Lemma 2.9.

LEMMA 2.9. *For any object $\mathcal{M} \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$, there exists a functorial exact sequence*

$$0 \longrightarrow \mathcal{M}^{\mathrm{m}} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}^{\mathrm{n}} \longrightarrow 0 \tag{2.10}$$

with $\mathcal{M}^{\mathrm{m}}$ a multiplicative submodule of $\mathcal{M}$ and $\mathcal{M}^{\mathrm{n}}$ being nilpotent.

Proof. Recall $I_+ = S \cap uK_0[\![u]\!]$, $S/I_+ \simeq W(k)$ and $\varphi(x) = c_1^{-h}\varphi_h(E^h x)$. Write $S_n := S/p^n S$ and assume that $\mathcal{M}$ is an S_n-module. We claim Lemma 2.2 still holds by replacing $\mathfrak{M}$ by $\mathcal{M}$, $M = \mathcal{M}/I_+$ and $q : \mathcal{M} \to M = \mathcal{M}/I_+$. Indeed, the same proof goes through because $\varphi^\ell(I_+) = 0$ in S_n for sufficient large ℓ. Now we can set $\mathcal{M}^{\mathrm{m}}$ as the S-submodule of $\mathcal{M}$ generated by $[M^{\mathrm{m}}]$ and $\mathcal{M}^{\mathrm{n}} := \mathcal{M}/\mathcal{M}^{\mathrm{m}}$. Using the same argument as in Lemma 2.3, the right exact sequence $S \otimes_{W(k)} [M^{\mathrm{m}}] \to \mathcal{M} \to \mathcal{M}^{\mathrm{n}} \to 0$ modulo I_+ becomes an exact sequence $0 \to M^{\mathrm{m}} \to M \to M^{\mathrm{n}} \to 0$. This forces $\mathcal{M}^{\mathrm{m}}/I_+ = M^{\mathrm{m}}$ and $\mathcal{M}^{\mathrm{n}}/I_+ = M^{\mathrm{n}}$. Set $\mathrm{Fil}^h \mathcal{M}^m = \mathrm{Fil}^h S \cdot \mathcal{M}^{\mathrm{m}}$ and $\mathrm{Fil}^h \mathcal{M}^{\mathrm{n}} = \mathrm{Fil}^h \mathcal{M}/\mathrm{Fil}^h \mathcal{M}^{\mathrm{m}}$. It is clear that $\varphi_h : \mathrm{Fil}^h \mathcal{M}^{\mathrm{m}} \to \mathcal{M}^{\mathrm{m}}$ and $\varphi_h : \mathrm{Fil}^h \mathcal{M}^{\mathrm{n}} \to \mathcal{M}^{\mathrm{n}}$ are well defined. So we obtain an exact sequence $0 \to \mathcal{M}^{\mathrm{m}} \to \mathcal{M} \to \mathcal{M}^{\mathrm{n}} \to 0$ in the category $^{\sim}\mathrm{Mod}_S^{\varphi,h}$.

To promote our exact sequence to the category $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$, we argue by induction on n where p^n kills $\mathcal{M}$. The base case $n = 1$ is most complicated and postponed to the end. For general n, by definition, $\mathcal{M}$ sits in the exact sequence in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$: $0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0$ with $\mathcal{M}_1$, $\mathcal{M}_2$ killed by p^{n-1} and p, respectively. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}_1^{\mathrm{m}} \arrow[r] \arrow[d] & \mathcal{M}_1 \arrow[r] \arrow[d] & \mathcal{M}_1^{\mathrm{n}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M}^{\mathrm{m}} \arrow[r] \arrow[d, "f"] & \mathcal{M} \arrow[r] \arrow[d] & \mathcal{M}^{\mathrm{n}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M}_2^{\mathrm{m}} \arrow[r] & \mathcal{M}_2 \arrow[r] & \mathcal{M}_2^{\mathrm{n}} \arrow[r] & 0
\end{tikzcd} \tag{2.11}$$

We need to show that the first columns is short exact. Note that $\mathcal{M}_2$ is finite S_1-free, and the exact sequence in the second column yields the exact sequence $0 \to M_1 \to M \to M_2 \to 0$ where $M_i := \mathcal{M}_i/I_+\mathcal{M}_i$ for $i = 1, 2$. So the sequence $0 \to \mathcal{M}_1^{\mathrm{m}}/I_+ \to \mathcal{M}^{\mathrm{m}}/I_+ \to \mathcal{M}_2^{\mathrm{m}}/I_+ \to 0$

is also exact as it is the same as the exact sequence $0 \to M_1^{\mathrm{m}} \to M^{\mathrm{m}} \to M_2^{\mathrm{m}} \to 0$. Note that $\mathcal{M}_i^{\mathrm{m}}$ is finite S-generated as they are generated by $[M_i^{\mathrm{m}}]$. Note that S_n is a coherent ring (see [LL20, Lemma 7.15]). By induction on n and [Sta21, Tag 05CW], we see that $\mathcal{M}$ is coherent and then $\mathcal{M}^{\mathrm{m}}$ is coherent. Since $\mathcal{M}_1^{\mathrm{m}}$ is coherent by induction, $\mathcal{L} = \mathcal{M}^{\mathrm{m}}/\mathcal{M}_1^{\mathrm{m}}$ is also coherent by [Sta21, Tag 05CW] again. Note that f induces a map $f' : \mathcal{L} \to \mathcal{M}_2^{\mathrm{m}}$. We need to show that f' is an isomorphism. Let $L = \mathcal{L}/I_+$. Note that $\overline{f}' := f' \bmod I_+ : L \to M_2^{\mathrm{m}}$ is an isomorphism. Nakayama's lemma shows that f' is surjective. Let $\mathcal{K} := \ker(f')$, which is still coherent. Since $\mathcal{M}_2^{\mathrm{m}}$ is finite S_1-free by induction, $\mathrm{Tor}_1^S(\mathcal{M}_2^{\mathrm{m}}, S/I_+) = 0$. So we obtain an exact sequence $0 \to \mathcal{K}/I_+ \to L \to M_2^{\mathrm{m}} \to 0$. Hence $\mathcal{K}/I_+ = 0$ as $\overline{f}'$ is an isomorphism. By Nakayama's lemma, $\mathcal{K} = 0$ and the first column is exact as a finite S-module. Using that $\mathcal{M}_2^{\mathrm{m}}$ is finite S_1-free, we see that the sequence $0 \to \mathcal{M}_1^{\mathrm{m}}/\operatorname{Fil}^h S \to \mathcal{M}^{\mathrm{m}}/\operatorname{Fil}^h S \to \mathcal{M}_2^{\mathrm{m}}/\operatorname{Fil}^h S \to 0$ is exact. So the sequence $0 \to \operatorname{Fil}^h S \cdot \mathcal{M}_1^{\mathrm{m}} \to \operatorname{Fil}^h S \cdot \mathcal{M}^{\mathrm{m}} \to \operatorname{Fil}^h S \cdot \mathcal{M}_2^{\mathrm{m}} \to 0$ is exact. Therefore, the first column of (2.11) is exact in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. Then it is standard to check that the last column is also exact sequence in $'\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. In particular, $\mathcal{M}^{\mathrm{n}}$ is an object in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ by induction on n. Once (2.10) is exact in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. Then φ on $\mathcal{M}^{\mathrm{m}}$, $\mathcal{M}$ and $\mathcal{M}^{\mathrm{n}}$ defined from $\varphi(x) = c_1^{-h}\varphi_h(x)$ are compatible with maps in the sequence. Since $\mathcal{M}^{\mathrm{m}}$ is generated by $[M^{\mathrm{m}}]$ and $\mathcal{M}^{\mathrm{n}}/I_+ = M^{\mathrm{n}}$, we see that $\mathcal{M}^{\mathrm{m}}$ is multiplicative and $\mathcal{M}^{\mathrm{n}}$ is nilpotent.

Now we discuss the case $n = 1$. We have already shown that $\mathcal{M}^{\mathrm{m}}$ is finite S-generated. Now the exact sequence $0 \to \mathcal{M}^{\mathrm{m}}/I_+ \to \mathcal{M}/I_+ \to \mathcal{M}^{\mathrm{n}}/I_+ \to 0$ is an exact sequence of k-vector spaces. Pick $m_i \in \mathcal{M}^{\mathrm{m}}$ and $n_j \in \mathcal{M}$ such that $m_i \bmod I_+$ and $n_j \bmod I_+$ are bases of $\mathcal{M}^{\mathrm{m}}/I_+$ and $\mathcal{M}^{\mathrm{n}}/I_+$ respectively. Using that $\mathcal{M}$ is finite S_1-free, it is easy to show that m_i, n_j forms a basis of $\mathcal{M}$ and then both $\mathcal{M}^{\mathrm{m}}$ and $\mathcal{M}^{\mathrm{n}}$ are finite S_1-free. Now it remains to show that $\operatorname{Fil}^h \mathcal{M} \cap \mathcal{M}^{\mathrm{m}} = \operatorname{Fil}^h S\mathcal{M}^{\mathrm{m}}$ such that $\operatorname{Fil}^h \mathcal{M}^{\mathrm{n}} = \operatorname{Fil}^h \mathcal{M}/\operatorname{Fil}^h \mathcal{M}^{\mathrm{m}}$ is a submodule of $\mathcal{M}^{\mathrm{n}}$. Then it is easy to check that $(\mathcal{M}^{\mathrm{n}}, \operatorname{Fil}^h \mathcal{M}^{\mathrm{n}}, \varphi_h)$ is an object in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ and thus the sequence $0 \to \mathcal{M}^{\mathrm{m}} \to \mathcal{M} \to \mathcal{M}^{\mathrm{n}} \to 0$ is in the category $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. To show that $\operatorname{Fil}^h \mathcal{M}^{\mathrm{n}} = \operatorname{Fil}^h \mathcal{M}/\operatorname{Fil}^h \mathcal{M}^{\mathrm{m}}$, consider $\mathcal{F} := \mathcal{M}^{\mathrm{m}}/\operatorname{Fil}^p S_1\mathcal{M}^{\mathrm{m}}$. Write $\widetilde{\operatorname{Fil}}^h \mathcal{F} := (\operatorname{Fil}^h \mathcal{M} \cap \mathcal{M}^{\mathrm{m}})/\operatorname{Fil}^p S_1$ and $\operatorname{Fil}^h \mathcal{F} = \operatorname{Fil}^h S\mathcal{M}^{\mathrm{m}}/\operatorname{Fil}^p S_1$. Since $\operatorname{Fil}^h \mathcal{F} = u^{eh}\mathcal{F} \subset \widetilde{\operatorname{Fil}}^h \mathcal{F} \subset \mathcal{F}$ which is a finite free $k[\![u]\!]/u^{pe}$-module, there exists a basis $e_1, \dots, e_d$ of $\mathcal{F}$ such that $\widetilde{\operatorname{Fil}}^h \mathcal{F}$ is generated by $u^{a_i}e_i$ with $0 \le a_i \le eh$. Suppose one of the a_i is less than eh, say $a_1 < eh$. Let $\hat{e}_i \in \mathcal{M}^{\mathrm{m}}$ be a basis which lifts e_i. Then $u^{a_1}\hat{e}_1 \in \operatorname{Fil}^h \mathcal{M} \cap \mathcal{M}^{\mathrm{m}}$. So $\varphi_h(u^{eh}\hat{e}_1) = \varphi_h(u^{eh-a_1}u_i^a\hat{e}_1) = \varphi(u^{eh-a_1})\varphi_h(u^{a_1}\hat{e}_1) \in I_+\mathcal{M}$. This contradicts that $\varphi_h(u^{eh}\hat{e}_i) \bmod I_+$ is a basis $M^m \subset M = \mathcal{M}/I_+$. So all the a_i equal eh and we have $\operatorname{Fil}^h \mathcal{M}^{\mathrm{m}} = \operatorname{Fil}^h S \cdot \mathcal{M}^{\mathrm{m}} = \operatorname{Fil}^h \mathcal{M} \cap \mathcal{M}^{\mathrm{m}}$ as required. □

Corollary 2.12. *The exact sequence* (2.10) *is canonical in the following sense. Suppose $\mathcal{M}$ admits another exact sequence in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$,*

$$0 \to \widetilde{\mathcal{M}}^{\mathrm{m}} \to \mathcal{M} \to \widetilde{\mathcal{M}}^{\mathrm{n}} \to 0,$$

with $\widetilde{\mathcal{M}}^{\mathrm{m}}$ being multiplicative and $\widetilde{\mathcal{M}}^{\mathrm{n}}$ being nilpotent. Then $\widetilde{\mathcal{M}}^{\mathrm{m}} = \mathcal{M}^{\mathrm{m}}$ and $\widetilde{\mathcal{M}}^{\mathrm{n}} = \mathcal{M}^{\mathrm{n}}$.

Proof. Since $\widetilde{\mathcal{M}}^{\mathrm{n}}$ is successive extension of finite free S_1-modules, $\mathrm{Tor}_S^1(\mathcal{M}^{\mathrm{n}}, S/I_+) = 0$. Hence, the sequence $0 \to \widetilde{\mathcal{M}}^{\mathrm{m}}/I_+ \to \mathcal{M}/I_+ \to \widetilde{\mathcal{M}}^{\mathrm{n}}/I_+ \to 0$ is exact. Since $\widetilde{\mathcal{M}}^{\mathrm{m}}$ is multiplicative, $\widetilde{\mathcal{M}}^{\mathrm{m}}/I_+ \subset M^{\mathrm{m}}$ and thus $\widetilde{\mathcal{M}}^{\mathrm{m}}/I_+ = M^{\mathrm{m}}$, otherwise φ on $\widetilde{\mathcal{M}}^{\mathrm{n}}/I_+$ cannot be nilpotent. So $[M^{\mathrm{m}}] \subset \widetilde{\mathcal{M}}^{\mathrm{m}}$. Hence, $\mathcal{M}^{\mathrm{m}} \subset \widetilde{\mathcal{M}}^{\mathrm{m}}$ as $\mathcal{M}^{\mathrm{m}}$ is constructed as S-submodule of $\mathcal{M}$ generated by $[M^{\mathrm{m}}]$. Since $\mathcal{M}^{\mathrm{m}}/I_+ = M^{\mathrm{m}}$, we have $\widetilde{\mathcal{M}}^{\mathrm{m}} = \mathcal{M}^{\mathrm{m}}$ by Nakayama's lemma. By the definition of

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exact sequence in the category $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$, we see that

$$\mathrm{Fil}^h \widetilde{\mathcal{M}}^{\mathrm{m}} = \widetilde{\mathcal{M}}^{\mathrm{m}} \cap \mathrm{Fil}^h \mathcal{M} = \mathcal{M}^{\mathrm{m}} \cap \mathrm{Fil}^h \mathcal{M} = \mathrm{Fil}^h \mathcal{M}^{\mathrm{m}},$$

where the last equality was proved by the end of the proof of Lemma 2.9. Therefore, we have the desired equality $(\widetilde{\mathcal{M}}^{\mathrm{m}}, \mathrm{Fil}^h \widetilde{\mathcal{M}}^{\mathrm{m}}, \varphi_h) = (\mathcal{M}^{\mathrm{m}}, \mathrm{Fil}^h \mathcal{M}^{\mathrm{m}}, \varphi_h)$ as a subobject of $\mathcal{M}$. □

2.3 Fontaine–Laffaille modules

Fix $h = p-1$ for this subsection. Let us review Fontaine–Laffaille theory from [FL82]. Let $\mathrm{FM}_{W(k)}$ denote the category whose objects are finite $W(k)$-modules M together with decreasing filtration $\{\mathrm{Fil}^i M\}_{i\geq 0}$ and a Frobenius semi-linear map $\varphi_i : \mathrm{Fil}^i M \to M$ satisfying:

(1) $\mathrm{Fil}^{i+1} M$ is a direct summand of $\mathrm{Fil}^i M$ for all $i \in \mathbb{N}$, and $\mathrm{Fil}^0 M = M$, $\mathrm{Fil}^{h+1} M = \{0\}$;[3]
(2) $\varphi_i|_{\mathrm{Fil}^{i+1} M} = p \cdot \varphi_{i+1}$;
(3) $\sum_{i\geq 0} \varphi_i(\mathrm{Fil}^i M) = M$.

Morphisms in $\mathrm{FM}_{W(k)}$ are $W(k)$-linear homomorphisms compatible with filtration and φ_i. It turns out that the category $\mathrm{FM}_{W(k)}$ is abelian (see [FL82, Proposition 1.8]); and any morphism is automatically strict with respect to the filtrations (see [FL82, 1.10(b)]). A sequence $0 \to M_1 \to M \to M_2 \to 0$ in $\mathrm{FM}_{W(k)}$ is *short exact* if the underlying $W(k)$-module is exact.[4] In this case, we call M_2 a quotient of M. An object $M \in \mathrm{FM}_{W(k)}$ is called *multiplicative* if $\mathrm{Fil}^1 M = \{0\}$ and M is called *nilpotent* if it does not have a multiplicative subobject. Just as in previous sections, we have the following lemma.

Lemma 2.13. *Let $(M, \mathrm{Fil}^\bullet M, \varphi_\bullet) \in \mathrm{FM}_{W(k)}$.*

(1) *It is multiplicative (respectively, nilpotent) if and only if φ_0 is bijective (respectively, nilpotent).*
(2) *There is a canonical multiplicative-nilpotent exact sequence in $\mathrm{FM}_{W(k),\mathrm{tor}}$,*

$$0 \longrightarrow M^{\mathrm{m}} \longrightarrow M \longrightarrow M^{\mathrm{n}} \longrightarrow 0, \tag{2.14}$$

such that M^{m} is the maximal multiplicative subobject in M and M^{n} is nilpotent.

Proof. (1) Condition (3) of being an object in $\mathrm{FM}_{W(k)}$ in the case of a multiplicative object translates to φ_0 being surjective, which is equivalent to being bijective due to length considerations. Conversely, if φ_0 is bijective, we let $M' \in \mathrm{FM}_{W(k)}$ be defined as follows: the underlying module is M itself, with $\mathrm{Fil}^0 M' = M \supset \mathrm{Fil}^1 M' = 0$ and φ_0. Then there is an evident morphism $M' \to M$ in $\mathrm{FM}_{W(k)}$, which is necessarily strict with respect to filtrations (see [FL82, 1.10(b)]), hence $\mathrm{Fil}^1 M = \mathrm{Fil}^1 M' = 0$. The proof for nilpotent objects is at end of the proof of (2).

(2) By the Fitting lemma, we have $M = M^{\mathrm{m}} \oplus M^{\mathrm{n}}$ only as φ-modules, such that φ_0 on M^{m} is bijective and φ_0 on M^{n} is nilpotent. Let $\mathrm{Fil}^1 M^{\mathrm{m}} = 0$; we get the desired sequence. The fact that the *quotient* M^{n} with the induced filtration is nilpotent follows from (1). By the exact sequence (2.14), M is nilpotent if and only if $M = M^{\mathrm{n}}$, whose φ_0 is nilpotent. □

For any object M in $\mathrm{FM}_{W(k)}$, we can attach a Breuil module $\underline{\mathcal{M}}_{\mathrm{FM}}(M) \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,\nabla}$ in the following ways. Let $\mathcal{M} = \underline{\mathcal{M}}_{\mathrm{FM}}(M) := S \otimes_{W(k)} M$, $\nabla_{\mathcal{M}} = \nabla_S \otimes \mathrm{id}_M$ and $\mathrm{Fil}^h \mathcal{M} := \sum_{i=0}^h \mathrm{Fil}^i S \otimes_{W(k)} \mathrm{Fil}^{h-i} M$. By definition $\mathrm{Fil}^h \mathcal{M}$ is a submodule of $\mathcal{M}$. We define

[3] It turns out that this condition follows from the next two conditions (see [Win84, Proposition 1.4.1 (ii)]).
[4] Note that by the above result of Fontaine–Laffaille, the sequence of filtrations is forced to be exact as well.

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$\varphi_{h,\mathcal{M}} : \mathrm{Fil}^h \mathcal{M} \to \mathcal{M}$ by $\varphi_{h,\mathcal{M}} := \sum_{i=0}^h \left(\varphi_i \mid_{\mathrm{Fil}^i S}\right) \otimes \left(\varphi_{h-i} \mid_{\mathrm{Fil}^{h-i} M}\right)$; this is well defined because $\mathrm{Fil}^{i+1} M$ is a direct summand of $\mathrm{Fil}^i M$. It is standard to check that $\underline{\mathcal{M}}_{\mathrm{FM}}(M)$ is a Breuil module in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,\nabla}$.

PROPOSITION 2.15.

(1) *Let $M \in \mathrm{FM}_{W(k),\mathrm{tor}}$. Then $\underline{\mathcal{M}}_{\mathrm{FM}}((2.14))$ is isomorphic to (2.10) with $\mathcal{M} = \underline{\mathcal{M}}(M)$. In particular, $\underline{\mathcal{M}}(M^{\mathrm{m}}) = \underline{\mathcal{M}}(M)^{\mathrm{m}}$.*
(2) *Given an $M \in \mathrm{FM}_{W(k),\mathrm{tor}}$ and suppose that there exists a classical Kisin module $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,\mathrm{c}}$ such that $\underline{\mathcal{M}}(\mathfrak{M}) \simeq \underline{\mathcal{M}}_{\mathrm{FM}}(M)$ in the category of $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. Then we have isomorphism $\underline{\mathcal{M}}_{\mathrm{FM}}((2.14)) \simeq \underline{\mathcal{M}}((2.4))$. In particular, $\underline{\mathcal{M}}(\mathfrak{M}^{\mathrm{n}}) = \underline{\mathcal{M}}(\mathfrak{M})^{\mathrm{n}} = \underline{\mathcal{M}}_{\mathrm{FM}}(M^{\mathrm{n}})$.*

Proof. It is easy to check that if $M \in \mathrm{FM}_{W(k),\mathrm{tor}}$ (respectively, $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,\mathrm{c}}$) is multiplicative or nilpotent then so is $\underline{\mathcal{M}}_{\mathrm{FM}}(M)$ (respectively, $\underline{\mathcal{M}}(\mathfrak{M})$). Then the proposition follows from Corollary 2.12. $\square$

For later use, let us prove the following technical lemma which says that one can test an object in $\mathrm{FM}_{W(k)}$ after looking at its 'Breuil's counterpart'. This is well known to experts.

LEMMA 2.16. *Let $(M, \mathrm{Fil}^\bullet M, \varphi_\bullet)$ be a filtered module with divided Frobenius; that is, only assuming the condition (2) in the definition of $\mathrm{FM}_{W(k)}$ is satisfied. Let $\mathcal{M} = \underline{\mathcal{M}}_{\mathrm{FM}}(M) := S \otimes_{W(k)} M$ and $\mathrm{Fil}^h \mathcal{M} := \sum_{i=0}^h \mathrm{Fil}^i S \otimes_{W(k)} \mathrm{Fil}^{h-i} M$. Suppose there is a semi-linear map $\varphi_h\colon \mathrm{Fil}^h \mathcal{M} \to \mathcal{M}$ satisfying*

$$\varphi_h = \sum_{i=0}^h \left(\varphi_i \mid_{\mathrm{Fil}^i S}\right) \otimes \left(\varphi_{h-i} \mid_{\mathrm{Fil}^{h-i} M}\right).$$

Then $(M, \mathrm{Fil}^\bullet M, \varphi_\bullet)$ is an object in $\mathrm{FM}_{W(k)}$ if and only if $\varphi_h(\mathrm{Fil}^h \mathcal{M})$ generates $\mathcal{M}$ as an S-module.

Proof. The 'only if' part follows from the standard direction of going from Fontaine–Laffaille modules to Breuil modules as discussed above. We prove the 'if' part, which is the only part that will be used later. To that end, we simply observe that $\mathcal{M}/(p, I_+) \cdot \mathcal{M} \cong M/p$. One checks that the induced map $\overline{\varphi_h}\colon \mathrm{Fil}^h \mathcal{M} \to M/p$ has image given by the image of $\sum_{i=0}^h \overline{\varphi_i}\colon \bigoplus_{i=0}^h \mathrm{Fil}^i M \to M/p$. Our condition now implies the reduction map is surjective. Since M is p-adically complete, it follows that the map $\sum_{i=0}^h \varphi_i\colon \bigoplus_{i=0}^h \mathrm{Fil}^i M \to M$ before mod p is also surjective, which is exactly what we need to show. $\square$

2.4 Relations to Galois representations

Fix $\pi_n \in \overline{K}$ such that $\underline{\pi} := (\pi_n) \in \mathcal{O}_{\mathbf{C}}^\flat$ and $\pi_0 = \pi$; also $K_\infty := \bigcup_{n \geq 0} K(\pi_n)$ and $G_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$. We embed $\mathfrak{S} \to A_{\mathrm{inf}}$ via $u \mapsto [\underline{\pi}]$. As discussed in [LL20, § 6.2], for a classical Kisin module $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,h}$, we can associate the Galois representation of G_∞ via $T_{\mathfrak{S}}(\mathfrak{M}) = (\mathfrak{M} \otimes_{\mathfrak{S}} W(\mathcal{O}_{\mathbf{C}}^\flat))^{\varphi=1}$ and $T_{\mathfrak{S}}^h(\mathfrak{M}) = (\mathrm{Fil}^h \varphi^*\mathfrak{M} \otimes A_{\mathrm{inf}})^{\varphi_h=1}$ where $\mathrm{Fil}^h \varphi^*\mathfrak{M} := \{x \in \varphi^*\mathfrak{M} \mid (1 \otimes \varphi)(x) \in E^h\mathfrak{M}\}$ and $\varphi_h : \mathrm{Fil}^h \varphi^*\mathfrak{M} \to \varphi^*\mathfrak{M}$ is given by $\varphi_h(x) = (1 \otimes \varphi)(x)/\varphi(a_0^{-1}E)^h$. See [LL20, § 6.2] for more details on $T_{\mathfrak{S}}^h$ and $T_{\mathfrak{S}}$; for example, $T_{\mathfrak{S}}^h(\mathfrak{M}) = T_{\mathfrak{S}}(\mathfrak{M})(h)$ and both $T_{\mathfrak{S}}$ and $T_{\mathfrak{S}}^h$ are exact.

Note that if $\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ has an A_{inf}-semi-linear G_K-action which extends the natural G_∞-action and commutes with φ, then $T_{\mathfrak{S}}(\mathfrak{M})$ is a G_K-representation. In particular, this is the case when $\mathfrak{M} = \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S}_n)$ modulo u^∞-torsion.

Now given a Breuil module $\mathcal{M} \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,\nabla}$, as explained around [LL20, Eq. (6.19)], we define $\mathrm{Fil}^h(\mathcal{M} \otimes_S A_{\mathrm{crys}}) := \mathrm{Fil}^h \mathcal{M} \otimes_S A_{\mathrm{crys}}$ and then φ_h extends to $\mathcal{M} \otimes_S A_{\mathrm{crys}}$ and we define a

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G_K-action on $\mathcal{M}\otimes_S A_{\mathrm{crys}}$: for any $\sigma\in G_K$, any $x\otimes a\in A_{\mathrm{crys}}\otimes_S\mathcal{M}$, define

$$\sigma(x\otimes a)=\sum_{i=0}^{\infty}\nabla^i(x)\otimes\gamma_i\big(\sigma([\underline{\pi}])-[\underline{\pi}]\big)\sigma(a). \tag{2.17}$$

The above G_K-action on $\mathcal{M}\otimes_S A_{\mathrm{crys}}$ extends the G_∞-action, preserves filtration and commutes with φ_h. As in [LL20, § 6.3], we define

$$T_S(\mathcal{M}):=(\mathrm{Fil}^h(\mathcal{M}\otimes_S A_{\mathrm{crys}}))^{\varphi_h=1},$$

which is a $\mathbb{Z}_p[G_K]$-module.

Now suppose $\mathfrak{M}\in\mathrm{Mod}^{\varphi,h,\mathrm{c}}_{\mathfrak{S},\mathrm{tor}}$ and let $\mathfrak{c}:=\prod_{n=1}^{\infty}\varphi^n(E/E(0))\in S^\times$. As explained in the proof of [LL20, Prop. 6.12], the map $m\mapsto\mathfrak{c}^h(1\otimes m)$ induces a natural map $\iota:T^h_{\mathfrak{S}}(\mathfrak{M})\to T_S(\underline{\mathcal{M}}(\mathfrak{M}))$.

Suppose that $\mathfrak{M}\otimes_{\mathfrak{S}}A_{\mathrm{inf}}$ has G_K-action which extends the G_∞-action and commutes with φ, and the natural map $\mathfrak{M}\otimes_{\mathfrak{S}}A_{\mathrm{inf}}\to\underline{\mathcal{M}}(\mathfrak{M})\otimes_S A_{\mathrm{crys}}$ is compatible with G_K-actions on both sides. Then, as explained in [LL20, Remark 6.14], the natural map $T_{\mathfrak{S}}(\mathfrak{M})(h)\simeq T^h_{\mathfrak{S}}(\mathfrak{M})\xrightarrow{\iota}T_S(\underline{\mathcal{M}}(\mathfrak{M}))$ is compatible with G_K-actions on both sides. In particular, this will happen (see the proof of Theorem 5.28) when $\mathfrak{M}=\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X},\mathbb{\Delta}_n)$ is an object in $\mathrm{Mod}^{\varphi,h,\mathrm{c}}_{\mathfrak{S},\mathrm{tor}}$ and $\underline{\mathcal{M}}(\mathfrak{M})$ is subobject of $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}/S_n)$ inside $\mathrm{Mod}^{\varphi,h,\nabla}_{S,\mathrm{tor}}$.

LEMMA 2.18. *If $\mathfrak{M}\in\mathrm{Mod}^{\varphi,h,\mathrm{c}}_{\mathfrak{S},\mathrm{tor}}$ is nilpotent then the natural map $\iota:T^h_{\mathfrak{S}}(\mathfrak{M})\to T_S(\underline{\mathcal{M}}(\mathfrak{M}))$ is an isomorphism.*

Proof. Write $\mathcal{M}:=\underline{\mathcal{M}}(\mathfrak{M})$. Then $\mathcal{M}$ is also nilpotent by Proposition 2.15. When $h\le p-2$, ι is known to be an isomorphism (without assuming nilpotency of $\mathfrak{M}$) by [LL20, Prop. 6.12]. So in the following, we assume $h=p-1$.

Since $T^h_{\mathfrak{S}}$ and $\underline{\mathcal{M}}$ are exact and T_S is left exact, we can assume that $\mathfrak{M}$ is killed by p such that $\mathfrak{M}$ is a finite free $k[\![u]\!]$-module with basis $e_1,\dots,e_d$. Write $\varphi(e_1,\dots,e_d)=(e_1,\dots,e_d)A$ with $AB=BA=a_0^{-h}u^{eh}I_d$. Let $\widetilde{e}_i:=1\otimes e_i$ be a basis of $\varphi^*\mathfrak{M}$ and an S_1-basis of $\mathcal{M}$. Then $\mathrm{Fil}^h\varphi^*\mathfrak{M}$ is generated by $(\alpha_1,\dots,\alpha_d)=(\widetilde{e}_1,\dots,\widetilde{e}_d)B$ and $\mathrm{Fil}^h\mathcal{M}$ is generated by $(\alpha_1,\dots,\alpha_d)$ and $\mathrm{Fil}^p S_1\mathcal{M}$. Note that $\iota(\widetilde{e}_1,\dots,\widetilde{e}_d)=\mathfrak{c}^h(\widetilde{e}_1,\dots,\widetilde{e}_d)$, and any $x\in(\mathrm{Fil}^h\varphi^*\mathfrak{M}\otimes_{\mathfrak{S}}A_{\mathrm{inf}})$ can be written as $x=(\alpha_1,\dots,\alpha_d)X$ with $X\in(\mathcal{O}^\flat_{\mathbf{C}})^d$ and any $y\in\mathrm{Fil}^h\mathcal{M}\otimes_S A_{\mathrm{crys}}$ can be written as $y=\mathfrak{c}^h(\alpha_1,\dots,\alpha_d)Y+\mathfrak{c}^h(\widetilde{e}_1,\dots,\widetilde{e}_d)Z$ with $Y\in(\mathcal{O}^\flat_{\mathbf{C}}/u^{ep})^d$ and $Z\in(\mathrm{Fil}^p A_{\mathrm{crys},1})^d$. Then ι is the same as

$$\{X\in(\mathcal{O}^\flat_{\mathbf{C}})^d\mid\varphi(X)=BX\}\longrightarrow\{(Y,Z)\mid Y\in(\mathcal{O}^\flat_{\mathbf{C}}/u^{ep})^d,Z\in(\mathrm{Fil}^p A_{\mathrm{crys},1})^d,$$
$$\varphi(Y)+\varphi(A)\varphi_h(Z)=BY+Z\}$$

by sending $X\mapsto(X,0)$. We must show the above map is bijective. For injectivity, note that $X\in\ker(\iota)$ if and only if $BX\in(u^{pe}\mathcal{O}^\flat_{\mathbf{C}})^d$. Then $a_0^{-h}u^{eh}X=ABX\in(u^{pe}\mathcal{O}^\flat_{\mathbf{C}})^d$. Hence, $Y=a_0u^{-e}X\in(\mathcal{O}^\flat_C)^d$. Note that $\varphi(X)=BX$ implies that $A\varphi(X)=a_0^{-h}u^{eh}X$ and then $Y=A\varphi(Y)$. So $Y=A\varphi(A)\cdots\varphi^m(A)\varphi^{m+1}(Y)$. Since $\mathfrak{M}$ is nilpotent, $A\varphi(A)\cdots\varphi^m(A)\to 0$ for $m\to\infty$, we see that $Y=0$. This proves the injectivity of ι.

To prove the surjectivity of ι, consider the equation $\varphi(Y)+\varphi(A)\varphi_h(Z)-BY=Z$. Note that $A_{\mathrm{crys},1}=(\mathcal{O}^\flat_{\mathbf{C}}/u^{pe})[\{z_i\}_{i\ge1}]/\{z_i^p,i\ge1\}$ with z_i the image of $\gamma_{p^i}(E)$ in $A_{\mathrm{crys},1}$. Since $\varphi_h(z_i)=a_0^{p^i}$ or 0, the left-hand side of the equation is in $(\mathcal{O}^\flat_{\mathbf{C}}/u^{pe})^d$; this forces the right-hand side $Z=0$ and we only have $\varphi(Y)=BY$ inside $(\mathcal{O}^\flat_{\mathbf{C}}/u^{pe})^d$. So it suffices to show there exists $\widetilde{Y}\in(\mathcal{O}^\flat_C)^d$ such that $\varphi(\widetilde{Y})=B\widetilde{Y}$ and $B\widetilde{Y}=BY$ inside $(\mathcal{O}^\flat_{\mathbf{C}}/u^{pe})^d$. To prove the existence of $\widetilde{Y}$, pick any lift $Y_0\in(\mathcal{O}^\flat_{\mathbf{C}})^d$ of Y. Then $\varphi(Y_0)=BY_0+u^{pe}W_0$. Since $u^{pe}I_d=BA(a_0u^e)^hI_d$, we have $\varphi(Y_0)=BY_1$ with $Y_1=Y_0+u^eAa_0^hW_0$. Then $\varphi(Y_1)=BY_1+u^{pe}\varphi(A)W_1$ for some $W_1\in(\mathcal{O}^\flat_{\mathbf{C}})^d$. Continuing to construct Y_n in this way, we have $\varphi(Y_n) = BY_n + u^{pe}A\varphi(A)\cdots\varphi^n(A)W_n$ for some $W_n \in (\mathcal{O}_{\mathbf{C}}^\flat)^d$ and then $Y_{n+1} = Y_n + u^e A\varphi(A)\cdots\varphi^n(A)W_n$. Then Y_n converges to $\widetilde{Y}$ as $A\varphi(A)\cdots\varphi^n(A) \to 0$. Since $\widetilde{Y} = Y_0 + u^e A\widetilde{W}$ for some $\widetilde{W} \in (\mathcal{O}_C^\flat)^d$, we see that $B\widetilde{Y} = BY_0 = BY$ inside $(\mathcal{O}_{\mathbf{C}}^\flat/u^{pe})^d$. $\square$

For $h = p-1$ the following example shows that the above lemma will fail without $\mathfrak{M}$ being nilpotent.

Example 2.19. Let $h = p-1$. Consider the rank 1 Kisin module $\mathfrak{M} = \mathfrak{S}\cdot e_1$ and $\varphi(e_1) = e_1$. Then $\widetilde{e}_1 = 1 \otimes e_1$ is a basis of $\varphi^*\mathfrak{M}$ with $\operatorname{Fil}^h \varphi^*\mathfrak{M} = E^h\varphi^*\mathfrak{M}$. We have $\mathcal{M} = \underline{\mathcal{M}}(\mathfrak{M}) = S\cdot\widetilde{e}_1$ with $\operatorname{Fil}^h \mathcal{M} = \operatorname{Fil}^h S\widetilde{e}_1$ and $\varphi_h(x\widetilde{e}_1) = \varphi_h(x)\widetilde{e}_1, \forall x \in \operatorname{Fil}^h S$. Hence,

$$T_{\mathfrak{S}}^h(\mathfrak{M}) = (E^h A_{\mathrm{inf}})^{\varphi_h=1}\widetilde{e}_1 = \{E^h x \in E^h A_{\mathrm{inf}} \,|\, \varphi(x) = a_0^{-h}E^h x\}\widetilde{e}_1 = E^h\mathfrak{t}^h\mathbb{Z}_p\widetilde{e}_1.$$

Here $\mathfrak{t} \in A_{\mathrm{inf}}$ is discussed in Example 3.2.3 in [Liu10] which also shows that $\varphi(\mathfrak{t}) = a_0^{-1}E\mathfrak{t}$ and $t = \mathfrak{c}\varphi(\mathfrak{t})$. On the other hand, $T_S(\mathcal{M}) = (\operatorname{Fil}^h A_{\mathrm{crys}})^{\varphi_h=1}\widetilde{e}_1 = (t^h/p)\mathbb{Z}_p\widetilde{e}_1$. Tracing the definition of $\iota : T_{\mathfrak{S}}^h(\mathfrak{M}) \to T_S(\mathcal{M})$, we see that $\iota(E^h\mathfrak{t}^h\widetilde{e}_1) = t^h\mathbb{Z}_p\widetilde{e}_1 \subset T_S(\mathcal{M}) = (t^h/p)\mathbb{Z}_p\widetilde{e}_1$. So ι is not a surjection in this case. By modulo p^n, we see $\ker(\iota) \cong \operatorname{coker}(\iota)$ is unramified and killed by p.

COROLLARY 2.20. *Let $h = p-1$ and $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,\mathrm{c}}$ be a classical Kisin module of height $p-1$. Then the kernel and cokernel of $\iota : T_{\mathfrak{S}}^h(\mathfrak{M}) \to T_S(\underline{\mathcal{M}}(\mathfrak{M}))$ are canonically isomorphic and are unramified representations killed by p.*

Proof. Note that $T_{\mathfrak{S}}^h$ is exact (see [LL20, § 6.2]). Since T_S is clearly left exact, by the exact sequence (2.4) and Lemma 2.18, it suffices to prove Corollary for $\mathfrak{M}$ being multiplicative. Clearly, we can assume $k = \overline{k}$ and then $\mathfrak{M}$ is direct sum of the $\mathfrak{S}_n \cdot e_1$ with $\varphi(e_1) = e_1$. Now our desired conclusion just follows from the above example. $\square$

Finally, given a Fontaine–Laffaille module $M \in \mathrm{FM}_{W(k)}$, set

$$T_{\mathrm{FM}}(M) := T_S(\underline{\mathcal{M}}_{\mathrm{FM}}(M)) = \operatorname{Fil}^h(M \otimes_{W(k)} A_{\mathrm{crys}})^{\varphi_h=1},$$

where $\operatorname{Fil}^h(M \otimes_{W(k)} A_{\mathrm{crys}}) = \sum_{i=0}^h \operatorname{Fil}^i M \otimes_{W(k)} \operatorname{Fil}^{h-i} A_{\mathrm{crys}}$.

3. Boundary degree prismatic cohomology

3.1 Structure of u^∞-torsion

Let $\mathcal{X}$ be a smooth proper formal scheme over $\mathcal{O}_K$ which is a degree e totally ramified extension of $W = W(k)$, the Witt ring of a perfect field k of characteristic $p > 0$. Let $\mathfrak{M}_n^i$ denote $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{A}_n)[u^\infty]$, where $n = \infty$ is to be understood as not modulo any power of p at all. Below we formulate the abstract structure shared by $\mathfrak{M}_n^i$, by considering the ith cohomology of the $(i-1)$th Nygaard filtration (and the mod p^n version).

DEFINITION 3.1. Let $i \in \mathbb{N}_{\geq 1}$. A *quasi-filtered BK module of height* i consists of:

- two $\mathfrak{S}$-modules $\mathfrak{M}$ and $\mathfrak{N}$; and
- four $\mathfrak{S}$-linear maps $f\colon \varphi^*\mathfrak{M} \to \mathfrak{N}$, $g\colon \mathfrak{N} \to \varphi^*\mathfrak{M}$, $h\colon \mathfrak{N} \to \mathfrak{M}$, and $h'\colon \mathfrak{M} \to \mathfrak{N}$ such that the following assertions hold.

(1) Composing f and g in both ways is equal to multiplication by E^{i-1}, and composing h and h' in both ways is equal to multiplication by E.
(2) h is injective.

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We shall summarize the above in the following diagram:

$$\begin{tikzcd} \varphi^*\mathfrak{M} \arrow[r, "f"] & \mathfrak{N} \arrow[r, "g"] \arrow[d, "h"] & \varphi^*\mathfrak{M} \\ & \mathfrak{M} \arrow[u, bend left, "h'"] & \end{tikzcd} \tag{$\boxdot$}$$

The composition $h \circ f$ exhibits a Frobenius structure on $\mathfrak{M}$.

Proposition 3.2. *The diagram*

$$\begin{tikzcd} \varphi^*\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, {\mathbb{\Delta}}_n) \otimes_{\mathfrak{S}} (E^{i-1}) \arrow[r] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n) \arrow[r] \arrow[d, "\varphi_{i-1}"'] & \varphi^*\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, {\mathbb{\Delta}}_n) \\ & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, {\mathbb{\Delta}}_n) & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i}_{\mathrm{N}}/p^n) \arrow[l, "\cong", "\varphi_i"'] \arrow[ul] \end{tikzcd}$$

gives rise to a quasi-filtered BK module of height i, as does the following diagram by passing to u^∞-torsion submodules:

$$\begin{tikzcd} \varphi^*\mathfrak{M}^i_n \otimes_{\mathfrak{S}} (E^{i-1}) \arrow[r] & \mathfrak{N} := \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n)[u^\infty] \arrow[r] \arrow[d, "\varphi_{i-1}"] & \varphi^*\mathfrak{M}^i_n \\ & \mathfrak{M}^i_n \arrow[u, bend left, "h'"] & \end{tikzcd}$$

Here the top row is given by (mod p^n of) inclusions of quasi-syntomic sheaves,

$${\mathbb{\Delta}}^{(1)} \otimes_{\mathfrak{S}} (E^{i-1}) \subset \mathrm{Fil}^{i-1}_{\mathrm{N}} \subset {\mathbb{\Delta}}^{(1)}.$$

φ_{i-1} and φ_i are the divided Frobenius. h' is defined as composing the inverse of φ_i, which is an isomorphism, with the natural inclusion of the ith Nygaard filtration in the $(i-1)$th Nygaard filtration. By definition, the induced Frobenius structure on $\mathfrak{M}^i_n$ and $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, {\mathbb{\Delta}}_n)$ is the usual one on prismatic cohomology.

Proof. For the first diagram, just apply [LL20, Lemma 7.8(3)]. We see that φ_{i-1} is injective in degree i and φ_i is an isomorphism in degree i. As for the second diagram, passage to u^∞-torsion submodule commutes with φ^* as $\varphi_{\mathfrak{S}}$ is flat and sends u to u^p, and the procedure of passing to a submodule clearly preserves injectivity. □

Below we shall see that one can say quite a lot about the module-theoretic structure of the underlying module $\mathfrak{M}$ of a quasi-filtered BK module. We shall retain the notation from Definition 3.1.

Proposition 3.3. *We have the following restriction on the annihilator ideal of $\mathfrak{M}$:*

$$E^{i-1} \cdot \mathrm{Ann}(\mathfrak{M}) \subset \mathrm{Ann}(\varphi^*\mathfrak{M}) = \mathrm{Ann}(\mathfrak{M}) \otimes_{\mathfrak{S}, \varphi_{\mathfrak{S}}} \mathfrak{S}.$$

Proof. The equality follows from the flatness of $\varphi_{\mathfrak{S}}$. The inclusion is because the multiplication by E^{i-1} on $\varphi^*\mathfrak{M}$ factors through $\mathfrak{N}$ (due to condition (1) of Definition 3.1) which is a submodule of $\mathfrak{M}$ (due to condition (2) of Definition 3.1). □

Corollary 3.4. *If there is an $\alpha \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{Ann}(\mathfrak{M}) + (p) = (u^\alpha, p)$, then we have*

$$\alpha \leq \frac{e(i-1)}{p-1}.$$

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Proof. Using Proposition 3.3, after modulo (p), we get the inclusion

$$E^{i-1} \cdot (u^\alpha) \subset \varphi^*(u^\alpha) = (u^{p\alpha})$$

in $\mathfrak{S}/p = k[\![u]\!]$. Since $E \equiv u^e$ modulo p, the above inclusion translates to the inequality

$$p\alpha \leq e(i-1) + \alpha,$$

which is exactly what we need to show. □

Later on we shall exhibit examples showing that the above bound is sharp for those coming from geometry as in Proposition 3.2 (see Remark 6.14(1)). We can say quite a lot when the α appearing above is ≤ 1.

PROPOSITION 3.5. *Assume that $\mathfrak{M}$ is finitely generated and u^∞-torsion.*

(1) *If $e \cdot (i-1) < p-1$, then $\mathfrak{M} = 0$.*
(2) *If $e \cdot (i-1) < 2(p-1)$, then $\mathrm{Ann}(\mathfrak{M}) + (u) \supset (p^{i-1}, u)$.*
(3) *If $e \cdot (i-1) = p-1$, then $\mathrm{Ann}(\mathfrak{M}) \supset (p, u)$. Moreover, the semi-linear Frobenius on $\mathfrak{M}$ is bijective.*

Proof. Our assumption implies the existence of α in Corollary 3.4. In the situation of (1), the inequality in Corollary 3.4 gives $\alpha = 0$, hence $\mathrm{Ann}(\mathfrak{M}) + (p)$ is the unit ideal. Since p is topologically nilpotent, this shows that $\mathrm{Ann}(\mathfrak{M})$ is already the unit ideal, hence $\mathfrak{M} = 0$.

In the situation of (2), the inequality in Corollary 3.4 gives $\alpha < 2$. Therefore, we have either $\mathfrak{M} = 0$ or $\mathrm{Ann}(\mathfrak{M}) + (p) = (u, p)$. Without loss of generality, we may assume $\mathfrak{M} \neq 0$, hence $\mathrm{Ann}(\mathfrak{M}) + (p) = (u, p)$. Let us pick an element $f = u + a \in \mathrm{Ann}(\mathfrak{M})$ with $a \in p \cdot W(k)$, and compute

$$E^{i-1} \cdot f = (E(u))^{i-1} \cdot (u+a) = (u^{e\cdot(i-1)} + \cdots + a_1 \cdot u + a_0) \cdot (u+a) = \sum_{j=0}^{p-1} \varphi_{\mathfrak{S}}(B_j) \cdot u^j.$$

Proposition 3.3 implies that all the B_i are in $\mathrm{Ann}(\mathfrak{M})$. Let us consider $C_1 = B_1(0)$: the above equation says $\varphi(C_1) = a_1 \cdot a + a_0$. Since we know $v_p(a_1) \geq i-1$ and $v_p(a_0) = i-1$, we see that $v_p(C_1) = i-1$, which implies $(B_1) + (u) \supset (u, p^{i-1})$.

Finally, we turn to (3). Similarly arguing as above, we may assume $\mathfrak{M} \neq 0$ and $\mathrm{Ann}(\mathfrak{M}) + (p) = (u, p)$, and our first task is to show $u \in \mathrm{Ann}(\mathfrak{M})$. To that end, pick again an element $f = u + a \in \mathrm{Ann}(\mathfrak{M})$ with $a \in p \cdot W(k)$. Next we compute

$$\begin{aligned} E^{i-1} \cdot f &= (u^e + p \cdot g_1)^{i-1} \cdot (u+a) = (u^{p-1} + p \cdot g_2) \cdot (u+a) \\ &= (u^p + p^{i-1} E(0)^{i-1} \cdot a) \cdot 1 + \sum_{j=1}^{p-1} b_j \cdot u^j. \end{aligned}$$

By Proposition 3.3, there is another element of the form $u + b \in \mathrm{Ann}(\mathfrak{M})$ with $b \in W(k)$ having a bigger p-adic valuation than that of a. Consequently, we have $u \in \mathrm{Ann}(\mathfrak{M})$, as $a - b$ and a differ by a unit in $W(k)$. Now we do the trick again:

$$E(u)^{i-1} \cdot u = (u^e + pg(u))^{i-1} \cdot u = u^p + \sum_{j=1}^{i-1} \binom{i-1}{j} u^{1+e(i-1-j)} (pg(u))^j = u^p + \sum_{j=1}^{p-1} B_j u^j,$$

with $B_j \in W(k)$. Since $u^p \in \mathrm{Ann}(\varphi^* \mathfrak{M}_n^i)$, we see that $\sum_{j=1}^{p-1} B_j u^j \in \mathrm{Ann}(\varphi^* \mathfrak{M}_n^i)$ and hence each $\varphi^{-1}(B_j)$ is in $\mathrm{Ann}(\mathfrak{M}_n^i)$. From the above expansion, we see that

$$E(u)^{i-1} \cdot u \equiv u^p + (i-1) u^{1+e(i-2)} (pg(u)) \mod p^2.$$

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Since $E(u)$ is an Eisenstein polynomial, we see that $g(0)$ is a p-adic unit. This implies that $v_p(B_{1+e(i-2)}) = 1$, so $p \in \mathrm{Ann}(\mathfrak{M}^i_n)$.

Finally, we need to show the semi-linear Frobenius on $\mathfrak{M}$ is a bijection. The previous paragraph tells us that $\mathfrak{M} \simeq k^{\oplus r}$. Let us consider the diagram ($\boxdot$):

$$\begin{tikzcd}
\varphi^*\mathfrak{M} \arrow[r, "f"] \arrow[d] & \mathfrak{N} \arrow[r, hook, "h"] \arrow[d, "g"] & \mathfrak{M} \\
E^{i-1} \cdot \varphi^*\mathfrak{M} \arrow[r, hook] & \varphi^*\mathfrak{M} &
\end{tikzcd}$$

We claim the first arrow in the top row is surjective, the middle vertical arrow is injective with image $E^{i-1} \cdot \varphi^*\mathfrak{M}$, and the map h is an isomorphism. We know $\varphi^*\mathfrak{M} \simeq (k[u]/u^p)^{\oplus r}$, hence $E^{i-1} \cdot \varphi^*\mathfrak{M}$ is also abstractly isomorphic to $k^{\oplus r}$. Let $\ell(\cdot)$ denote the k-length. The above diagram gives a chain of inequalities of lengths

$$r \leq \ell(\mathfrak{N}) \leq r = \ell(\mathfrak{M}),$$

where the first inequality follows from the previous sentence. So the above inequalities are both equalities, and the claim follows easily. The composition, which we have shown to be surjective, of

$$\mathrm{Frob}_k^*\mathfrak{M} \cong \varphi_{\mathfrak{S}}^*\mathfrak{M}/u \xrightarrow{f} \mathfrak{N} \xrightarrow{h} \mathfrak{M}$$

is the linearization of the semi-linear Frobenius on $\mathfrak{M}$. This shows that the semi-linear Frobenius on $\mathfrak{M}$ is surjective, hence bijective by length/dimension considerations. ☐

The above gives our current knowledge on the u^∞-torsion submodules in prismatic cohomology.

Theorem 3.6. *Recall* $\mathfrak{M}^i_n := \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty]$.

(1) *If* $e \cdot (i-1) < p-1$, *then* $\mathfrak{M}^i_n = 0$.
(2) *If* $e \cdot (i-1) < 2(p-1)$, *then* $\mathrm{Ann}(\mathfrak{M}^i_n) + (u) \supset (p^{i-1}, u)$.
(3) *If* $e \cdot (i-1) = p-1$, *then* $\mathrm{Ann}(\mathfrak{M}^i_n) \supset (p, u)$. *Moreover, the semi-linear Frobenius on* $\mathfrak{M}^i_n$ *is bijective. In particular,* $\mathfrak{M}^i_n$ *gives rise to an étale* φ*-module on* k*, hence an* $\mathbb{F}_p$*-representation of* G_k *or equivalently an unramified* $\mathbb{F}_p$*-representation of* G_K.

Later we shall give an interpretation of the G_k-representation in (3) above (see Theorem 4.14 and Corollary 4.15).

Proof. Combining Propositions 3.2 and 3.5 gives us what we want. ☐

Below let us remark on results in the literature concerning u^∞-torsion in Breuil–Kisin prismatic cohomology.

Remark 3.7. (1) Under the assumption $e \cdot i < p-1$, Min [Min21, Theorem 0.1] showed that the ith prismatic cohomology has no u-torsion and 'looks like' the étale cohomology of the geometric generic fiber. His strategy is to exploit the fact that Frobenius map in degree i has height i. Note that his method also shows that in the same range, the ith (derived) mod p^n prismatic cohomology also has no u-torsion. But as far as we can tell, the method stops outside the above range.

(2) Philosophically speaking, the u^∞-torsion in the ith (derived) mod p^n prismatic cohomology is surjected on by the $(i-1)$th cohomology of the sheaf Δ_n/u^N for some large N, hence

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it should secretly have height $i-1$. Our Proposition 3.3 may be taken as a manifestation of this philosophy. Later on we show this philosophy is literally true for u^∞-torsion in the integral prismatic cohomology (see Corollary 3.16).

(3) In our previous work, we showed a close relation between u-torsion in prismatic cohomology and the structure of Breuil's crystalline cohomology [LL20, Theorem 7.22]. Using this relation, together with Caruso's result [Car08, Theorem 4.1.24], we obtained the same conclusion as in Theorem 3.6(1) and an improvement of Caruso's result [Car08, Theorems 4.1.24 and 4.2.1] (see [LL20, Corollary 7.25]). Note that our bound on the cohomological index is 1 higher than Caruso's result.

(4) Our control of u-torsion in this paper bypasses Caruso's result. Hence, we obtain a proof of Caruso's result and its improvement simultaneously (cf. [LL20, Theorem 7.22 and Corollary 7.25]).

(5) Later on, we shall see that our bound is in some sense sharp by exhibiting an example having (u,p)-torsion with $e=p-1$ and $i=2$. See § 6.

Let us give an application by showing that the module structure of prismatic cohomology in low range looks like a $\mathbb{Z}_p$-module.

COROLLARY 3.8. *Let i be an integer satisfying $e\cdot(i-1)<p-1$. Then there exists a (non-canonical) isomorphism of $\mathfrak{S}$-modules,*

$$\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \simeq \mathrm{H}^i_{\text{ét}}(\mathcal{X}_C,\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathfrak{S}.$$

Proof. We always have inclusions $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})/p^n \subset \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X},\Delta_n)$. In the specified range, we know the latter has no u-torsion by Theorem 3.6(1). Applying Proposition 2.6 shows that there exists an isomorphism of $\mathfrak{S}$-modules

$$\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \simeq N_i\otimes_{\mathbb{Z}_p}\mathfrak{S}$$

for some $\mathbb{Z}_p$-module N_i. To obtain $N_i\simeq \mathrm{H}^i_{\text{ét}}(\mathcal{X}_C,\mathbb{Z}_p)$, we simply use the étale comparison of Bhatt and Scholze (see [BMS18, Theorem 1.8(iv)] and [BS22, Theorem 1.8(4)]). Here we are using the fact that the isomorphism class of a finitely generated $\mathbb{Z}_p$-module is determined by its base change to $W(C^\flat)$. □

One should compare this with Min's result [Min21, Theorem 5.11]. Our bound on the cohomological degree i here is 1 better than Min's. Below we recall a useful result in [BMS18] assuring nice behavior of prismatic cohomology when crystalline cohomology has no torsion, which is a condition often cited in the literature.

Remark 3.9. If $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)$ is torsion-free, then $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$ is free. This follows from [BMS18, Corollary 4.17] and crystalline comparison. We sketch a proof below (see also [KP21, Lemma 4.3.28.(1)]).

Proof. Denote $\mathfrak{M} \coloneqq \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$. We shall use the two short exact sequences appearing in the proof of Proposition 2.6. Crystalline comparison implies $\mathfrak{M}/u\mathfrak{M}\hookrightarrow \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)$. Let us derived modulo the sequence

$$0\to\mathfrak{M}_{\mathrm{tor}}\to\mathfrak{M}\to\mathfrak{M}_{\mathrm{tf}}\to 0$$

by u. Since $\mathfrak{M}_{\mathrm{tf}}$ is torsion-free, we have $\mathfrak{M}_{\mathrm{tor}}/u\hookrightarrow\mathfrak{M}/u$. The target is p-torsion-free by assumption whereas $\mathfrak{M}_{\mathrm{tor}}$ consists of p-power torsion, therefore $\mathfrak{M}_{\mathrm{tor}}=0$ and $\mathfrak{M}_{\mathrm{tf}}=\mathfrak{M}$. Now we again

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derived modulo the sequence

$$0 \to \mathfrak{M} \to \mathfrak{M}_{\mathrm{fr}} \to \mathfrak{M}_0 \to 0$$

by u to get $\mathfrak{M}_0[u] \hookrightarrow \mathfrak{M}/u$; the same argument as above shows $\mathfrak{M}_0[u] = 0$, whereas $\mathfrak{M}_0$ is supported at the maximal ideal of $\mathfrak{S}$. Therefore, we again conclude $\mathfrak{M}_0 = 0$ and $\mathfrak{M} = \mathfrak{M}_{\mathrm{fr}}$. $\square$

Let us conclude this subsection by posing some questions.

Question 3.10. Recall $\mathfrak{M}_n^i \coloneqq \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n)[u^\infty]$.

(1) Let β be the smallest exponent such that $p^\beta \in \mathrm{Ann}(\mathfrak{M}_n^i)$, and let γ be the exponent such that $\mathrm{Ann}(\mathfrak{M}_n^i) + (u) = (u, p^\gamma)$. Is there a bound on β and γ in terms of e and i?
(2) In light of the example in § 6, is β and/or γ bounded above by a polynomial in $\log_p$ of a polynomial in e and i, perhaps simply bounded above by $\log_p((e \cdot (i-1))/(p-1)) + 1$ when p is odd?[5]

3.2 Comparing Frobenius and Verschiebung

Given a smooth proper formal scheme $\mathcal{X}$ over $\mathcal{O}_K$, for each degree i, we have a natural inclusion $\mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}/u \hookrightarrow \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)$ from the crystalline comparison of prismatic cohomology theory. Here the superscript $(-)^{(1)}$ denotes the Frobenius twist, so

$$\mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)} \coloneqq \varphi^*_{\mathfrak{S}} \mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S}) \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}).$$

The map is compatible with Frobenius and Verschiebung, hence induces Frobenius and Verschiebung maps on the cokernel $\mathrm{H}^{i+1}_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$. How to understand these maps? That is the question we shall answer in this subsection.

Given any algebra R which is quasi-syntomic over $\mathcal{O}_K$, we may take its mod π reduction R_0 which is quasi-syntomic over k. This way we obtain a natural map of sites $i\colon k_{\mathrm{qSyn}} \to (\mathcal{O}_K)_{\mathrm{qSyn}}$. Note that the functor $R_0 \mapsto \mathrm{Cris}(R_0/W)$ is a quasi-syntomic sheaf on k_{qSyn}. Here by abuse of notation we use $\mathrm{Cris}(R_0/W)$ to denote the left Kan extended crystalline cohomology. The sheaf $i_* \mathrm{Cris}$ takes an algebra R in $(\mathcal{O}_K)_{\mathrm{qSyn}}$ to $i_* \mathrm{Cris}(R) \coloneqq \mathrm{Cris}(R_0/W)$. The base change property and the crystalline comparison of prismatic cohomology [BS22, Theorem 1.8.(1)&(5)] give us the following exact triangles of sheaves on $(\mathcal{O}_K)_{\mathrm{qSyn}}$:

$$\mathbb{\Delta} \xrightarrow{\cdot u} \mathbb{\Delta} \to i_* \mathrm{Cris}^{(-1)}$$

and

$$\mathbb{\Delta}^{(1)} \xrightarrow{\cdot u} \mathbb{\Delta}^{(1)} \to i_* \mathrm{Cris},$$

where $i_* \mathrm{Cris}^{(-1)}(R) = \mathrm{Cris}(R_0/W) \otimes_{W, \varphi^{-1}} W$ is the Frobenius inverse twist.

Proposition 3.11.

(1) *The linear Frobenius maps* $\mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)} \to \mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ *and* $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W) \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)^{(-1)}$ *induce a linear map* $\mathrm{H}^{i+1}_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}[u] \to \mathrm{H}^{i+1}_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})[u]$ *which agrees with the linear Frobenius* $\mathrm{H}^{i+1}_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)} \to \mathrm{H}^{i+1}_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ *restricted to u-torsion.*
(2) *The semi-linear Frobenius maps* $\mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)} \to \mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}$ *and* $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W) \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)$ *induce a semi-linear map* $\mathrm{H}^{i+1}_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}[u] \to \mathrm{H}^{i+1}_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$. *This map is* u^{p-1} *times the semi-linear Frobenius on* $\mathrm{H}^{i+1}_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}$ *restricted to u-torsion.*

[5] After considering the image of Whitehead's J-homomorphism, we suspect the above bound should be higher by 1 when $p = 2$ and $e \cdot (i-1) \geq 2$.

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Note that semi-linearity means u-torsion is only sent to u^p-torsion under the semi-linear Frobenius; after multiplying u^{p-1} we land in u-torsion again.

Proof. Below we use lin–Frob (respectively, sl–Frob) to denote the linearized Frobenius (respectively, semi-linear Frobenius) on $\mathbb{\Delta}^{(1)}$.

(1) This follows from the commutative diagram

$$\begin{tikzcd} \mathbb{\Delta}^{(1)} \arrow[r, "\cdot u"] \arrow[d, "\mathrm{lin-Frob}"'] & \mathbb{\Delta}^{(1)} \arrow[r] \arrow[d, "\mathrm{lin-Frob}"'] & i_* \operatorname{Cris} \arrow[d, "i_*(\mathrm{lin-Frob})"] \\ \mathbb{\Delta} \arrow[r, "\cdot u"] & \mathbb{\Delta} \arrow[r] & i_* \operatorname{Cris}^{(-1)} \end{tikzcd}$$

(2) This follows from the analogous commutative diagram

$$\begin{tikzcd} \mathbb{\Delta}^{(1)} \arrow[r, "\cdot u"] \arrow[d, "u^{p-1}\cdot(\mathrm{sl-Frob})"'] & \mathbb{\Delta}^{(1)} \arrow[r] \arrow[d, "\mathrm{sl-Frob}"'] & i_* \operatorname{Cris} \arrow[d, "i_*(\mathrm{sl-Frob})"] \\ \mathbb{\Delta}^{(1)} \arrow[r, "\cdot u"] & \mathbb{\Delta}^{(1)} \arrow[r] & i_* \operatorname{Cris} \end{tikzcd}$$

□

Remark 3.12. Comparing the above two formulas, the appearance of an extra u^{p-1} factor has a natural explanation. Let M be an $\mathfrak{S}$-module. Then by flatness of $\varphi_{\mathfrak{S}}$ we know that $\big(M \otimes_{\mathfrak{S},\varphi_{\mathfrak{S}}} \mathfrak{S}\big)[u] \cong \big(M[u] \otimes_{\mathfrak{S},\varphi_{\mathfrak{S}}} \mathfrak{S}\big)[u]$. We may expand the right-hand side as $\big(M[u] \otimes_{W,\varphi_W} W\big) \otimes_W (\mathfrak{S}/u^p[u])$. Under this identification, one checks that there is a semi-linear bijection $M[u] \xrightarrow{\simeq} \big(M[u] \otimes_{W,\varphi_W} W\big) \otimes_W (\mathfrak{S}/u^p[u])$ given by $m \mapsto (m \otimes 1) \otimes u^{p-1}$. Applying this to $M = \mathrm{H}^{i+1}_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ gives the relation between (1) and (2) above.

Next we turn to the map on $\mathrm{H}^{i+1}_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$ induced from Verschiebung maps. We need the following fact on Nygaard filtration.

Lemma 3.13. *The divided Frobenius $\varphi_{i-1} \colon \mathrm{Fil}^{i-1}_{\mathrm{N}} \to \mathbb{\Delta}$ induces an isomorphism*

$$\varphi_{i-1} \colon \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}})_{\mathrm{tors}} \xrightarrow{\cong} \mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X})_{\mathrm{tors}}.$$

Proof. Note that we have a commutative diagram of quasi-syntomic sheaves:

$$\begin{tikzcd} \mathrm{Fil}^{i-1}_{\mathrm{N}} \otimes_{\mathfrak{S}} (E) \arrow[rr, "\mathrm{incl}"] \arrow[dr, "\varphi_{i-1}"'] & & \mathrm{Fil}^i_{\mathrm{N}} \arrow[dl, "\varphi_i"] \\ & \mathbb{\Delta} & \end{tikzcd}$$

By [LL20, Lemma 7.8(3)] we know the ith divided Frobenius map in degree i is an isomorphism for any bounded prism. Therefore, we only need to show that the map $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}) \otimes_{\mathfrak{S}} (E) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{N}})$ induces an isomorphism on the torsion submodule.

We claim these modules have the property that their torsion submodule coincides with the p^∞-torsion submodule. To see this, just use the fact that both φ_{i-1} and φ_i are injective in degree i, thanks to [LL20, Lemma 7.8.(3)]. The torsion submodule in prismatic cohomology is well known to coincide with p^∞-torsion submodule.

Therefore, we are reduced to showing the above map induces an isomorphism on the p^∞-torsion submodule. To that end, we use the exact sequence of quasi-syntomic

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sheaves: $\mathrm{Fil}_{\mathrm{N}}^{i-1} \otimes_{\mathfrak{S}} (E) \to \mathrm{Fil}_{\mathrm{N}}^{i} \to \mathrm{Fil}_{\mathrm{H}}^{i}$. Finally, just note that $\mathrm{H}^i(\mathcal{X}, \mathrm{Fil}_{\mathrm{H}}^i) \cong \mathrm{H}^0(\mathcal{X}, \Omega^i_{\mathcal{X}/\mathcal{O}_K})$ is p-torsion-free. □

COROLLARY 3.14. *If $e \cdot (i-1) = p-1$, then the map* $\mathrm{incl}\colon \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}_{\mathrm{N}}^{i-1})[u] \to \mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$ *is an isomorphism.*

Proof. We consider the following diagram and consider taking H^i:

$$\begin{tikzcd} & \mathbb{\Delta}^{(1)} \otimes (E)^{\otimes i-1} \arrow[dl, "\varphi\otimes id"'] \arrow[d, "\mathrm{incl}"] \arrow[dr, "\mathrm{incl}"] & \\ \mathbb{\Delta} \otimes (E)^{\otimes i-1} & \mathrm{Fil}_{\mathrm{N}}^{i-1} \arrow[l, "\varphi"'] \arrow[r, "\mathrm{incl}"] & \mathbb{\Delta}^{(1)} \end{tikzcd}$$

By Theorem 3.6(3), applied to $n=\infty$, combined with Lemma 3.13 we know that the map

$$\mathrm{incl}\colon \left(\mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}[u^\infty]\right)/u \otimes (E)^{\otimes i-1} \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}_{\mathrm{N}}^{i-1})[u]$$

is an isomorphism. Using Theorem 3.6(3) again, we know that the map

$$\mathrm{incl}\colon \left(\mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}[u^\infty]\right)/u \otimes (E)^{\otimes i-1} \to \mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$$

is also an isomorphism. Therefore, we get the desired result. □

The relevance of Nygaard filtration when discussing the Verschiebung map follows from [LL20, Corollary 7.9]. We recall its statement below:

LEMMA 3.15. *Let (A, I) be a bounded prism, and let $\mathcal{X}$ be a smooth formal scheme over* $\mathrm{Spf}(A/I)$. *The ith Verschiebung map (see [BS22, Corollary 15.5])*

$$V_i\colon \tau^{\le i}\mathbb{\Delta}_{\mathcal{X}/A} \otimes_A I^{\otimes i} \to \tau^{\le i}\mathbb{\Delta}^{(1)}_{\mathcal{X}/A}$$

can be functorially identified with $\mathrm{incl} \circ \varphi^{-1}$*:*

$$\tau^{\le i}\mathbb{\Delta}_{\mathcal{X}/A} \otimes_A I^{\otimes i} \xleftarrow[\cong]{\varphi} \tau^{\le i}\,\mathrm{Fil}_{\mathrm{N}}^i(\mathcal{X}/A) \xrightarrow{\mathrm{incl}} \tau^{\le i}\mathbb{\Delta}^{(1)}_{\mathcal{X}/A}.$$

Proof sketch. This follows from the following commutative diagram:

$$\begin{tikzcd} \tau^{\le i}\,\mathrm{Fil}_{\mathrm{N}}^i(\mathcal{X}/A) \arrow[r, "\varphi", "\cong"'] \arrow[d, "\mathrm{incl}"'] & \tau^{\le i}\mathbb{\Delta}_{\mathcal{X}/A} \otimes_A I^{\otimes i} \arrow[d] \arrow[dl, dashed, "V_i"'] \\ \tau^{\le i}\mathbb{\Delta}^{(1)}_{\mathcal{X}/A} \arrow[r, "\varphi"] & \tau^{\le i}\mathbb{\Delta}_{\mathcal{X}/A} \end{tikzcd}$$

Here the top arrow is an isomorphism due to [LL20, Lemma 7.8(3)], and the diagonal map is defined affine locally and follows from the description $\mathbb{\Delta}^{(1)}_{\mathcal{Y}/A} \cong L\eta_I \mathbb{\Delta}_{\mathcal{Y}/A} \xrightarrow[\cong]{\varphi} \mathbb{\Delta}_{\mathcal{Y}/A}$ for any smooth affine $\mathcal{Y}$ over $\mathrm{Spf}(A/I)$ (see [BS22, Theorem 15.3]). □

Consequently, we see that the torsion and u^∞-torsion in the ith prismatic cohomology is canonically a (generalized) Kisin module of height $i-1$.

COROLLARY 3.16. *The restriction of the Verschiebung map $V_i\colon \mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)})$ to either the torsion submodule or u^∞-torsion submodule of the source is canonically divisible*

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by E. The division is given by

$$\mathrm{H}^i_{\Delta}(\mathcal{X})_{\mathrm{tors}} \xleftarrow[\cong]{\varphi_{i-1}} \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}})_{\mathrm{tors}} \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)})_{\mathrm{tors}},$$

which, together with the usual prismatic Frobenius, makes the torsion submodule and u^∞-torsion submodule in $\mathrm{H}^i_{\Delta}(\mathcal{X})$ a (generalized) Kisin module of height $i-1$.

Proof. This follows from combining Lemmas 3.13 and 3.15. ☐

We can finally understand the induced 'Verschiebung' map.

Corollary 3.17. *The ith linear Verschiebung maps $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \to \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}$ and $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)^{(-1)} \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)$ induce a linear map $V_i\colon \mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})[u] \to \mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$ which fits into the following diagram:*

$$\begin{tikzcd}
\mathrm{H}^{i+1}_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{N}})[u] \arrow[rr, "\varphi_i", "\cong"'] \arrow[dr, "\mathrm{incl}"'] & & \mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})[u] \arrow[dl, "V_i"] \\
& \mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u] &
\end{tikzcd}$$

In particular, the induced map V_i is identified with $\mathrm{incl} \circ \varphi_i^{-1}$.

In other words, the induced V_i is the restriction of 'V_{i+1} divided by E' (from Corollary 3.16) to the u-torsion submodule.

Proof. This follows from combining Lemmas 3.13 and 3.15. ☐

Corollary 3.18. *If $e \cdot (i-1) = p-1$, then the induced Verschiebung $V_{i-1}\colon \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})[u] \to \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$ is an isomorphism.*

Proof. This follows from Corollary 3.17, Lemma 3.13 and Corollary 3.14. ☐

Summary

Let us summarize our knowledge on the structure of prismatic cohomology, with the auxiliary $(i-1)$th Nygaard filtration in mind. Fix the cohomological degree i and $n \in \mathbb{N} \cup \{\infty\}$. The relevant diagram is:

$$\begin{tikzcd}
\Delta^{(1)} \arrow[r, "\otimes(E^{i-1})"] \arrow[dr, "\varphi"'] & \mathrm{Fil}^{i-1}_{\mathrm{N}} \arrow[r, "\otimes(E)"] \arrow[d, "\varphi_{i-1}"] \arrow[l, dashed, bend right, "\mathrm{incl}"'] & \mathrm{Fil}^i_{\mathrm{N}} \arrow[dl, "\varphi_i"] \arrow[l, dashed, bend right, "\mathrm{incl}"'] \\
& \Delta &
\end{tikzcd}$$

If we drop the dashed arrows, then the diagram commutes. On the other hand, the two circles on top have the property that composing the two arrows either way gives multiplication by E^{i-1} and E separately.

The above diagram induces a diagram

$$\begin{tikzcd}
\varphi^*_{\mathfrak{S}}\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n) \arrow[r, "f"] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n) \arrow[r, "h"] \arrow[l, bend right] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n) \arrow[l, bend right]
\end{tikzcd}$$

We know the following facts about the above diagram.

(1) The two arrows in the first circle composed either way give multiplication by E^{i-1}.
(2) The two arrows in the second circle composed either way give multiplication by E.

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(3) Composing the rightward arrows gives the prismatic Frobenius.
(4) Composing the leftward arrows gives the prismatic Verschiebung V_i (see [BS22, Corollary 15.5], [LL20, Corollary 7.9] and Lemma 3.15).
(5) The map h is injective (see [LL20, Lemma 7.8.(3)]).
(6) If $n = \infty$, then h induces an isomorphism of torsion submodules (see Lemma 3.13). Hence, as far as torsion or u^∞-torsion in $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$ is concerned, we may focus on the first circle and see that these Frobenius modules are canonically (generalized) Kisin modules of height $i-1$ (see Corollary 3.16).

3.3 Induced Nygaard filtration

Finally, let us discuss the induced Nygaard filtration on u^∞-torsion in the boundary degree prismatic cohomology.

Lemma 3.19. *Assume $e \cdot (i-1) = p-1$ and let $n \in \mathbb{N} \cup \{\infty\}$. For any $j \in \mathbb{N}$, consider the induced map on $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, -/p^n)$ of the maps of quasi-syntomic sheaves $\mathrm{Fil}^j_{\mathrm{N}} \to \Delta^{(1)}$, the following two submodules of $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)[u^\infty]$ agree:*

- $\mathrm{Im}\big(\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)\big) \cap \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)[u^\infty]$;
- $\mathrm{Im}\big(\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n)[u^\infty] \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)[u^\infty]\big)$.

Proof. We consider the following diagram of $\mathfrak{S}$-modules, with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n)[u^\infty] \arrow[r] \arrow[d] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n) \arrow[r] \arrow[d, "f"] & Q_1 \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)[u^\infty] \arrow[r] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n) \arrow[r] & Q_2 \arrow[r] & 0
\end{tikzcd}$$

By [LL20, Proposition 7.12], $\mathrm{Ker}(f)$ has finite length, hence must be contained in $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n)[u^\infty]$. The snake lemma implies that $\mathrm{Ker}(g)$ embeds inside a quotient of $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)[u^\infty]$. Since $\mathrm{Ker}(g)$, being a submodule of Q_1, is u-torsion-free, we see it must be zero, which is exactly what we need to show. ☐

If no confusion would arise, when $e \cdot (i-1) = p-1$, we shall refer to the submodule in the above lemma as the *(induced) jth Nygaard filtration* on $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)[u^\infty]$. The following proposition reveals what this filtration is.

Proposition 3.20. *Assume $e \cdot (i-1) = p-1$ and let $n \in \mathbb{N} \cup \{\infty\}$.*

(1) *The Nygaard filtration on $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)[u^\infty]$ as above is the $E(u) \equiv u^e$-adic filtration.*
(2) *The map $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)$ is injective.*
(3) *For any $j \geq 0$, the map $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i+j}_{\mathrm{N}}/p^n) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)$ has kernel given by u^∞-torsion of the source.*

Remark 3.21. (1) We recall that, under the hypothesis of this lemma, Theorem 3.6(3) gives a canonical isomorphism of $\mathfrak{S}$-modules

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)[u^\infty] \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty] \otimes_{\mathfrak{S}, \varphi_{\mathfrak{S}}} \mathfrak{S} \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u, p] \otimes_k \mathfrak{S}/(p, u^p).$$

Therefore, the $E(u)$-adic filtration is the same as u^e-filtration.

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(2) Also note that $u^{e\cdot(i+j)} = u^{p-1+e\cdot(j+1)} \in (u^p)$ if $j \geq 0$, hence (1) implies (3).

(3) To put Proposition 3.20(3) into context, let us point to [LL20, Corollary 7.9] which says that the divided Frobenius $\varphi_{i+j}\colon \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i+j}_{\mathrm{N}}/p^n) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)$ is an isomorphism for all $j \geq 0$.

Proof of Proposition 3.20. Throughout this proof, all filtrations referred to are filtrations on $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n^{(1)})[u^\infty]$.

Since we have a containment of quasi-syntomic sheaves, $E^j \cdot \Delta^{(1)} \subset \mathrm{Fil}^j_{\mathrm{N}} \subset \Delta^{(1)}$, one easily sees that the Nygaard filtration contains the u^e-adic filtration. All we need to show is the converse containment.

Let us first show that (1) holds for the $(i-1)$th Nygaard filtration and (2). As discussed above, since $u^{e\cdot(i-1)} = u^{p-1}$, we see that the $(i-1)$th Nygaard filtration has length at least that of $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty]$.[6] To finish, it suffices to show that the u^∞-torsion in $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n)$ has length at most that. This follows from the fact that the divided Frobenius, which is $\mathfrak{S}$-linear,

$$\varphi_{i-1}\colon \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n),$$

is injective (see [LL20, Lemma 7.8(3)]).

Next we show that (1) holds for the jth filtration whenever $0 \leq j \leq i-1$. We consider another containment of quasi-syntomic sheaves: $E^{i-1-j} \cdot \mathrm{Fil}^j_{\mathrm{N}} \subset \mathrm{Fil}^{i-1}_{\mathrm{N}} \subset \mathrm{Fil}^j_{\mathrm{N}}$. Therefore, we see the jth filtration can differ from the $(i-1)$th filtration by at most $u^{e\cdot(i-1-j)}$; this gives the desired converse containment by what we proved in the previous paragraph.

Finally, we show that (1) holds for the $(i+j)$th filtration for any $j \geq 0$; note that this implies (3), as remarked right after the statement of this proposition. We want to show that the map $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i+j}_{\mathrm{N}}/p^n)[u^\infty] \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n^{(1)})[u^\infty]$ is the zero map when $j \geq 0$. Since this map factors through the $j=0$ case, it suffices to prove the $j=0$ case. To that end, we shall utilize [LL20, Corollary 7.9], according to which we need to show that the prismatic Verschiebung annihilates $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty]$. Now we consider the following sequence of arrows:

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n^{(1)})[u^\infty] \xrightarrow{\varphi} \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty] \xrightarrow{\psi} \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n^{(1)})[u^\infty].$$

Here $\psi = V_i$ is the ith Verschiebung as in [BS22, Corollary 15.5]. The composition of these two arrows is multiplication by $E^i = u^{p-1+e} = 0$, as the module is abstractly several copies of $\mathfrak{S}/(p, u^p)$. We finish the proof by recalling that the map φ above is surjective (Theorem 3.6(3)). □

As a consequence, in the boundary degree, we can use torsion in the cohomology of $\mathcal{O}_\mathcal{X}$ to bound u^∞-torsion.

Corollary 3.22. *Assume $e \cdot (i-1) = p-1$ and let $n \in \mathbb{N} \cup \{\infty\}$. The natural map $\Delta^{(1)} \to \mathrm{gr}^0_{\mathrm{N}} \Delta^{(1)} \cong \mathcal{O}$ gives rise to a canonical injection:*

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty] \otimes_k (\mathcal{O}_K/p) \hookrightarrow \mathrm{H}^i(\mathcal{X}, \mathcal{O}_\mathcal{X}/p^n).$$

Proof. The exact sequence $\mathrm{Fil}^1_{\mathrm{N}} \to \Delta^{(1)} \to \mathcal{O}_X$ tells us that the kernel of the map

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n^{(1)})[u^\infty] \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty] \otimes_k k[u]/(u^p) \to \mathrm{H}^i(\mathcal{O}_\mathcal{X}/p^n)$$

is given by the induced first Nygaard filtration on the source, which we know is exactly u^e times the source, thanks to Proposition 3.20(1). Notice that, as an $\mathcal{O}_K$-algebra, we have

[6] Note that here we are not twisting the prismatic cohomology by Frobenius.

$k[u]/(u^e) \cong \mathcal{O}_K/p$. Therefore, we get the desired injection

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, {\Delta}_n)[u^\infty] \otimes_k k[u]/(u^e) \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, {\Delta}_n)[u^\infty] \otimes_k \big(\mathcal{O}_K/p\big) \hookrightarrow \mathrm{H}^i(\mathcal{O}_\mathcal{X}/p^n). \qquad \square$$

4. Geometric applications

4.1 The discrepancy of Albanese varieties

In this subsection, we give a geometric interpretation of u-torsion in the second Breuil–Kisin prismatic cohomology. Our main application in this subsection has partly been obtained by Raynaud in [Ray79]; our method is of course quite different. Without loss of generality, we assume our smooth proper (formal) scheme $\mathcal{X}$ has an $\mathcal{O}_K$-point. This can be arranged after an unramified extension of $\mathcal{O}_K$.

The generic fiber of $\mathcal{X}$ is a smooth proper rigid space X over $\mathrm{Sp}(K)$ admitting a K-point. Specializing the main result of [HL00] to our case where X has a smooth proper formal model, we know that $\mathrm{Pic}^0(X)$ is an abeloid variety which has good reduction, namely it is the rigid generic fiber of a formal abelian scheme over $\mathcal{O}_K$. In the algebraic situation, the existence of the abelian scheme integral model follows from Serre and Tate's generalization [ST68] of the Néron–Ogg–Shafarevich's criterion. For the general theory of the Néron model of abeloid variety, we refer readers to [Lüt95]. Now we can form the Albanese of X, which is a universal map

$$g_K \colon X \to A$$

from X to abeloid varieties (see [HL20, § 4]). Since in this case A is the dual of $\mathrm{Pic}^0(X)$, we know it also has good reduction: the Néron model of A is a formal abelian scheme $\mathcal{A}$ over $\mathcal{O}_K$. Finally, since $\mathcal{X}$ is smooth over $\mathcal{O}_K$, the Néron mapping property implies that the map $X \to A$ extends uniquely to

$$g \colon \mathcal{X} \to \mathcal{A}$$

over $\mathcal{O}_K$. Taking the special fiber of the above map, we get

$$g_0 \colon \mathcal{X}_0 \to \mathcal{A}_0.$$

Now the Albanese theory tells us that the above map factors:

$$\begin{tikzcd} \mathcal{X}_0 \arrow[rr, "g_0"] \arrow[dr, "h"'] & & \mathcal{A}_0 \\ & \mathrm{Alb}(\mathcal{X}_0) \arrow[ur, "f"'] & \end{tikzcd}$$

where $\mathrm{Alb}(\mathcal{X}_0)$ is the Albanese of $\mathcal{X}_0$. Therefore, out of a pointed smooth proper formal scheme $\mathcal{X}$ over $\mathcal{O}_K$, we can cook up a map $f \colon \mathrm{Alb}(\mathcal{X}_0) \to \mathcal{A}_0$ of abelian varieties over k. What can we say about this map?

Proposition 4.1. *The map $f \colon \mathrm{Alb}(\mathcal{X}_0) \to \mathcal{A}_0$ above is an isogeny of p-power degree.*

Proof. It suffices to show that f induces an isomorphism of the first ℓ-adic étale cohomology for all primes $\ell \neq p$. From now on we fix such an ℓ. Standard Albanese theory tells us that the Albanese maps $h \colon \mathcal{X}_0 \to \mathrm{Alb}(\mathcal{X}_0)$ and $g_K \colon X \to A$ induce an isomorphism of the first ℓ-adic étale cohomology. To finish the proof, we just use the smooth and proper base change theorems in étale cohomology theory to see that the map g_0, being a reduction of the 'smooth proper model' g of g_K, also induces an isomorphism of the first ℓ-adic étale cohomology. Since $h^* \circ f^* = g_0^*$ and

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both h^* and g_0^* induce an isomorphism of the first ℓ-adic étale cohomology, we conclude that f^* also does. $\square$

Let us denote the finite p-power order group scheme $\ker(f)$ by G. The Dieudonné module of G is related to $\mathcal{X}$ in the following way.

Theorem 4.2. *We have an isomorphism of W-modules*

$$\mathbb{D}(G) \cong \mathrm{H}^2_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}[u].$$

Under this identification, the semi-linear Frobenius F on the left-hand side and the semi-linear Frobenius φ on the left-hand side are related via $F = u^{p-1} \cdot \varphi$, and the linear Verschiebung on the left-hand side can be understood as

$$\mathrm{H}^2_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})[u] \underset{\cong}{\xleftarrow{\varphi_1}} \mathrm{H}^2_{\mathrm{qSyn}} \mathrm{Fil}^1_{\mathrm{N}}(\mathcal{X}/\mathfrak{S})[u] \xrightarrow{incl} \mathrm{H}^2_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}[u].$$

Proof. The Dieudonné module of G in our situation is given by

$$\mathbb{D}(G) \cong \mathrm{Coker}\big(f^*\colon \mathrm{H}^1_{\mathrm{crys}}(\mathcal{A}_0/W) \to \mathrm{H}^1_{\mathrm{crys}}(\mathrm{Alb}(\mathcal{X}_0)/W)\big),$$

so we need to understand the above map f^*.

We want to relate everything to $\mathcal{X}$. First, by [Ill79, Remarque 3.11.2] we know that the map

$$h^*\colon \mathrm{H}^1_{\mathrm{crys}}(\mathrm{Alb}(\mathcal{X}_0)/W) \to \mathrm{H}^1_{\mathrm{crys}}(\mathcal{X}_0/W)$$

is an isomorphism. Therefore, by composing with h^* we have

$$\mathbb{D}(G) \cong \mathrm{Coker}\big(g_0^*\colon \mathrm{H}^1_{\mathrm{crys}}(\mathcal{A}_0/W) \to \mathrm{H}^1_{\mathrm{crys}}(\mathcal{X}_0/W)\big).$$

Next, we use the crystalline comparison of prismatic cohomology [BS22, Theorem 1.8(1)], and get the following diagram:

$$\begin{tikzcd}
\mathrm{H}^1_{\mathbb{\Delta}}(\mathcal{A}/\mathfrak{S})^{(1)} \arrow[r, two heads] \arrow[d, "g^*", "\cong"'] & \mathrm{H}^1_{\mathbb{\Delta}}(\mathcal{A}/\mathfrak{S})^{(1)}/u \arrow[r, "\cong"] \arrow[d, "g^*", "\cong"'] & \mathrm{H}^1_{\mathrm{crys}}(\mathcal{A}_0/W) \arrow[d, "g_0^*"] \\
\mathrm{H}^1_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)} \arrow[r, two heads] & \mathrm{H}^1_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}/u \arrow[r, hook] & \mathrm{H}^1_{\mathrm{crys}}(\mathcal{X}_0/W)
\end{tikzcd}$$

We postpone the proof of the left (and therefore the middle) vertical arrow being an isomorphism of φ-modules over $\mathfrak{S}$ to the next proposition. The right horizontal arrows are injective because of the standard sequence $0 \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathbb{\Delta}^{(1)})/u \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathbb{\Delta}^{(1)}/u) \to \mathrm{H}^{i+1}_{\mathrm{qSyn}}(\mathbb{\Delta}^{(1)})[u] \to 0$. The top right horizontal arrow is an isomorphism, as the (Breuil–Kisin) prismatic cohomology of abelian schemes is finite free, which in turn follows from the torsion-freeness of the crystalline cohomology of abelian varieties and Remark 3.9. The above diagram and sequence tell us that g_0^* is injective with cokernel given by $\mathrm{H}^2_{\mathbb{\Delta}^{(1)}}(\mathcal{X}/\mathfrak{S})[u]$. The description of (the semi-linear) Frobenius follows from Proposition 3.11(2), and the description of the linear Verschiebung follows from Corollary 3.17. $\square$

The following proposition was mentioned in the above proof.

Proposition 4.3.

(1) *The underlying $\mathfrak{S}$-module of $\mathrm{H}^1_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ is finite free.*
(2) *The map $g^*\colon \mathrm{H}^1_{\mathbb{\Delta}}(\mathcal{A}/\mathfrak{S}) \to \mathrm{H}^1_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ is an isomorphism of Kisin modules. Therefore, the Frobenius-twisted version $g^*\colon \mathrm{H}^1_{\mathbb{\Delta}}(\mathcal{A}/\mathfrak{S})^{(1)} \to \mathrm{H}^1_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})^{(1)}$ is also an isomorphism.*

Proof. (1) This follows from Corollary 3.8. Alternatively we can prove it using Remark 3.9 [Ill79, Remarque 3.11.2] and the fact that the crystalline cohomology of abelian varieties is torsion-free.

(2) Since étale realization of finite free Kisin modules is fully faithful (see [Kis06, Proposition 2.1.12] and also [BS21, Theorem 7.2]), we are reduced to checking that the étale realization of g^* is an isomorphism. Since the map $\mathfrak{S} \to A_{\mathrm{inf}}$ sending u to $[\underline{\pi}]$ is p-completely faithfully flat, it remains so after p-completely inverting u and $[\underline{\pi}]$, respectively. Therefore, we are further reduced to proving it for the $W(C^\flat)$-étale realizations. Now the étale comparison [BS22, Theorem 1.8(4)] translates the above to the statement that g_K induces an isomorphism of the first p-adic étale cohomology, which follows from the usual Kummer sequence together with the fact that the Picard variety of X is an abeloid.[7] □

We get two consequences from Theorem 4.2.

Corollary 4.4. *The finite group scheme G is connected.*

Proof. Since the induced Frobenius on $\mathbb{D}(G)$, when identified with $\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u] \subset \mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty]$, is divisible by u^{p-1}, powers of Frobenius will gain more and more u-divisibility. We see that the Frobenius is nilpotent as $\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u] \subset \mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty]$ and there is a power of u which kills the latter. Now Theorem 4.2 implies the Frobenius on $\mathbb{D}(G)$ is nilpotent, therefore G is connected. □

Remark 4.5. The above fact can actually be seen directly. Let us quotient out $\mathrm{Alb}(\mathcal{X}_0)$ by the neutral component subgroup scheme of G, denoted by $\mathcal{A}'_0$. Then we get a factorization $\mathcal{X}_0 \to \mathcal{A}'_0 \xrightarrow{f'_0} \mathcal{A}_0$ of g_0. Now f'_0 is finite étale by construction. Hence, deformation theory implies the above sequence lifts to $\mathcal{X} \to \mathcal{A}' \xrightarrow{f'} \mathcal{A}$, with $\mathcal{A}'$ being a formal abelian scheme finite étale above $\mathcal{A}$. Now the composition of the above map is the universal map from $\mathcal{X}$ to formal abelian schemes as pointed formal schemes,[8] and we conclude that the map f' has to be an isomorphism, hence the neutral component subgroup scheme of G is G itself.

Combining Theorem 4.2, Theorem 3.6 (with $i = 2$) and Corollary 3.18, we immediately obtain the following result.

Corollary 4.6.

(1) *If $e < p-1$ then the map $f\colon \mathrm{Alb}(\mathcal{X}_0) \to \mathrm{Alb}(X)_0$ is an isomorphism.*
(2) *If $e < 2(p-1)$ then $\ker(f)$ is p-torsion.*
(3) *If $e = p-1$ then $\ker(f)$ is p-torsion and of multiplicative type, hence must be a form of several copies of μ_p. Moreover, there is a canonical injection of $\mathcal{O}_K$-modules $\mathbb{D}(\ker(f)) \otimes_k (\mathcal{O}_K/p) \hookrightarrow \mathrm{H}^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.*

Proof. Parts (1) and (2) follow from Theorems 4.2 and 3.6(1)–(2) (with $i = 2$), respectively. As for the multiplicativity claim in (3), recall that a finite flat group scheme over k is of multiplicative type if and only if its Dieudonné module has bijective Verschiebung, hence the claim follows from Theorem 4.2 and Corollary 3.18. The last sentence follows from Corollary 3.22. □

When $e = 1$ and $p = 2$ the above says that although f need not be an isomorphism, the kernel is always a 2-torsion, such an interesting example can be found in [BMS18, § 2.1], and one

[7] Note that in general the Albanese of smooth proper rigid spaces (granting its existence) always induces an injective but not necessarily surjective map of the first étale cohomology, no matter whether ℓ-adic or p-adic (see [HL20, Proposition 4.4, Example 5.2 and Example 5.8]). The surjectivity is equivalent to the Picard variety being an abeloid (assuming p is invertible in the ground non-Archimedean field).

[8] We use the Néron mapping property and the fact that the generic fiber map is the Albanese map.

can check directly that the example there does satisfy our prediction here. In fact the f in their example can be identified with the relative Frobenius of an ordinary elliptic curve (which is the reduction of the E in their notation) over $\mathbb{F}_2$. For a generalization of this example to the case when $p \neq 2$, we refer readers to § 6 below, and specifically Remarks 6.11 and 6.14(3).

Remark 4.7. If $\mathrm{Pic}^0(\mathcal{X}_0)$ is reduced, then the relative (formal) Picard scheme of $\mathcal{X}/\mathcal{O}_K$ is a formal abelian scheme which is the Néron model of the Picard variety of X/K. The base change property of relative Picard functor now guarantees that the f we have been studying is an isomorphism in this case. Therefore, taking Theorem 4.2 into account, we see that $\mathcal{X}_0$ having reduced Picard scheme implies that the second prismatic cohomology of $\mathcal{X}$ has no u-torsion.

The dual question to what we have discussed here was studied by Raynaud [Ray79]. Below we recall some of the main results in that work and compare them with ours.

Remark 4.8. Using determinant construction [KM76], the universal line bundle on $\mathcal{X}_K \times_K \mathrm{Pic}^0(\mathcal{X}_K)$ extends to a line bundle on $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{P}$, where $\mathcal{P}$ is the formal Néron model of $\mathrm{Pic}^0(\mathcal{X}_K)$ which is itself a formal abelian scheme over $\mathcal{O}_K$. Here we used the regularity of $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{P}$ so that any coherent sheaf on it can be presented as a perfect complex, in order to perform the determinant construction. Moreover, if we rigidify using the given point $x \in \mathcal{X}(\mathcal{O}_K)$, then the extension as a rigidified line bundle is unique. Taking the special fiber, we get an induced map $\mathcal{P}_0 \to \mathrm{Pic}^0(\mathcal{X}_0)$ which necessarily factors through the reduced subvariety of the target $f^\vee \colon \mathcal{P}_0 \to \mathrm{Pic}^0(\mathcal{X}_0)_{\mathrm{red}}$. By construction, the map $f^\vee$ is dual to the map f we considered before.

Raynaud has studied the question of whether $f^\vee$ is an isomorphism in [Ray79]. His main result makes the following assertions.

(1) When $e < p - 1$, then $f^\vee$ is an isomorphism [Ray79, Thèoréme 4.1.3.(2)].
(2) When $e = p - 1$, then $\ker(f^\vee)$ is p-torsion and unramified [Ray79, Thèoréme 4.1.3.(3)].

We see that his results are the same as Corollary 4.6(1) and first half of (3); our slight improvement is Corollary 4.6(2) and second half of (3). We prove the map $f^\vee$ has p-torsion kernel in a larger range of ramifications, and when $e = p - 1$ the second cohomology of structure sheaf needs to have 'actual' p-torsion in order for $\ker(f)$ to be non-zero. On the other hand, Raynaud's result allows $\mathcal{X}$ to be singular: for instance, he just needs $\mathcal{X}_0$ to be normal. Our method crucially relies on prismatic theory, which only seems to work well with local complete intersection singularities. Whether our Corollary 4.6 can be extended to the generality considered by Raynaud remains unclear.

Remark 4.9. One of the key ingredients allowing Raynaud to prove the aforementioned results in [Ray79] is an earlier result of his [Ray74] concerning prolongations of finite flat commutative group schemes. In § 6.1 we shall see a way to go backward: applying these structural results on Picard/Albanese varieties to a marvelous construction due to Bhatt, Morrow and Scholze [BMS18], one deduces Raynaud's prolongation theorem.

4.2 The p-adic specialization maps

Another reason why one might care about u^∞-torsion is because it appears naturally in understanding the specialization map of the p-adic étale cohomology or, phrased differently, the p-adic vanishing cycle.

Let us introduce some notation. Fix a complete algebraically closed non-Archimedean extension C of K, with ring of integers $\mathcal{O}_C$. Denote the perfect prism associated with $\mathcal{O}_C$, which is known to be oriented, by $(A_{\mathrm{inf}}, (\xi))$. Given a p-adic formal scheme $\mathcal{X}$ over $\mathrm{Spf}(\mathcal{O}_K)$, we denote its base change to $\mathcal{O}_C$ (respectively, C) by $\mathcal{X}_{\mathcal{O}_C}$ (respectively, $\mathcal{X}_C$). Denote the central fiber of $\mathcal{X}_{\mathcal{O}_C}$ by $\mathcal{X}_{\overline{k}}$. We continue to assume $\mathcal{X}$ to be smooth and proper over $\mathrm{Spf}(\mathcal{O}_K)$.

Recall that the proper base change theorem gives, for any prime ℓ, a specialization map [Sta21, Tag 0GJ2]

$$\mathrm{Sp}\colon \mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}_\ell) \to \mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}_\ell).$$

The cone of the specialization map above is called the vanishing cycle (of $\mathbb{Z}_\ell$). The smooth base change theorem says that the above map is an isomorphism for any $\ell \neq p$ [Sta21, Tag 0GKD], in other words the ℓ-adic vanishing cycle vanishes in our setting. On the other hand, one may ask what happens when $\ell = p$. Fix a cohomological degree i and $n \in \mathbb{N} \cup \{\infty\}$. Let us consider

$$\mathrm{Sp}^i_n\colon \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \to \mathrm{H}^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p^n);$$

when $n = \infty$, then by $\mathbb{Z}/p^n$ we mean $\mathbb{Z}_p$ and we simply denote it by Sp^i. It is well known that Sp^i is almost never surjective unless for trivial reasons such as the target being 0. We shall consider $\ker(\mathrm{Sp}^i)$ in this subsection.

In [BS22, § 9] one finds a prismatic interpretation of the p-adic specialization map.

THEOREM 4.10 [BS22, Theorem 9.1 and Remark 9.3]. *There are canonical identifications*

$$\mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p^n) \cong \big((\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^{\wedge}/p^n\big)^{\varphi=1}$$

and

$$\mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \cong \big(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)^{\varphi=1},$$

fitting in the following diagram, which is commutative up to coherent homotopy:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \arrow[r, "\cong"] \arrow[d, "\mathrm{Sp}"'] & \big(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)^{\varphi=1} \arrow[d, "\mathrm{incl}"] \\
\mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p^n) \arrow[r, "\cong"] & \big((\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^{\wedge}/p^n\big)^{\varphi=1}
\end{tikzcd}$$

Here $(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^{\wedge}$ denotes the p-completion of the localization, which is only relevant in the statement when $n = \infty$. This theorem is true with no smooth or proper assumption on $\mathcal{X}$: one may safely replace $\mathcal{X}_{\mathcal{O}_C}$ over $\mathrm{Spf}(\mathcal{O}_C)$ with any p-adic formal scheme $\mathcal{Y}$ over a perfectoid base ring as in loc. cit.

Sketch of proof following that of [BS22]. The first identification is [BS22, Theorem 9.1], the second identification is [BS22, Remark 9.3] with details left to readers, so let us fill in some of the details.

We follow the proof of [BS22, Theorem 9.1]. First we see that both $\mathrm{R}\Gamma_{\text{ét}}((-)_{\overline{k}}, \mathbb{Z}/p^n)$ and $\big(\Delta_{-/A_{\mathrm{inf}}}/p^n\big)^{\varphi=1}$ are arc-sheaves (see [BM21] for more details on this notion) on $\mathrm{fSch}_{/\,\mathrm{Spf}(\mathcal{O}_C)}$. The former is [BM21, Theorem 5.4], the latter follows from the same argument as in [BS22, Theorem 9.1]: using [BS22, Lemma 9.2], one has an identification $\big(\Delta_{-/A_{\mathrm{inf}}}/p^n\big)^{\varphi=1} \cong \big(\Delta_{-/A_{\mathrm{inf}},\mathrm{perf}}/p^n\big)^{\varphi=1}$, and then one again uses [BS22, Corollary 8.10] to see that the latter is an arc-sheaf.

Since everything involved is an arc-sheaf and is arc-locally supported in cohomological degree 0, the relevant maps (of arc-sheaves) live in mapping spaces with contractible components.

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Altogether we get the following diagram which commutes up to coherent homotopy:

$$\begin{tikzcd} \mathrm{R}\Gamma_{\text{ét}}((-)_{\overline{k}}, \mathbb{Z}/p^n) \arrow[r] \arrow[d, "\mathrm{Sp}"'] & \left(\Delta_{-/A_{\inf}}/p^n\right)^{\varphi=1} \arrow[d, "\mathrm{incl}"] \\ \mathrm{R}\Gamma_{\text{ét}}((-)_C, \mathbb{Z}/p^n) \arrow[r, "\cong"] & \left((\Delta_{-/A_{\inf}}[1/\xi])^{\wedge}/p^n\right)^{\varphi=1} \end{tikzcd}$$

Finally, we need to show the top horizontal arrow is an isomorphism. We may localize in the arc-topology, reducing to the case of Spf of a perfectoid ring S, which follows from applying Artin–Schreier–Witt and the fact that perfection does not change the étale site (of a characteristic p scheme). $\square$

In order to pass from the derived statement above to concrete cohomology groups, we need the following lemma.

Lemma 4.11. *For any i and n, the $\mathbb{Z}_p$-linear operator $\varphi - 1$ is surjective on both $\mathrm{H}^i\big((\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\inf})[1/\xi])^{\wedge}/p^n\big)$ and $\mathrm{H}^i\big(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\inf})/p^n\big)$.*

Proof. We observe that, since $\varphi([a]) = [a]^p$ for any $a \in \mathfrak{m}_C^\flat$, we know that φ acts topologically nilpotently on

$$[\mathfrak{m}_C^\flat] \cdot \mathrm{H}^i\big(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\inf})/p^n\big).$$

Therefore, to check surjectivity of $\varphi - 1$ on $\mathrm{H}^i\big(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\inf})/p^n\big)$ we may quotient out $W(\mathfrak{m}_C^\flat) \cdot \mathrm{H}^i\big(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\inf})/p^n\big)$. Since $\mathcal{X}$ is smooth and proper over $\mathcal{O}_K$, we know the relevant groups are finitely generated modules over $W(C^\flat)$ and $W(\overline{k})$. Both of $C^\flat$ and $\overline{k}$ are algebraically closed fields of characteristic p, hence we are reduced to [Cha98, Exposé III, Lemma 3.3]. $\square$

Using the same proof, we may identify the p-adic étale cohomology of $\mathcal{X}_{\overline{k}}$ as Frobenius fixed points in various prismatic cohomologies of $\mathcal{X}$, after suitably base-changing to $W(\overline{k})$.

Porism 4.12. *Consider the $\mathfrak{S}$-algebra $W(\overline{k})[\![u]\!]$. We have an identification of G_k-modules:*

$$\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \cong \Big(\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n) \otimes_{\mathfrak{S}} W(\overline{k})[\![u]\!]\Big)^{\varphi=1} \cong \Big(\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n^{(1)}) \otimes_{\mathfrak{S}} W(\overline{k})[\![u]\!]\Big)^{\varphi=1}.$$

Proof. As showed in the proof of Lemma 4.11, we may compute Frobenius fixed points after quotienting out $W(\mathfrak{m}_C^\flat)$ (for the $A_{\inf}$-module) or u for the Frobenius module appearing in this porism. Now the first identification is reduced to Theorem 4.10 and an equality of $\mathfrak{S}$-algebras: $A_{\inf}/W(\mathfrak{m}_C^\flat) \cong W(\overline{k}) \cong W(\overline{k})[\![u]\!]/(u)$. The second identification is reduced to the fact that, given a Frobenius module M on $W(\overline{k})$, the natural map $M \to M \otimes_{W(\overline{k}),\varphi} W(\overline{k})$ given by $m \mapsto m \otimes 1$ induces an isomorphism of Frobenius fixed points. $\square$

Remark 4.13. Assume that the residue field $\overline{k}$ of $\mathcal{O}_K$ is separably closed. Porism 4.12 above induces a map

$$\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \cong \Big(\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n^{(1)})\Big)^{\varphi=1} \hookrightarrow \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n^{(1)}) \to \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n).$$

This map can be seen at the level of étale sheaves on $\mathrm{fSch}_{/\,\mathrm{Spf}(\mathcal{O}_K)}$: $\mathbb{Z}_p/p^n \to \Delta_n^{(1)} \to \mathcal{O}_{\mathcal{X}}/p^n$. Therefore, we get a canonical map

$$\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \otimes_{\mathbb{Z}_p} W \to \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n).$$

In general, we just base-change along $W(k) \to W(\overline{k})$ and get a G_k-equivariant map

$$\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \otimes_{\mathbb{Z}_p} W(\overline{k}) \to \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n) \otimes_W W(\overline{k}).$$

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Later in Corollary 4.15(3) we shall see a peculiar result concerning this map in the boundary degree. Now we return to the relation between kernel of specialization map and u^∞-torsion in prismatic cohomology.

Theorem 4.14. *Let $\mathcal{X}$ be a smooth proper formal scheme over* $\mathrm{Spf}(\mathcal{O}_K)$. *Recall* $\mathfrak{M}^i_n := \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty]$. *There is a canonical isomorphism of G_k-modules*

$$\ker(\mathrm{Sp}^i_n) \cong \left(\mathfrak{M}^i_n \otimes_{\mathfrak{S}} A_{\mathrm{inf}}\right)^{\varphi=1} \cong \left(\mathfrak{M}^i_n/u \otimes_{W(k)} W(\overline{k})\right)^{\varphi=1}$$

for any $n \in \mathbb{N} \cup \{\infty\}$.

Proof. Combining Theorem 4.10 and Lemma 4.11, we get the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \arrow[r] \arrow[d, "\mathrm{Sp}^i_n"] & \mathrm{H}^i\big(\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big) \arrow[r, "\varphi-1"] \arrow[d, "\mathrm{incl}"] & \mathrm{H}^i\big(\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big) \arrow[r] \arrow[d, "\mathrm{incl}"] & 0 \\
0 \arrow[r] & \mathrm{H}^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p^n) \arrow[r] & \mathrm{H}^i\big((\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^\wedge/p^n\big) \arrow[r, "\varphi-1"] & \mathrm{H}^i\big((\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^\wedge/p^n\big) \arrow[r] & 0
\end{tikzcd}$$

We shall apply the snake lemma to the above. First, we claim

$$\mathrm{H}^i\big(\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)[\xi^\infty] \cong \ker(\mathrm{H}^i\big(\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big) \xrightarrow{\mathrm{incl}} \mathrm{H}^i\big((\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^\wedge/p^n\big).$$

When $n \in \mathbb{N}$ the map is localization with respect to ξ, hence tautological. We need to see this when $n = \infty$, that is, we need to show injectivity of $\mathrm{H}^i\big(\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi]\big) \to \mathrm{H}^i(\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^\wedge$. Here the latter completion is the classical p-adic completion: our assumption implies all cohomology groups $\mathrm{H}^i_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})$ have bounded p-torsion, hence derived p-completion agrees with derived p-completion. Since the $\mathrm{H}^i_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})$ are finitely presented over A_{inf}, its localization with respect to ξ has separated p-adic topology, hence the p-adic completion map is injective.

Next, applying the base change property of prismatic cohomology to the p-completely faithfully flat map $\mathfrak{S} \to A_{\mathrm{inf}}$ and [BMS18, Proposition 4.3], we get an identification of Frobenius modules:

$$\mathfrak{M}^i_n \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \cong \mathrm{H}^i\big(\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)[\xi^\infty].$$

Now we get the first identification. To finish, just observe that $\varphi([a]) = [a]^p$, for any $a \in \mathfrak{m}^\flat_C$, which acts nilpotently on $\mathrm{H}^i\big(\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)[\xi^\infty]$. Hence, the map $\varphi - 1$ is necessarily an isomorphism (of $\mathbb{Z}_p$-modules) on $[\mathfrak{m}^\flat] \cdot \mathrm{H}^i\big(\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)[\xi^\infty]$. Therefore, we may quotient this part, as far as Frobenius fixed points are concerned, which leads to the second identification. □

Corollary 4.15. *Let $\mathcal{X}$ be a smooth proper formal scheme over* $\mathrm{Spf}(\mathcal{O}_K)$ *with ramification index e, and let $i \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$. We have the following understanding of the kernel of the specialization map* $\mathrm{Sp}^i_n \colon \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \to \mathrm{H}^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p^n)$.

(1) *If $e \cdot (i-1) < p-1$, then Sp^i_n is injective.*
(2) *If $e \cdot (i-1) < 2(p-1)$, then $\ker(\mathrm{Sp}^i_n)$ is annihilated by p^{i-1}.*
(3) *If $e \cdot (i-1) = p-1$, then $\ker(\mathrm{Sp}^i_n)$ is p-torsion, and corresponds to the étale-φ module $\mathfrak{M}^i_n$ over k. Moreover, the natural G_k-equivariant map in Remark 4.13,*

$$\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \otimes_{\mathbb{Z}_p} W(\overline{k}) \to \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n) \otimes_W W(\overline{k}),$$

induces a G_k-equivariant injection,

$$\ker(\mathrm{Sp}^i_n) \otimes_{\mathbb{F}_p} \big(\mathcal{O}_K \otimes_W W(\overline{k})\big)/p \hookrightarrow \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n) \otimes_W W(\overline{k}).$$

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Proof. All but the last statement immediately follow from Theorems 3.6 and 4.14. The last statement is a Galois-theoretic analog of Corollary 3.22. To prove this, we may base-change $\mathcal{X}$ from $\mathcal{O}_K$ to $\mathcal{O}_K \otimes_W W(\overline{k})$ and it suffices to prove the statement there. Hence, it suffices to assume that $\mathcal{O}_K$ has algebraically closed residue field $\overline{k}$.

Let us analyze the sequence of maps of $\mathfrak{S}$-modules

$$\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \cong \left(\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}_n)\right)^{\varphi=1} \hookrightarrow \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}_n) \to \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n).$$

By Corollary 4.15(3), we see the first map induces an isomorphism:

$$\ker(\mathrm{Sp}^i_n) \otimes_{\mathbb{F}_p} \overline{k}[u]/(u^p) \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}_n)[u^\infty].$$

The exact sequence $\mathrm{Fil}^1_{\mathrm{N}} \to \mathbb{\Delta}^{(1)} \to \mathcal{O}_{\mathcal{X}}$ tells us that the kernel of the map

$$\ker(\mathrm{Sp}^i_n) \otimes_{\mathbb{F}_p} \overline{k}[u]/(u^p) \to \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n)$$

is given by the induced first Nygaard filtration on the source, which we know is exactly u^e times the source, thanks to Proposition 3.20(1). Notice that, as an $\mathcal{O}_K$-algebra, we have $\overline{k}[u]/(u^e) \cong \mathcal{O}_K/p$. Therefore, we get the desired injection

$$\ker(\mathrm{Sp}^i_n) \otimes_{\mathbb{F}_p} \overline{k}[u]/(u^e) \hookrightarrow \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n). \qquad \square$$

We refer readers to § 6, especially Remarks 6.11 and 6.14(4), for a related interesting example.

4.3 Revisiting the integral Hodge–de Rham spectral sequence

In this subsection, we revisit the question discussed in [Li22]: what mild condition on $\mathcal{X}$ guarantees that the Hodge numbers of the generic fiber X can be read off from the special fiber $\mathcal{X}_0$?

Let us introduce some notation, which is the threshold of cohomological degree for which we can say something about the integral Hodge–de Rham spectral sequence, based on knowledge of the integral Hodge–Tate spectral sequence.

Notation 4.16. Let T be the largest integer such that $e \cdot (T-1) \leq p-1$.

The main result in this subsection is the following theorem.

Theorem 4.17 (Improvement of [Li22, Theorem 1.1]). *Let $\mathcal{X}$ be a smooth proper p-adic formal scheme over* $\mathrm{Spf}(\mathcal{O}_K)$.

(1) *Assume there is a lift of $\mathcal{X}$ to $\mathfrak{S}/(E^2)$. Then for all i, j satisfying $i+j < T$, we have equalities*

$$h^{i,j}(X) = \mathfrak{h}^{i,j}(\mathcal{X}_0)$$

where the latter denotes virtual Hodge numbers of $\mathcal{X}_0$, defined as in [Li22, Definition 3.1].

(2) *Assume, furthermore, that $e \cdot (\dim \mathcal{X}_0 - 1) \leq p-1$. Then the special fiber $\mathcal{X}_0$ knows the Hodge numbers of the rigid generic fiber X.*

For instance, in the unramified case $e=1$, condition (1) is automatic and condition (2) says we allow $\mathcal{X}$ to be at most dimension p. From the proof, we shall see that the Hodge numbers of X can be computed using the virtual Hodge numbers of $\mathcal{X}_0$ (see [Li22, § 3.2]) together with Euler characteristics of the $\Omega^i_{\mathcal{X}_0}$ in an algorithmic way.

We largely follow the proof of [Li22, Theorem 1.1]. Just like there, we need to first analyze the integral Hodge–de Rham spectral sequence, hence the title of this subsection.

Theorem 4.18. *Let $\mathcal{X}$ be a smooth proper p-adic formal scheme over* $\mathrm{Spf}(\mathcal{O}_K)$ *liftable to* $\mathfrak{S}/(E^2)$. *Let $n \in \mathbb{N} \cup \{\infty\}$.*

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(1) *The Hodge–de Rham spectral sequence for $\mathcal{X}_n$ has no non-zero differentials with source of total degree $< T$.*
(2) *If $e > 1$, then $\mathfrak{M}_n^T := \mathrm{H}^T_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty] = 0$. In particular, the prismatic cohomology $\mathrm{H}^m_\Delta(\mathcal{X}/\mathfrak{S}) \simeq M_m \otimes_{\mathbb{Z}_p} \mathfrak{S}$ is of the shape of a $\mathbb{Z}_p$-module M_m for all $m \le T$.*
(3) *If $e = 1$, The induced Hodge filtrations $\mathrm{H}^i(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{H}}) \subset \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X})$ are split for any $i \le p$ and any j.*
(4) *If $e > 1$, The induced Hodge filtrations $\mathrm{H}^i(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{H}}) \subset \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X})$ are split for any $i < T$ and any j.*

Here $\mathcal{X}_n$ denotes the mod p^n fiber. We do not know if the split statement in (3) above holds at the mod p^n level. Mimicking the terminology in [Li22], we may say the Hodge–de Rham sequence for $\mathcal{X}_n$ is split degenerate up to degree T. We need some preparation.

LEMMA 4.19.

(1) *If $e = 1$, we have $\ell\big(\mathrm{Tor}_1^{\mathfrak{S}}(k, \mathcal{O}_K)\big) = \ell\big(\mathrm{Tor}_1^{\mathfrak{S}}(k, \varphi_{\mathfrak{S},*}\mathcal{O}_K)\big)$.*
(2) *If $e > 1$, we have $\ell\big(\mathrm{Tor}_1^{\mathfrak{S}}(k, \mathcal{O}_K)\big) < \ell\big(\mathrm{Tor}_1^{\mathfrak{S}}(k, \varphi_{\mathfrak{S},*}\mathcal{O}_K)\big)$.*
(3) *Let M be a finitely generated p^∞-torsion $\mathfrak{S}$-module without u-torsion. Then*

$$\ell\big(M \otimes_{\mathfrak{S}} \mathcal{O}_K\big) = \ell\big(M \otimes_{\mathfrak{S},\varphi_{\mathfrak{S}}} \mathcal{O}_K\big).$$

Here $\ell(-)$ denotes length of the $\mathcal{O}_K$-module.

Proof. For (1) and (2), simply note that $\mathrm{Tor}_1^{\mathfrak{S}}(k, \mathcal{O}_K)$ is the u^e-torsion in $k = \mathfrak{S}/(p, u)$, whereas the module $\mathrm{Tor}_1^{\mathfrak{S}}(k, (\varphi_{\mathfrak{S}})_*\mathcal{O}_K)$ is the u^e-torsion in $k \otimes_{\mathfrak{S},\varphi_{\mathfrak{S}}} \mathfrak{S} = \mathfrak{S}/(p, u^p)$.

For (3), it is easy to see that the condition guarantees a finite filtration on M with graded pieces given by $\mathfrak{S}/p \cong k[\![u]\!]$. Indeed, we just argue by induction on the exponent of powers of p that annihilates M and consider the sequence

$$0 \to M[p] \to M \to M/M[p] \to 0.$$

Hence, the equality of lengths follows from the equality of $\mathfrak{S}/p \otimes_{\mathfrak{S},\varphi_{\mathfrak{S}}} \mathfrak{S} \simeq \mathfrak{S}/p$. □

LEMMA 4.20. *Let $F \subset M$ be an inclusion of finitely generated $W(k)$-modules. If the induced maps $F/p^n \to M/p^n$ are injective for any $n \in \mathbb{N}$, then F is a direct summand in M.*

Proof. Denoting M/F by C, the condition implies that $M[p^n] \twoheadrightarrow C[p^n]$ for all n. Writing the torsion submodule C_{tor} as direct sums of cyclic torsion $W(k)$-modules, and using the condition, we see that each cyclic summand admits a section back to M. This way we see that the extension class restricts to 0 in $\mathrm{Ext}^1_{W(k)}(C_{\mathrm{tor}}, F)$, hence it must come from a class in $\mathrm{Ext}^1_{W(k)}(C/C_{\mathrm{tor}}, F)$. But now C/C_{tor} is finitely generated torsion-free $W(k)$-module, which is well known to be a free $W(k)$-module, hence the extension group is 0. □

Proof of Theorem 4.18. Let us show (1) and (2). The case of $n = \infty$ follows from the finite n case: for (1) this is by left exactness of taking inverse limit, for (2) it follows from Proposition 2.6. Now assuming $n \in \mathbb{N}$, the degeneration statement is equivalent to equality of lengths

$$\ell(\mathrm{H}^m_{\mathrm{dR}}(\mathcal{X}_n)) = \sum_{i+j=m} \ell(\mathrm{H}^{i,j}(\mathcal{X}_n)),$$

for any $m < T$. Note that by the mere existence of the Hodge–de Rham spectral sequence, we have the inequality

$$\ell(\mathrm{H}^m_{\mathrm{dR}}(\mathcal{X}_n)) \le \sum_{i+j=m} \ell(\mathrm{H}^{i,j}(\mathcal{X}_n))$$

for free for any m. Below we shall try to show the converse inequality for $m < T$.

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To that end, by the same argument as in the first paragraph of [Li22, Proof of Theorem 1.1], the liftability condition implies that the Hodge–Tate complex in degrees $\leq p-1$ splits into direct sums of its cohomology sheaves (see [BS22, Remark 4.13 and Proposition 4.14], [ALB23, Proposition 3.2.1], and [LL20, Corollary 4.23]). In particular, since $T-1 \leq p-1$ we have a splitting of $\mathcal{O}_K$-modules: $\mathrm{H}^m_{\mathrm{HT}}(\mathcal{X}_n) \simeq \bigoplus_{i+j=m} \mathrm{H}^{i,j}(\mathcal{X}_n)$ for any $m < T$. Here the Hodge–Tate cohomology of $\mathcal{X}_n$ is defined to be the quasi-syntomic cohomology of the mod p^n of the Hodge–Tate sheaf $\overline{\mathcal{O}}_{\mathbb{\Delta}}$. What remains to be shown is an inequality of length

$$\ell(\mathrm{H}^m_{\mathrm{HT}}(\mathcal{X}_n)) \leq \ell(\mathrm{H}^m_{\mathrm{dR}}(\mathcal{X}_n)).$$

By the Hodge–Tate and de Rham comparisons of prismatic cohomology [BS22, Theorem 4.10 and Corollary 15.4], we have equalities

$$\ell(\mathrm{H}^m_{\mathrm{HT}}(\mathcal{X}_n)) = \ell\big(\mathrm{H}^m_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n) \otimes_{\mathfrak{S}} \mathcal{O}_K\big) + \ell\big(\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}_n^{m+1}, \mathcal{O}_K)\big)$$

and

$$\ell(\mathrm{H}^m_{\mathrm{dR}}(\mathcal{X}_n)) = \ell\big(\mathrm{H}^m_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n) \otimes_{\mathfrak{S},\varphi_{\mathfrak{S}}} \mathcal{O}_K\big) + \ell\big(\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}_n^{m+1}, (\varphi_{\mathfrak{S}})_*\mathcal{O}_K)\big).$$

Now the desired inequality between the length of Hodge–Tate and de Rham cohomology follows from the definition of T, the inequality $m < T$, Theorem 3.6, and Lemma 4.19. This finishes the proof of (1).

Turning to (2), note that by Theorem 3.6(3), if $\mathfrak{M}_n^T$ were non-zero, it would necessarily be a direct sum of k as an $\mathfrak{S}$-module. Then Lemma 4.19(2) shows that when $e > 1$, the strict inequality

$$\ell(\mathrm{H}^{T-1}_{\mathrm{HT}}(\mathcal{X}_n)) < \ell(\mathrm{H}^{T-1}_{\mathrm{dR}}(\mathcal{X}_n))$$

holds, which violates the fact that the left-hand side is the same as the sum of the lengths of Hodge cohomology groups whereas the right-hand side is at most that sum. Hence, we arrive at a contradiction. The vanishing of $\mathfrak{M}_n^m$ when $m < T$ already follows from Theorem 3.6(1). The statement concerning structure of prismatic cohomology now follows from Proposition 2.6.

Now we turn to (3): $e = 1$, hence $T = p$. In this case, the statement (1) we proved above implies that for any $i \leq p$ and any j, the map $\mathrm{H}^i(\mathcal{X}_n, \mathrm{Fil}^j_{\mathrm{H}}) \to \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}_n)$ is injective. Hence, the submodule $\mathrm{H}^i(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{H}}) \subset \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X})$ has the property that it induces an injection modulo any p^n. The desired splitness follows from Lemma 4.20.

Finally, we show (4): when $e > 1$. We follow the argument of [Li22, Corollary 3.9]. Using the vanishing statement established in (2), it follows that we have abstract isomorphism $\mathrm{H}^m_{\mathrm{HT}}(\mathcal{X}) \simeq \mathrm{H}^m_{\mathrm{dR}}(\mathcal{X})$ whenever $m < T$. Hence, the argument of [Li22, Corollary 3.9] shows that in the range $m < T$, the splitting of the Hodge–Tate filtration on $\mathrm{H}^m_{\mathrm{HT}}(\mathcal{X})$ is equivalent to the splitting of the Hodge filtration on $\mathrm{H}^m_{\mathrm{dR}}(\mathcal{X})$. We can then finish our proof, as liftability to $\mathfrak{S}/(E^2)$ gives the desired splitting of the Hodge–Tate filtration in the range $m < T \leq p$. □

Remark 4.21. Comparing our Theorem 4.18(1) with what Fontaine and Messing obtained [FM87, II.2.7.(i)] (assuming the existence of a lifting over W), we seemingly get a stronger statement. Fontaine and Messing only proved a degeneration statement when the differential has a target of degree $< p$ whereas ours allows the differential to have source of degree $< p$ (so the target can have degree p). However, this is due to Fontaine and Messing not trying to squeeze their method to the most optimal, which is understandable given how many indices they needed to take care of. Indeed, their [FM87, II.2.6.(ii)] implies the map in the next degree (following their notation) $\bigoplus_{r=1}^t \mathrm{H}^{m+1}(J_n^{[r]}) \to \bigoplus_{r=0}^t \mathrm{H}^{m+1}(J_n^{[r]})$ is injective, which can be used to strengthen their [FM87, II.2.7.(i)], hence also gaining the extra degeneration statement we obtained here.

We are now in a position to prove the main theorem in this subsection.

Proof of Theorem 4.17. Fix an $m < T$ and a $j \in \mathbb{N}$. We consider the map of two $\mathcal{O}_K$-complexes

$$\mathrm{R}\Gamma(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{H}}) \to \mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K).$$

Our Theorem 4.18(1), (3) and (4) implies that this map in degree m satisfies the assumption of [Li22, Lemma 2.16] (with our m being the n in that lemma). We finish the proof of (1) by combining the conclusion of [Li22, Lemma 2.16] with the definition of Hodge numbers of X and virtual Hodge numbers of $\mathcal{X}_0$.

The fact that (1) implies (2) is rather a brain teaser. In the Hodge diamond of X, all numbers below the middle row, which is the row with total degree given by $\dim(X)$ ($\leq T$ by assumption), are given by the corresponding virtual Hodge number of $\mathcal{X}_0$. Serre duality implies that $\mathcal{X}_0$ also knows all numbers above the middle row. Now for the middle row, simply use the fact that Euler characteristic is locally constant for any flat family of coherent sheaves. □

For the rest of this subsection, let us specialize to the case of $e = 1$. Using knowledge on the Hodge–de Rham spectral sequence, we have a similar degeneration of the 'Nygaard–Prism' spectral sequence up to cohomological degree p.

Theorem 4.22. *Assume $e = 1$ (so $\mathcal{O}_K = W$), and let $n \in \mathbb{N} \cup \{\infty\}$. The map*

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}_n)$$

is injective, when $i < p$ or $i = p, j \leq p-1$.

Recall that when $i = p$ and $j \geq p$, kernels of these maps have been studied in Proposition 3.20(3).

Proof. We shall induct on j, the case of $j = 0$ being trivial. We need to consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n) \otimes_{\mathfrak{S}} (E) \arrow[r] \arrow[d, hook] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{j+1}_{\mathrm{N}}/p^n) \arrow[r] \arrow[d] & \mathrm{H}^i(\mathcal{X}, \Omega^{\geq j+1}_{\mathcal{X}/W}/p^n) \arrow[d, hook] \\
\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}_n) \otimes_{\mathfrak{S}} (E) \arrow[r] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}_n) \arrow[r] & \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}_n/W_n)
\end{tikzcd}$$

The rows are exact as they are part of long exact sequences, coming from exact sequences of sheaves on $\mathcal{X}_{\mathrm{qSyn}}$. The right vertical arrow is injective for all $i \leq p$ thanks to Theorem 4.18(1); note that $T = p$ as $e = 1$. The left vertical arrow is injective by induction hypothesis.

Let us first show the statement for $i < p$. Take an element in the kernel of the middle vertical arrow; by diagram-chasing we see that the element comes from an element α in $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n) \otimes_{\mathfrak{S}} (E)$. Now it suffices to show that the image of α in $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}_n) \otimes_{\mathfrak{S}} (E)$ is zero. Finally, we note that the further image in $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}_n)$ is zero, therefore it suffices to know $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}_n)$ has no E-torsion, or equivalently it has no u-torsion, thanks to Theorem 3.6(1).

Finally, let us show the statement when $i = p$, and let $j + 1 \leq p - 1$. Arguing as in the previous paragraph, we are reduced to showing that, given an element $\beta_j \in \mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n)$ whose image γ in $\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}_n)$ is an E-torsion, the image of $\beta_j \otimes E$ in $\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{j+1}_{\mathrm{N}}/p^n)$ is

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already zero. To that end, we need the help of another diagram:

$$\begin{tikzcd}
\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{p-1}_{\mathrm{N}}/p^n) \otimes_{\mathfrak{S}} (E) \arrow[r] \arrow[d] & \mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{p-1}_{\mathrm{N}}/p^n) \arrow[d] \\
\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{j}_{\mathrm{N}}/p^n) \otimes_{\mathfrak{S}} (E) \arrow[r] \arrow[d, hook] & \mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{j+1}_{\mathrm{N}}/p^n) \arrow[d] \\
\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n) \otimes_{\mathfrak{S}} (E) \arrow[r] & \mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)
\end{tikzcd}$$

We point out that the two vertical arrows in the top square are both injective because of Proposition 3.20(2), although we make no further use of this. Since γ is an E-torsion, we know it is (u,p)-torsion (see Theorem 3.6(3)). Therefore we see γ is the image of a (u,p)-torsion β_{p-1} in $\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{p-1}_{\mathrm{N}}/p^n)$ thanks to Proposition 3.20(1). By the induction hypothesis, we see that the image of β_{p-1} in $\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{j}_{\mathrm{N}}/p^n)$ is precisely β_j. Now we are done as $E \cdot \beta_{p-1} = 0$ in $\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{p-1}_{\mathrm{N}}/p^n)$. ∎

5. Crystalline cohomology in boundary degree

Notation 5.1. Throughout this section let us fix $n \in \mathbb{Z} \cup \{\infty\}$, and fix e, i such that $e \cdot i = p-1$. Let S be the PD envelope of $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$, and let $c_1 = \varphi(E)/p \in S^\times$. Denote $\mathfrak{S}_n \coloneqq \mathfrak{S}/p^n$ and $S_n \coloneqq S/p^n$. Let $\mathcal{X}$ be a smooth proper formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$. Let $\mathfrak{M} \coloneqq \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)$, let $\mathcal{M} \coloneqq \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{O}_{\mathrm{crys}}) \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{dR}_{-/\mathfrak{S}}/p^n)$, and finally let $V \coloneqq \mathrm{H}^{i+1}_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty]$. We use Frob_k to denote the Frobenius on k.

Recall that, by Theorem 3.6, the module $\mathfrak{M}$ is u-torsion-free and the Frobenius $\mathfrak{S}$-module V is an étale φ-module over k. Also recall ([BS22, Theorem 5.2] and [LL20, Theorem 3.5 and Lemma 7.16]) that we have a short exact sequence of Frobenius S-modules:

$$0 \to \mathfrak{M} \otimes_{\mathfrak{S}_n} S_n \to \mathcal{M} \to \mathrm{Tor}_1^{\mathfrak{S}_n}(V, \varphi_* S_n) \cong \mathrm{Tor}_1^{\mathfrak{S}_1}(V, \varphi_* S_1) =: \overline{M} \to 0, \qquad (\boxdot)$$

where the last equality follows from the fact that $S_1 = \mathfrak{S}_1 \otimes^{\mathbb{L}}_{\mathfrak{S}_n} S_n$. Here, by assumption on $\mathcal{X}$, we know $\mathfrak{M}$ is finitely generated over $\mathfrak{S}$ and we can replace completed tensor with tensor to ease notation a little bit.

Let us give a functorial description of $\overline{M}$.

Lemma 5.2. *Let N be an $\mathfrak{S}_1$-module. Then we have identifications of $\mathfrak{S}_1$-modules:*

(1) $\mathrm{Tor}_1^{\mathfrak{S}_1}(V, N) \cong V \otimes_k \big(N[u]\big)$*; and*
(2) $\mathrm{Tor}_1^{\mathfrak{S}_1}(V, \varphi_* N) \cong \mathrm{Frob}_k^*(V) \otimes_k \big(N[u^p]\big)$.

Here the $\mathfrak{S}$-module structures on the right-hand sides are via the second factor.

In particular, we have $\overline{M} \cong \mathrm{Frob}_k^*(V) \otimes_k S_1[u^p]$.

Proof. Let us prove (2) here as the proof of (1) follows a similar argument. Note that

$$V \otimes^{\mathbb{L}}_{\mathfrak{S}_1,\varphi} N = V \otimes^{\mathbb{L}}_{k,id} k \otimes^{\mathbb{L}}_{\mathfrak{S}_1,\varphi} N = V \otimes^{\mathbb{L}}_{k,id} k \otimes^{\mathbb{L}}_{\mathfrak{S}_1,\varphi} \mathfrak{S}_1 \otimes^{\mathbb{L}}_{\mathfrak{S}_1} N.$$

Then one simply computes

$$k \otimes^{\mathbb{L}}_{\mathfrak{S}_1,\varphi} \mathfrak{S}_1 \cong \mathfrak{S}_1/u^p,$$

https://doi.org/10.1112/S0010437X23007261 Published online by Cambridge University Press

with k module structure via Frobenius on k. Therefore, the tensor derived above becomes

$$\mathrm{Frob}_k^* V \otimes_k \mathrm{Tor}_1^{\mathfrak{S}_1}(\mathfrak{S}_1/u^p, N) \cong \mathrm{Frob}_k^* V \otimes_k \big(N[u^p]\big). \qquad \square$$

In the following we shall describe the induced filtrations, divided Frobenii and connections on all terms of the sequence (⚁).

5.1 Understanding filtrations

Recalling [LL20, Theorem 4.1] (and references therein) we have filtered isomorphisms

$$\mathrm{R\Gamma}(\mathcal{X}, \mathrm{Fil}^\bullet_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}}) \xrightarrow{\cong} \mathrm{R\Gamma}_{\mathrm{crys}}(\mathcal{X}/S, \mathcal{I}^\bullet_{\mathrm{crys}}).$$

By the above identification, we need to understand the Hodge filtration on the derived de Rham cohomology of $\mathcal{X}/\mathfrak{S}$.

Lemma 5.3. *We have the following assertions.*

(1) *The map* $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{N}} \Delta_n^{(1)}) \to \mathfrak{M}$ *is injective.*
(2) *The map* $\mathrm{H}^i(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}}/p^n) \to \mathcal{M}$ *is injective.*

This fact has appeared in the proof of [LL20, Theorem 7.22]; for the convenience of readers we reproduce its proof below. The key point is that the inequality $e \cdot (i-1) < p-1$ implies the ith prismatic cohomology is u-torsion-free, which in turn guarantees injectivity.

Proof. By [LL20, Corollary 4.23] and diagram-chasing, we know the kernel of

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{N}} \Delta_n^{(1)}) \to \mathfrak{M}$$

surjects onto the kernel of

$$\mathrm{H}^i(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}}/p^n) \to \mathcal{M}.$$

Hence, it suffices to prove (1).

By [LL20, Lemma 7.8] we know the ith divided Frobenius $\varphi_i \colon \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{N}} \Delta_n^{(1)}) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)$ is an isomorphism. Combining this with Theorem 3.6(1) we see that the cohomology of Nygaard filtration has no finite length sub-$\mathfrak{S}$-module. Finally, [LL20, Proposition 7.12] says the kernel of the map in (1) must be a finite length sub-$\mathfrak{S}$-module, thus zero. $\square$

Notation 5.4. We denote the images of the above injections by $\mathrm{Fil}^i \mathfrak{M}$ and $\mathrm{Fil}^i \mathcal{M}$, respectively.

The submodule $\mathrm{Fil}^i \mathcal{M} \subset \mathcal{M}$ induces filtrations on the first and third terms in the sequence (⚁). For instance,

$$\mathrm{Fil}^i \big(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\big) \coloneqq \big(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\big) \cap \mathrm{Fil}^i \mathcal{M}$$

where the intersection happens inside $\mathcal{M}$, and

$$\mathrm{Fil}^i \overline{M} \coloneqq \mathrm{Im}(\mathrm{Fil}^i \mathcal{M} \to \overline{M}).$$

Let us investigate these filtrations.

Let $\mathcal{I}^{[i]} \subset S$ be the ith PD filtration ideal, which is p-completely generated by $\geq i$th divided powers of $E(u)$ in S. Note that the quotient $S/\mathcal{I}^{[i]}$ is p-torsion-free, hence the ideal $\mathcal{I}^{[i]}_n \coloneqq \mathcal{I}^{[i]}/p^n \subset S_n$ can be regarded as the ith PD filtration ideal on S_n.

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Recall [LL20, § 4] that we have a commutative diagram of sheaves on $(\mathcal{O}_K)_{\mathrm{qSyn}}$:

$$\begin{tikzcd}
(E(u)^j) \otimes_{\mathfrak{S}} \Delta^{(1)} \arrow[r] \arrow[d] & \mathrm{Fil}^j_{\mathrm{N}} \Delta^{(1)} \arrow[r] \arrow[d] & \Delta^{(1)} \arrow[d] \\
\mathcal{I}^{[j]} \otimes_S \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \arrow[r] & \mathrm{Fil}^j_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \arrow[r] & \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}
\end{tikzcd}$$

Lemma 5.5. *The diagram above induces the following commutative diagram of sheaves on $(\mathcal{O}_K)_{\mathrm{qSyn}}$:*

$$\begin{tikzcd}
0 \arrow[r] & \frac{\mathrm{Fil}^j_{\mathrm{N}} \Delta^{(1)}}{(E(u)^j)\otimes_{\mathfrak{S}} \Delta^{(1)}} \arrow[r] \arrow[d] & \frac{\Delta^{(1)}}{(E(u)^j)\otimes_{\mathfrak{S}} \Delta^{(1)}} \arrow[r] \arrow[d] & \frac{\Delta^{(1)}}{\mathrm{Fil}^j_{\mathrm{N}} \Delta^{(1)}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{\mathrm{Fil}^j_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}}{\mathcal{I}^{[j]} \hat{\otimes}_S \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}} \arrow[r] & \frac{\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}}{\mathcal{I}^{[j]} \hat{\otimes}_S \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}} \arrow[r] & \frac{\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}}{\mathrm{Fil}^j_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}} \arrow[r] & 0
\end{tikzcd}$$

which has short exact rows, and vertical arrows are isomorphisms if $j \le p$. Moreover, all these statements remain true after derived mod p^n is an exact functor.

Proof. The derived mod p^n statement follows from the fact that derived mod p^n is exact. It suffices to show two of the three vertical arrows are isomorphisms.

Using $\mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \cong S \hat{\otimes}_{\mathfrak{S}} \Delta^{(1)}$ (see [BS22, Theorem 5.2] and [LL20, Theorem 3.5]), the middle vertical arrow is identified with

$$\Delta^{(1)} \hat{\otimes}_{\mathfrak{S}} \left(\frac{\mathfrak{S}}{(E(u)^j)} \longrightarrow \frac{S}{\mathcal{I}^{[j]}} \right),$$

hence it suffices to note that the ring map $\mathfrak{S}/(E(u)^j) \to S/\mathcal{I}^{[j]}$ is an isomorphism.

The right vertical arrow is an isomorphism (thanks to [LL20, Corollary 4.23]). □

Proposition 5.6. *The map $\mathrm{Fil}^i \mathcal{M} \to \overline{M}$ is surjective. Hence, $\mathrm{Fil}^i \overline{M} = \overline{M}$.*

Proof. We consider the following map between long exact sequences:

$$\begin{tikzcd}
\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{N}} \Delta^{(1)}/p^n) \arrow[r] \arrow[d] & \mathfrak{M} \arrow[r] \arrow[d, "\iota"] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, (\Delta^{(1)}/\mathrm{Fil}^i_{\mathrm{N}} \Delta^{(1)})/p^n) \arrow[d, "\simeq"] \\
\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n) \arrow[r] & \mathcal{M} \arrow[r] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, (\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/\mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}})/p^n)
\end{tikzcd}$$

Chasing the diagram, we see that it suffices to show the top right horizontal arrow is a surjection. Indeed, granting the surjectivity assertion, we get that the summation map

$$\mathrm{Fil}^i \mathcal{M} \oplus \mathfrak{M} \to \mathcal{M}$$

is a surjection. Projection further to $\overline{M}$ kills the second factor above, hence we get the desired surjectivity.

Finally, we prolong the top long exact sequence:

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, (\Delta^{(1)}/\mathrm{Fil}^i_{\mathrm{N}} \Delta^{(1)})/p^n) \to \mathrm{H}^{i+1}_{\mathrm{qSyn}}(X_n, \mathrm{Fil}^i_{\mathrm{N}} \Delta^{(1)}/p^n) \xrightarrow{\iota} \mathrm{H}^{i+1}_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}/p^n).$$

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We are reduced to showing ι is injective, which is exactly Proposition 3.20(2); note that

$$e \cdot \big((i+1)-1\big) = e \cdot i = p-1. \qquad \square$$

Using what is proved in the above proposition, we can also understand $\mathrm{Fil}^i\left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right)$. The diagram before Lemma 5.5 implies that we have a natural map $\mathfrak{M} \otimes_{\mathfrak{S}_n} \mathcal{I}_n^{[i]} \to \mathrm{Fil}^i\left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right)$. Since the map $\Delta^{(1)} \to \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}$ of quasi-syntomic sheaves is filtered, we also have a natural map $\mathrm{Fil}^i \mathfrak{M} \to \mathrm{Fil}^i\left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right)$. We recall (Notation 5.4) that the source denotes H^i of the ith mod p^n Nygaard filtration.

Proposition 5.7. *The summation map* $\mathrm{Fil}^i \mathfrak{M} \oplus \left(\mathfrak{M} \otimes_{\mathfrak{S}_n} \mathcal{I}_n^{[i]}\right) \to \mathrm{Fil}^i\left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right)$ *is surjective.*

Proof. Note that

$$\frac{\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n}{\mathfrak{M} \otimes_{\mathfrak{S}_n} \mathcal{I}_n^{[i]}} = \mathfrak{M} \otimes_{\mathfrak{S}_n} \frac{S_n}{\mathcal{I}_n^{[i]}} = \mathfrak{M}/(E^i).$$

In the last equality, we use the fact that $i<p$ implies $S_n/\mathcal{I}_n^{[i]} = \mathfrak{S}_n/(E^i)$. Therefore, any element x in $\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n$ can be written as $x = y+z$ with $y \in \mathfrak{M}$ and z is in the image of $\mathfrak{M} \otimes_{\mathfrak{S}_n} \mathcal{I}_n^{[i]}$. Hence, we have

$$\mathrm{Fil}^i\left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right) = \Big(\mathrm{Fil}^i\left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right) \cap \mathfrak{M}\Big) + \mathrm{Im}(\mathfrak{M} \otimes_{\mathfrak{S}_n} \mathcal{I}_n^{[i]}).$$

It suffices to show

$$\mathrm{Fil}^i\left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right) \cap \mathfrak{M} = \mathrm{Fil}^i \mathfrak{M} := \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{N}} \Delta^{(1)}/p^n),$$

which indeed follows from chasing the diagram in the proof of Proposition 5.6. $\square$

Corollary 5.8. *Let $e=1$ and $i=p-1$. Then the triple $(\mathcal{M}, \mathrm{Fil}^i \mathcal{M}, \varphi_i)$ is an object in* $\mathrm{Mod}^{\varphi, p-1}_{S,\mathrm{tor}}$.

Proof. Note that the map $\mathrm{Fil}^i \mathcal{M} \to \mathcal{M}$ is injective by Lemma 5.3. We need to show admissibility, that is, the image φ_i generates $\mathcal{M}$. To that end, we shall explain why both images of $\varphi_i \colon \mathrm{Fil}^i\left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right) \to \mathrm{Fil}^i\left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right)$ and $\overline{\varphi_i} \colon \mathrm{Fil}^i \overline{M} = \overline{M} \to \overline{M}$ generate the target. For the latter, it follows from the $e=1$ case of Proposition 5.13. For the former, just note that the Nygaard filtration, $\mathrm{Fil}^i \mathfrak{M}$ (see Notation 5.4), already has its image of φ_i generating the module, thanks to [LL20, Lemma 7.8(3)]. $\square$

5.2 Computing the divided Frobenius

Next we discuss the divided Frobenius on Fil^i of terms in the sequence (⚀). We will use φ_i to denote the divided Frobenius on both Nygaard and Hodge filtrations; hopefully readers can tell them apart by looking at the source of the arrow to see which divided Frobenius we are using.

Recall [LL20, Remark 4.24] that when $j \le p-1$, the semi-linear Frobenius φ on $\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}$ becomes uniquely divisible by p^j when restricted to the sub-quasi-syntomic sheaf $\mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}$ (cf. [Bre98, p. 10]), which we denote by φ_j. The divided Frobenii on Nygaard and Hodge filtrations are related by

$$\begin{tikzcd}
\mathrm{Fil}^j_{\mathrm{N}} \Delta^{(1)} \arrow[r, "\varphi_j"] \arrow[d, "\iota"] & \Delta \arrow[d, "1 \otimes c_1^j"] \\
\mathrm{Fil}^j_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \arrow[r, "\varphi_j"] & \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \cong \Delta \hat{\otimes}_{\mathfrak{S}, \varphi} S
\end{tikzcd}$$

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as one computes $\varphi/\varphi(E)^j\cdot(\varphi(E)/p)^j=\varphi/p^j$. Restricting further to $\mathcal{I}^{[j]}\hat{\otimes}_{\varphi,\mathfrak{S}}\mathbb{\Delta}\subset \mathrm{Fil}^j_{\mathrm{H}}\,\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}$, the divided Frobenius is related to the (semi-linear) prismatic Frobenius via

$$\begin{tikzcd}
\mathcal{I}^{[j]}\hat{\otimes}_{\varphi,\mathfrak{S}}\mathbb{\Delta} \arrow[r, "\varphi_j\otimes\varphi"] \arrow[d, "\iota"] & S\hat{\otimes}_{\varphi,\mathfrak{S}}\mathbb{\Delta} \arrow[d, "\cong"] \\
\mathrm{Fil}^j_{\mathrm{H}}\,\mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \arrow[r, "\varphi_j"] & \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}
\end{tikzcd}$$

where the φ_j and φ on the top arrow are respectively the divided Frobenius on $\mathcal{I}^{[j]}\subset S$ and the semi-linear Frobenius on $\mathbb{\Delta}$. Since we assumed $e\cdot i=p-1$, in particular $i\le p-1$. From the discussion, we immediately get the following lemma.

Lemma 5.9. *Restricting the divided Frobenius* $\varphi_i\colon \mathrm{Fil}^i\,\mathcal{M}\to\mathcal{M}$ *to* $\mathrm{Fil}^i\left(\mathfrak{M}\otimes_{\mathfrak{S}_n}S_n\right)$, *the image lands in the submodule* $\mathfrak{M}\otimes_{\mathfrak{S}_n}S_n$.

Proof. By the above discussion, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fil}^i\,\mathfrak{M}\oplus\left(\mathcal{I}_n^{[i]}\otimes_{\mathfrak{S}_n}\mathfrak{M}\right) \arrow[r] \arrow[d] & \mathfrak{M}\otimes_{\mathfrak{S}_n}S_n \arrow[d, "\iota"] \\
\mathrm{Fil}^i\,\mathcal{M} \arrow[r, "\varphi_i"] & \mathcal{M}
\end{tikzcd}$$

where the top arrow is given by $\left(\varphi_i\otimes c_1^i\right)\oplus\left(\varphi_i\otimes\varphi\right)$. Our claim follows from Proposition 5.7 which says the image of the left vertical arrow is precisely $\mathrm{Fil}^i\left(\mathfrak{M}\otimes_{\mathfrak{S}_n}S_n\right)$. □

Consequently, the divided Frobenius $\varphi_i\colon \mathrm{Fil}^i\,\mathcal{M}\to\mathcal{M}$ descends to a semi-linear map $\mathrm{Fil}^i\,\overline{M}=\overline{M}\to\overline{M}$ (see Proposition 5.6), which we refer to as the *residual divided Frobenius*. Our next task is to relate this residual divided Frobenius to the Frobenius on V.

To that end, we factorize the divided Frobenius on the ith Hodge filtration as

$$\mathrm{Fil}^i_{\mathrm{H}}\,\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}\xrightarrow{\alpha}\mathbb{\Delta}\widehat{\otimes}_{\mathfrak{S}}\mathcal{I}^{[i]}\xrightarrow{\mathrm{id}\otimes\varphi_i}\mathbb{\Delta}\widehat{\otimes}_{\mathfrak{S}}\varphi_*S. \tag{⚁}$$

Here α is S-linear and is defined at the level of sheaves in $\left(\mathcal{O}_K\right)_{\mathrm{qSyn}}$. Recall [LL20, Remark 4.24] that on the basis of large quasi-syntomic algebras,

$$\mathrm{Fil}^i_{\mathrm{H}}\,\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}=\sum_{0\le j\le i}\mathcal{I}^{[i-j]}\hat{\otimes}_{\mathfrak{S}}\mathrm{Fil}^j_{\mathrm{N}}\,\mathbb{\Delta}^{(1)}.$$

Therefore, the linear Frobenius

$$\mathbb{\Delta}\widehat{\otimes}_{\mathfrak{S}}\varphi_*S\cong\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}\xrightarrow{\beta}\mathbb{\Delta}\widehat{\otimes}_{\mathfrak{S}}S$$

restricted to the ith Hodge filtration lands in $\mathbb{\Delta}\widehat{\otimes}_{\mathfrak{S}}\mathcal{I}^{[i]}$, and composition further with the ith divided Frobenius on the second factor gives the semi-linear divided Frobenius.

Lemma 5.10. *The map* $\mathrm{Fil}^i_{\mathrm{H}}\,\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}\xrightarrow{\alpha}\mathbb{\Delta}\widehat{\otimes}_{\mathfrak{S}}\mathcal{I}^{[i]}$ *induces a commutative diagram*

$$\begin{tikzcd}
\mathrm{Fil}^i\left(\mathfrak{M}\otimes_{\mathfrak{S}_n}S_n\right) \arrow[r] \arrow[d] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X},\mathbb{\Delta}_n)\otimes_{\mathfrak{S}_n}\mathcal{I}_n^{[i]} \arrow[d] \\
\mathrm{Fil}^i\,\mathcal{M} \arrow[r, "\alpha"] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X},\mathbb{\Delta}_n\hat{\otimes}_{\mathfrak{S}_n}\mathcal{I}_n^{[i]})
\end{tikzcd}$$

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The content of this lemma is that when we first derived mod α by p^n, then took $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X},-)$, and finally restricted it to the submodule $\mathrm{Fil}^i\left(\mathfrak{M}\otimes_{\mathfrak{S}_n} S_n\right)$, it landed in the submodule $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X},{\mathbb{\Delta}}_n)\otimes_{\mathfrak{S}_n}\mathcal{I}_n^{[i]}$ of the target. This is proved exactly the same way as in Lemma 5.9 so we omit it. From the above lemma, we know the map α descends to a map

$$\mathrm{Fil}^i\,\overline{M}=\overline{M}=\mathrm{Frob}_k^*(V)\otimes_k S_1[u^p]\xrightarrow{\overline{\alpha}}\mathrm{Tor}_1^{\mathfrak{S}_n}(V,\mathcal{I}_n^{[i]})=V\otimes_k\mathcal{I}_1^{[i]}[u].$$

Here we use $\mathcal{I}_n^{[i]}\otimes^{\mathbb{L}}_{\mathfrak{S}_n}\mathfrak{S}_1=\mathcal{I}_1^{[i]}$ and Lemma 5.2(1) to obtain the identification of the target.

Proposition 5.11. *Let $F\colon V\to V$ denote the semi-linear prismatic Frobenius on V, which induces the linearized Frobenius $\widetilde{F}\colon\mathrm{Frob}_k^*(V)\to V$. Then the map*

$$\mathrm{Frob}_k^*(V)\otimes_k S_1[u^p]\xrightarrow{\overline{\alpha}}V\otimes_k\mathcal{I}_1^{[i]}[u]$$

is given by $\widetilde{F}\otimes u^{p-1}$.

Note that given a u^p-torsion in S_1, multiplication by u^{p-1} gives us a u-torsion in S_1, implicitly in the statement we have used the fact that the inclusion $\mathcal{I}_1^{[i]}[u]\subset S_1[u]$ is a bijection because $i\leq ep-1$.

Proof. We consider the following commutative diagram of sheaves on $\left(\mathcal{O}_K\right)_{\mathrm{qSyn}}$:

$$\begin{tikzcd}
\mathrm{Fil}^i_{\mathrm{H}}\,\mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \arrow[r, "\alpha"] \arrow[d] & {\mathbb{\Delta}}\widehat{\otimes}_{\mathfrak{S}}\mathcal{I}^{[i]} \arrow[d] \\
{\mathbb{\Delta}}\widehat{\otimes}_{\mathfrak{S}}\varphi_*S\cong\mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \arrow[r, "\beta"] & {\mathbb{\Delta}}\widehat{\otimes}_{\mathfrak{S}}S
\end{tikzcd}$$

This induces the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fil}^i\,\overline{M} \arrow[r, "\overline{\alpha}"] \arrow[d, "\cong"'] & V\otimes_k\mathcal{I}_1^{[i]}[u] \arrow[d, "\cong"] \\
\overline{M} \arrow[r, "\beta"] & V\otimes_k S_1[u]
\end{tikzcd}$$

The left vertical arrow is an isomorphism. As explained right after the statement, the right vertical arrow is also an isomorphism. Therefore, we are reduced to computing the effect on H^{-1} of the map

$$(V\otimes^{\mathbb{L}}_{\mathfrak{S}_1,\varphi}\mathfrak{S}_1)\otimes^{\mathbb{L}}_{\mathfrak{S}_1}S_1\to V\otimes^{\mathbb{L}}_{\mathfrak{S}_1}S_1$$

induced by the linearized Frobenius $V\otimes^{\mathbb{L}}_{\mathfrak{S}_1,\varphi}\mathfrak{S}_1\cong\mathrm{Frob}_k^*(V)\otimes_k\mathfrak{S}_1/u^p\xrightarrow{\widetilde{F}\otimes\mathrm{proj}}V\otimes_k\mathfrak{S}_1/u$. We can choose the following explicit resolution of the above map of $\mathfrak{S}_1$-modules:

$$\begin{tikzcd}
\mathrm{Frob}_k^*(V)\otimes_k\mathfrak{S}_1 \arrow[r, "\mathrm{id}\otimes u^p"] \arrow[d, "\widetilde{F}\otimes u^{p-1}"'] & \mathrm{Frob}_k^*(V)\otimes_k\mathfrak{S}_1 \arrow[r, "\mathrm{id}\otimes\mathrm{proj}"] \arrow[d, "\widetilde{F}\otimes\mathrm{id}"'] & \mathrm{Frob}_k^*(V)\otimes_k\mathfrak{S}_1/u^p \arrow[d, "\widetilde{F}\otimes\mathrm{proj}"] \\
V\otimes_k\mathfrak{S}_1 \arrow[r, "\mathrm{id}\otimes u"] & V\otimes_k\mathfrak{S}_1 \arrow[r, "\mathrm{id}\otimes\mathrm{proj}"] & V\otimes_k\mathfrak{S}_1/u
\end{tikzcd}$$

Tensoring the above with S_1 over $\mathfrak{S}_1$ and considering the induced map on H^{-1} yields the conclusion. ∎

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The effect of the second arrow in (⚂) is very easy to understand: we only need to understand the divided Frobenius $\varphi_i\colon \mathcal{I}_1^{[i]}[u] \xrightarrow{\varphi_i} S_1[u^p]$. Note that we assumed $e \cdot i = p-1$, hence $e=1$ means $i = p-1$.

Lemma 5.12. *The S_1-module $\mathcal{I}_1^{[i]}[u] = S_1[u]$ is generated by u^{ep-1}, and we have*

$$\varphi_i(u^{ep-1}) = \begin{cases} c_1^{p-1} \in S_1 = S_1[u^p], & \text{when } e=1, \\ 0, & \text{when } e>1. \end{cases}$$

Proof. The description of $\mathcal{I}_1^{[i]}[u]$ is well known. It follows from the explicit description of $\mathcal{I}_1^{[i]} \subset S_1$, given in the proof of Proposition 5.7.

Let us choose a lift of $u^{ep-1} \equiv E(u)^{p-1} \cdot u^{e-1}$ to $\mathcal{I}^{[i]}$ and compute

$$\varphi_i(E(u)^{p-1} \cdot u^{e-1}) = c_1^{p-1} \cdot p^{p-1-i} \cdot u^{ep-p}.$$

After reduction mod p, the right-hand side is 0 if $0 < p-1-i$ which is equivalent to $e>1$, and the right-hand side is c_1^{p-1} when $e=1$. □

Putting everything together, we arrive at the following proposition.

Proposition 5.13. *The divided Frobenius $\mathrm{Fil}^i \mathcal{M} \to \mathcal{M}$ descends to a residual divided Frobenius*

$$\overline{\varphi_i}\colon \mathrm{Fil}^i \overline{M} = \overline{M} \to \overline{M}.$$

After identifying $\overline{M} \cong \mathrm{Frob}_k^*(V) \otimes_k S_1[u^p]$, we have

$$\overline{\varphi_i} = \begin{cases} F \otimes c_1^{p-1} \cdot \varphi_{S_1}, & \text{when } e=1, \\ 0, & \text{when } e>1. \end{cases}$$

Here we abuse notation a little by writing the induced Frobenius on $\mathrm{Frob}_k^*(V)$ as F.

Proof. The first sentence is Lemma 5.9. As for the computation of the residual divided Frobenius, we consider the sequence (⚂), which gives rise to

$$\mathrm{Frob}_k^*(V) \otimes_k S_1[u^p] \xrightarrow{\overline{\alpha}} V \otimes_k \mathcal{I}_1^{[i]}[u] \xrightarrow{\mathrm{id}\otimes\varphi_i} V \otimes_{k,\varphi} S_1[u^p].$$

Combining Proposition 5.11 and Lemma 5.12 yields the result. □

5.3 The connection

In [LL20, § 5.1] we explained how one gets a natural connection on the derived de Rham complex relative to $\mathfrak{S}$. Consequently, we see that there is a connection $\nabla\colon \mathcal{M} \to \mathcal{M}$ satisfying $\nabla(f \cdot m) = f' \cdot m + f \cdot \nabla(m)$ for any $f \in S$ and $m \in \mathcal{M}$. In this section, we shall see that in a strong sense there is a unique such connection. As a corollary, the connection ∇ preserves the sequence (⚂). Moreover, the compatibility between ∇ and the divided Frobenius [LL20, § 5.2] will determine the residual connection on $\overline{M}$.

Notation 5.14. Let $S[\epsilon] \coloneqq S[x]/(x^2)$ and let $S \xrightarrow{\iota_1} S[\epsilon]$ and $S \xrightarrow{\iota_2} S[\epsilon]$ be two ring homomorphisms defined as $\iota_1(f) = f \otimes 1$ and $\iota_2(f) = f \otimes 1 + f' \otimes \epsilon$.

Proposition 5.15. *There is a unique $\mathbb{E}_\infty$-$S[\epsilon]$-algebra isomorphism $\mathrm{dR}^\wedge_{R/\mathfrak{S}} \otimes_{S,\iota_1} S[\epsilon] \to \mathrm{dR}^\wedge_{R/\mathfrak{S}} \otimes_{S,\iota_2} S[\epsilon]$ which reduces to the identity modulo ϵ and is functorial in the formally smooth $\mathcal{O}_K$-algebra R.*

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Proof. One observes that the formula $\nabla \mapsto \big(g(m\otimes 1) = m\otimes 1 + \nabla(m)\otimes\epsilon\big)$ gives a bijection between functorial connections on $\mathrm{dR}^{\wedge}_{R/\mathfrak{S}}$ and the said functorial isomorphisms. Therefore, the existence follows from [LL20, § 5.1].

To show uniqueness, we follow the same argument as in the proof of [LL20, Theorem 3.13]. Firstly, by left Kan extension and quasi-syntomic descent, it suffices to check the uniqueness when viewing both sides as quasi-syntomic sheaves of $S[\epsilon]$-algebras. Secondly, by the same argument in [LL20, Theorem 3.13], one sees that restricting to the category of quasi-syntomic $\mathcal{O}_K$-algebras of the form $\mathcal{O}_K\langle X_j^{1/p^\infty}; j\in J\rangle$ for some set J determines such morphisms of $S[\epsilon]$-algebras. Finally, when $\widetilde{R} = \mathcal{O}_K\langle X_j^{1/p^\infty}; j\in J\rangle$, both the source and the target are given by $S[\epsilon]\langle X_j^{1/p^\infty}; j\in J\rangle$. Now we need to show that $g(X)$ must be X.

To that end, let us assume $g(X^{1/p^n}) = X^{1/p^n} + Y_n\otimes\epsilon$. Then we compute $g(X) = g(X^{1/p^n})^{p^n} = (X^{1/p^n} + Y_n\otimes\epsilon)^{p^n} \equiv X$ modulo p^n. Therefore we conclude that $g(X) - X$ is divided by arbitrary powers of p, hence must be 0 by the p-adic separatedness of $S[\epsilon]\langle X_j^{1/p^\infty}; j\in J\rangle$. ☐

Remark 5.16. For any quasi-compact and quasi-separated smooth formal scheme $\mathcal{Y}$ over $\mathrm{Spf}(\mathcal{O}_K)$, the crystal nature of $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{Y}/S)$ gives a connection on $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{Y}/S)$ (see [BdJ11, p. 2 and Lemma 2.8]). Note that although in [BdJ11] the authors were talking about crystals in quasi-coherent modules, their argument works in our setting of crystals in perfect complexes as $\Omega^{1,pd}_{S/W}$ is finite free over S, so there is no subtlety when derived tensoring it with any other S-module. Consequently, one gets a connection on $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{Y}/S)$, and when identifying $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{Y}/S) \cong \mathrm{dR}^{\wedge}_{\mathcal{Y}/\mathfrak{S}}$, our Proposition 5.15 shows the 'crystalline' connection agrees with our 'derived de Rham' connection.

Below we explain yet another way to get the connection, via the prismatic crystal nature of prismatic cohomology. Recall [BS21, Construction 7.13] that there is a cosimplicial prism $\big(\mathfrak{S}^{(\bullet)}, J^{(\bullet)}\big) \to \mathcal{O}_K \cong \mathfrak{S}^{(\bullet)}/J^{(\bullet)}$. Let $S^{(\bullet)} \to \mathcal{O}_K$ be the similarly defined cosimplicial ring obtained by taking divided power envelopes of $\mathfrak{S}^{\hat{\otimes}_W n} \twoheadrightarrow \mathcal{O}_K$ where $[n]\in\Delta$. Note that there is a map of these cosimplicial rings induced by the Frobenius $\varphi_{\mathfrak{S}}^{\otimes\bullet}\colon \mathfrak{S}^{\otimes\bullet} \to \mathfrak{S}^{\otimes\bullet}$. Let us explicate this for $\bullet = 0, 1$ as we will need it later:

$$\begin{tikzcd}
\mathfrak{S}\cong W[\![u]\!] \arrow[r, "\iota_1"] \arrow[d, "u\mapsto u^p"'] & W[\![u,v]\!]\{\frac{u-v}{E(u)}\}^\wedge \cong \mathfrak{S}^{(1)} \cong W[\![u,v]\!]\{\frac{u-v}{E(v)}\}^\wedge \arrow[d, "u\mapsto u^p"', "v\mapsto v^p"] & W[\![v]\!]\cong\mathfrak{S} \arrow[l, "\iota_2"'] \arrow[d, "v\mapsto v^p"] \\
S\cong W[\![u]\!]\langle\!\langle E(u)\rangle\!\rangle \arrow[r, "\iota_1"] & W[\![u,v]\!]\langle\!\langle E(u), u-v\rangle\!\rangle \cong S^{(1)} \cong W[\![u,v]\!]\langle\!\langle E(v), u-v\rangle\!\rangle & W[\![v]\!]\langle\!\langle E(v)\rangle\!\rangle \cong S \arrow[l, "\iota_2"']
\end{tikzcd}$$

where $\langle\!\langle - \rangle\!\rangle$ denotes p-completely adjoining divided powers of the designated elements. To see the middle arrow is well defined we use the fact that $\varphi(E(u))$ and $\varphi(E(v))$ in $S^{(1)}$ is p times a unit, and adjoining $\varphi(u-v)/p$ as a δ-ring is the same as adjoining divided powers of $u-v$ (see [BS22, Corollary 2.39]).

Now for any p-adically smooth $\mathcal{O}_K$-algebra R, we have a functorial isomorphism of $\mathbb{E}_\infty$-$\mathfrak{S}^{(1)}$-algebras,

$$\Delta_{R/\mathfrak{S}} \hat{\otimes}_{\mathfrak{S},\iota_1} \mathfrak{S}^{(1)} \cong \mathfrak{S}^{(1)} \hat{\otimes}_{\iota_2,\mathfrak{S}} \Delta_{R/\mathfrak{S}},$$

by base change of prismatic cohomology. Base-changing the above along the aforesaid map $\mathfrak{S}^{(1)} \to S^{(1)}$ (and use either [BS22, Theorem 5.2] or [LL20, Theorem 3.5]) identifies the left

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(respectively, right) hand side with

$$\Delta_{R/\mathfrak{S}} \hat{\otimes}_{\mathfrak{S},\iota_1} \mathfrak{S}^{(1)} \hat{\otimes}_{\mathfrak{S}^{(1)},\varphi} S^{(1)} \cong \Delta_{R/\mathfrak{S}} \hat{\otimes}_{\mathfrak{S},\varphi} S \hat{\otimes}_{S,\iota_1} S^{(1)} \cong \mathrm{dR}^{\wedge}_{R/\mathfrak{S}} \hat{\otimes}_{S,\iota_1} S^{(1)}$$

(respectively, $S^{(1)} \hat{\otimes}_{\iota_2,S} \mathrm{dR}^{\wedge}_{R/\mathfrak{S}}$). This gives rise to another description of the 'crystalline' connection.

Proposition 5.17. *The following diagram commutes functorially in the p-adically smooth $\mathcal{O}_K$-algebra R:*

$$\begin{tikzcd}
\Delta_{R/\mathfrak{S}} \hat{\otimes}_{\mathfrak{S},\iota_1} \mathfrak{S}^{(1)} \arrow[r, "\cong"] \arrow[d, "\hat{\otimes}_{\mathfrak{S}^{(1)}} S^{(1)}"'] & \mathfrak{S}^{(1)} \hat{\otimes}_{\iota_2,\mathfrak{S}} \Delta_{R/\mathfrak{S}} \arrow[d, "\hat{\otimes}_{\mathfrak{S}^{(1)}} S^{(1)}"] \\
\mathrm{dR}^{\wedge}_{R/\mathfrak{S}} \hat{\otimes}_{S,\iota_1} S^{(1)} \arrow[r, "\cong"] & S^{(1)} \hat{\otimes}_{\iota_2,S} \mathrm{dR}^{\wedge}_{R/\mathfrak{S}}
\end{tikzcd}$$

Proof. Base-changing the top arrow along $\mathfrak{S}^{(1)} \to S^{(1)}$ gives a potentially different functorial isomorphism at the bottom. Therefore, it suffices to show that there is no non-trivial automorphism of the quasi-syntomic sheaf of $S^{(1)}$-algebras $R \mapsto \mathrm{dR}^{\wedge}_{R/\mathfrak{S}} \hat{\otimes}_{S,\iota_1} S^{(1)}$. The same argument as in [LL20, Theorem 3.13] does the job. $\square$

As a consequence, we know the sequence (⚀) is stable under the connection. In fact more generally we have the following result.

Corollary 5.18. *For any $j \in \mathbb{N}$ and any $n \in \mathbb{N} \cup \{\infty\}$, the connection on $\mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n)$ preserves the submodule $\mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}/p^n) \otimes_{\mathfrak{S}_n} S_n$.*

Proof. In line with the dictionary between connections and crystals [BdJ11, Lemma 2.8], we need to show that the isomorphism (note that both of $\iota_i \colon S \to S^{(1)}$ are p-completely flat)

$$\mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n) \otimes_{S_n,\iota_1} S_n^{(1)} \cong S_n^{(1)} \otimes_{\iota_2,S_n} \mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n)$$

preserves the submodule $\mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}/p^n) \otimes_{\mathfrak{S}_n} S_n$. This immediately follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \Delta/p^n) \hat{\otimes}_{\mathfrak{S}_n,\iota_1} \mathfrak{S}_n^{(1)} \arrow[r, "\cong"] \arrow[d, "- \hat{\otimes}_{\mathfrak{S}_n^{(1)}} S_n^{(1)}"'] & \mathfrak{S}^{(1)} \hat{\otimes}_{\iota_2,\mathfrak{S}_n} \mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \Delta/p^n) \arrow[d, "S_n^{(1)} \hat{\otimes}_{\mathfrak{S}_n^{(1)}} -"] \\
\mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n) \hat{\otimes}_{S_n,\iota_1} S_n^{(1)} \arrow[r, "\cong"] & S_n^{(1)} \hat{\otimes}_{\iota_2,S_n} \mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n)
\end{tikzcd}$$

induced by Proposition 5.17. $\square$

Therefore, we see that there is a *residual connection* $\overline{\nabla} \colon \overline{M} \to \overline{M}$. Recall [LL20, § 5.2] that the connection ∇ and divided Frobenius φ_i are related by the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fil}^i \mathcal{M} \arrow[r, "\varphi_i"] \arrow[d, "E(u)\cdot\nabla"'] & \mathcal{M} \arrow[d, "c_1\cdot\nabla"] \\
\mathrm{Fil}^i \mathcal{M} \arrow[r, "u^{p-1}\varphi_i"] & \mathcal{M}
\end{tikzcd}$$

Since all maps descend to $\mathrm{Fil}^i \overline{M} = \overline{M}$, we have the following proposition.

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PROPOSITION 5.19. *There is a commutative diagram:*

$$\begin{tikzcd} \overline{M} \arrow[r, "\varphi_i"] \arrow[d, "E(u)\cdot\overline{\nabla}"'] & \overline{M} \arrow[d, "c_1\cdot\overline{\nabla}"] \\ \overline{M} \arrow[r, "u^{p-1}\overline{\varphi_i}"] & \overline{M} \end{tikzcd}$$

Consequently, when $e=1$, after identifying $\overline{M} \cong \mathrm{Frob}_k^*(V)\otimes_k S_1[u^p]$, we have $\overline{\nabla}(v\otimes 1) = v\otimes \mathrm{d}\log(c_1)$.

Here $\mathrm{d}\log(c_1) = c_1'/c_1 = u^{p-1}/c_1$.

Proof. The existence of such a commutative diagram follows from the preceding discussion and the fact that both φ_i and ∇ descend to $\overline{M}$ by Proposition 5.13 and Corollary 5.18, respectively.

Starting with $v\otimes 1$ at the top left corner and comparing the end results of the two routes, we arrive at an identity,

$$\overline{\nabla}(F(v)\otimes 1)\cdot c_1^p + F(v)\otimes(p-1)c_1^{p-1}\cdot c_1' = 0,$$

where we used the description of φ_i in Proposition 5.13. Now we use the fact that $\overline{M}$ is p-torsion and the fact that F is a bijection to yield the desired conclusion. □

COROLLARY 5.20. *Let $e=1$ and $h=p-1$. Then the quadruple $(\overline{M}, \mathrm{Fil}^{p-1}\overline{M}, \varphi_{p-1}, \nabla)$ is a Breuil module and there is a canonical isomorphism $T_S(\overline{M}) \xrightarrow{\cong} (V\otimes_{W(k)} W(\overline{k}))^{\varphi=1}$ of representation of G_K. In particular the resulting Galois representation $T_S(\overline{M})$ is the unramified $\mathbb{F}_p$-representation associated with the étale φ-module V.*

Proof. The first part of statement follows from Lemma 5.2 and Propositions 5.13 and 5.19. To compute $T_S(\overline{M})$, let $I_+A_{\mathrm{crys}} \subset A_{\mathrm{crys}}$ be the ideal such that I_+A_{crys} contains $W(\mathfrak{m}_{\mathcal{O}_{\mathbf{C}}^\flat})$ and $A_{\mathrm{crys}}/I_+A_{\mathrm{crys}} = W(\overline{k})$. It is clear that $\varphi^n(a)\to 0$ for any $a\in I_+A_{\mathrm{crys}}$ and $I_+A_{\mathrm{crys}}\cap S = I_+$. By (2.17), $\overline{M}\otimes_S I_+A_{\mathrm{crys}}$ is stable under the G_K-action. So we have a canonical map of G_K-representations

$$\begin{aligned} T_S(\overline{M}) &= (\mathrm{Fil}^h\,\overline{M}\otimes_S A_{\mathrm{crys}})^{\varphi_h=1} = (\overline{M}\otimes_S A_{\mathrm{crys}})^{\varphi_h=1} \\ &\to (\overline{M}\otimes_A A_{\mathrm{crys}}/I_+A_{\mathrm{crys}})^{\varphi_h=1} = (\overline{M}/I_+\otimes_k \overline{k})^{\varphi_h=1}. \end{aligned}$$

By Proposition 5.13, if we identify $\overline{M} = \mathrm{Frob}^*V\otimes_k S_1[u^p]$ then $\forall x\otimes 1\in \overline{M}/I_+$, $\varphi(x\otimes 1) = F(x)\otimes a_0^{p-1}$ with $a_0 = E(0)/p\in W(k)^\times$. So $\varphi_h : \overline{M}/I_+\to\overline{M}/I_+$ is bijective. Using that $\lim_{n\to\infty}\varphi^n(a) = 0$, $\forall a\in I_+A_{\mathrm{crys}}$, we conclude that the above map is an isomorphism $T_S(\overline{M}) \xrightarrow{\cong} (\overline{M}/I_+\otimes_k\overline{k})^{\varphi_h=1}$ of G_K-representations. Finally, we have to check that $\overline{M}/I_+\simeq \mathrm{Frob}^*V$ as φ-modules. Indeed, $\mathrm{Frob}^*V\to\mathrm{Frob}^*V\otimes_k S_1[u^p]/I_+S = \overline{M}/I_+$ via $x\mapsto a_0(x\otimes 1)$ is the required isomorphism of φ-modules. □

5.4 Fontaine–Laffaille and Breuil modules

In this subsection we assume $e=1$. For simplicity we pick the uniformizer p, but all results in this subsection hold true with any other uniformizer. We shall compare the two approaches to understanding étale cohomology, as a Galois representation, from linear algebraic data on certain crystalline cohomologies, which are due to Fontaine, Messing and Kato, and to Breuil and Caruso.

First we need a reminder of the filtered comparison between the derived de Rham cohomology and crystalline cohomology (see [LL20, Theorem 4.1] and references therein).

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Remark 5.21. Let $\mathcal{X}$ be a smooth p-adic formal scheme over $\mathrm{Spf}(W)$. We have filtered isomorphisms

$$\mathrm{R\Gamma}(\mathcal{X}, \mathrm{Fil}^{\bullet}_{\mathrm{H}} \, \mathrm{dR}^{\wedge}_{-/W}) \xrightarrow{\cong} \mathrm{R\Gamma}_{\mathrm{crys}}(\mathcal{X}/W, \mathcal{I}^{\bullet}_{\mathrm{crys}})$$

and

$$\mathrm{R\Gamma}(\mathcal{X}, \mathrm{Fil}^{\bullet}_{\mathrm{H}} \, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}) \xrightarrow{\cong} \mathrm{R\Gamma}_{\mathrm{crys}}(\mathcal{X}/S, \mathcal{I}^{\bullet}_{\mathrm{crys}}).$$

In classical references by Fontaine and Messing, Kato, and Breuil and Caruso, they considered the right-hand-side objects of the above isomorphisms. However, we will be thinking about the derived de Rham side, as it is compatible with various techniques developed by Bhatt, Morrow and Scholze and by Bhatt and Scholze.

For the remainder of this subsection we let $\mathcal{X}$ be a quasi-compact quasi-separated p-adic formal scheme over $\mathrm{Spf}(W)$. At the derived level, we have the following comparisons.

PROPOSITION 5.22. *Consider the diagram*

$$\begin{tikzcd}
\mathcal{X} \arrow[d] \arrow[dr] \arrow[drr] & & \\
\mathrm{Spf}(W) \arrow[r, "u \mapsto p"] & \mathrm{Spf}(\mathfrak{S}) \arrow[r] & \mathrm{Spf}(W)
\end{tikzcd}$$

For any $n \in \mathbb{Z} \cup \{\infty\}$, we have the following assertions.

(1) *The canonical maps of p-complete cotangent complexes $\mathbb{L}^{\wedge}_{\mathcal{X}/W} \to \mathbb{L}^{\wedge}_{\mathcal{X}/\mathfrak{S}}$ (from the right triangle) and $\mathbb{L}^{\wedge}_{W/\mathfrak{S}} \to \mathbb{L}^{\wedge}_{\mathcal{X}/\mathfrak{S}}$ (from the left triangle) induce an isomorphism*

$$\mathbb{L}^{\wedge}_{\mathcal{X}/W}/p^n \oplus \big(\mathbb{L}^{\wedge}_{W/\mathfrak{S}} \hat{\otimes}_W \mathcal{O}_{\mathcal{X}}\big)/p^n \xrightarrow{\cong} \mathbb{L}^{\wedge}_{\mathcal{X}/\mathfrak{S}}/p^n,$$

functorial in $\mathcal{X}/W$.

(2) *The canonical filtered maps of p-complete de Rham complexes $\mathrm{dR}^{\wedge}_{\mathcal{X}/W} \to \mathrm{dR}^{\wedge}_{\mathcal{X}/\mathfrak{S}}$ (from the right triangle) and $\mathrm{dR}^{\wedge}_{W/\mathfrak{S}} \to \mathrm{dR}^{\wedge}_{\mathcal{X}/\mathfrak{S}}$ (from the left triangle) induce a filtered isomorphism*

$$\big(\mathrm{dR}^{\wedge}_{\mathcal{X}/W} \hat{\otimes}_W \mathrm{dR}^{\wedge}_{W/\mathfrak{S}}\big)/p^n \xrightarrow{\cong} \mathrm{dR}^{\wedge}_{\mathcal{X}/\mathfrak{S}}/p^n,$$

functorial in $\mathcal{X}/W$.

(3) *Moreover, the identification in (2) is compatible with divided Frobenii φ_j on the jth filtration of both sides for any $j \leq p-1$.*

In case readers are worried that we do not put any smoothness assumption on $\mathcal{X}$, just notice that both sides of these equalities are left Kan extended from smooth $\mathcal{X}$s, therefore it suffices to prove these statements for smooth affine $\mathcal{X}$s. That said, we will prove the statement without the smoothness assumption as the proof just works in this generality.

Proof. The finitary n cases follow from the case of $n = \infty$. Henceforth, we assume $n = \infty$.

(1) This follows from the exact triangle of cotangent complexes associated with a triangle of morphisms.

(2) Let $\mathcal{X}_{\mathfrak{S}} := \mathcal{X} \times_{\mathrm{Spf}(W)} \mathrm{Spf}(\mathfrak{S})$ be the base change. Then we have $\mathcal{X} \cong \mathcal{X}_{\mathfrak{S}} \times_{\mathrm{Spf}(\mathfrak{S})} \mathrm{Spf}(W)$. These objects fit in a commutative diagram:

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d] & \mathcal{X}_{\mathfrak{S}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathrm{Spf}(W) \arrow[r, "u \mapsto p"] & \mathrm{Spf}(\mathfrak{S}) \arrow[r] & \mathrm{Spf}(W)
\end{tikzcd}$$

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Using the Künneth formula for the derived de Rham complex, we obtain a filtered isomorphism:

$$\mathrm{dR}^{\wedge}_{\mathcal{X}_{\mathfrak{S}}/\mathfrak{S}} \hat{\otimes}_{\mathfrak{S}} \mathrm{dR}^{\wedge}_{W/\mathfrak{S}} \xrightarrow{\cong} \mathrm{dR}^{\wedge}_{\mathcal{X}/\mathfrak{S}}.$$

The base change formula for the derived de Rham complex gives us a filtered isomorphism:

$$\mathrm{dR}^{\wedge}_{\mathcal{X}/W} \hat{\otimes}_W \mathfrak{S} \xrightarrow{\cong} \mathrm{dR}^{\wedge}_{\mathcal{X}_{\mathfrak{S}}/\mathfrak{S}}.$$

In both filtered isomorphisms above we put the derived Hodge filtration on the derived de Rham complex, and a trivial filtration on the coefficient ring W and $\mathfrak{S}$. Combining these two filtered isomorphisms gives our desired filtered isomorphism.

(3) This follows from the fact that the two maps in (2) are compatible with divided Frobenii. ☐

Remark 5.23. Since $\mathfrak{S} \xrightarrow{u \mapsto p} W$ is a complete intersection, the p-adic derived de Rham complex $\mathrm{dR}^{\wedge}_{W/\mathfrak{S}} \cong S$ is given by Breuil's ring S with the Hodge filtration given by divided powers of $(u-p)$ and the usual Frobenius $u \mapsto u^p$. Similarly, the mod p^n derived de Rham complex is S/p^n with the induced filtration. This is because the rings S and $S/\mathcal{I}^{[j]}$ are all p-torsion-free for any $j \in \mathbb{N}$.

To obtain consequences at the level of cohomology groups, we need the following abstract lemma.

Lemma 5.24. *Let $\mathcal{C}$ be a stable ∞-category. Let $F \colon \mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}} \to \mathcal{C}$ be a simplicial diagram of simplicial sets. Then, for any $0 < m \leq n$, we have a pushout diagram*

$$\begin{tikzcd} F(n+1-m,m) \arrow[r] \arrow[d] & F(n+1-m,m-1) \arrow[d] \\ \mathrm{colim}_{i+j \geq n, j \geq m} F(i,j) \arrow[r] & \mathrm{colim}_{i+j \geq n, j \geq m-1} F(i,j) \end{tikzcd}$$

In particular, the two inclusions $F(i,j) \to F(i-1,j)$ and $F(i,j) \to F(i,j-1)$ give rise to a pushout diagram:

$$\bigoplus_{i+j=n+1, i>0, j>0} F(i,j) \rightrightarrows \bigoplus_{i+j=n} F(i,j) \longrightarrow \mathrm{colim}_{i+j \geq n} F(i,j).$$

Proof. The second statement follows from repeatedly applying the first statement and observing that

$$\mathrm{colim}_{i+j \geq n, j \geq n} F(i,j) = \mathrm{colim}_{(i,j) \geq (0,n)} F(i,j) = F(0,n)$$

as $(0,n)$ is the final object in $\{(i,j) \geq (0,n)\} \subset \mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}$. The first statement follows from [Lur09, Proposition 4.4.2.2]: applying the statement to

$$\{i+j \geq n, j \geq m-1\} = \{i+j \geq n, j \geq m\} \bigsqcup_{\{(i,j) \geq (n+1-m,m)\}} \{(i,j) \geq (n+1-m, m-1)\}$$

yields the desired pushout diagram. ☐

Combining the previous two general statements yield the following corollary.

Corollary 5.25. *Let $\mathcal{I}^{[\bullet]} \subset S$ be the filtration given by divided powers of $(u-p)$. The natural maps, for any $q+j \geq m$,*

$$\mathrm{R}\Gamma(\mathcal{X}, \mathrm{Fil}^q_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}) \hat{\otimes}_W \mathcal{I}^{[j]} \to \mathrm{R}\Gamma(\mathcal{X}, \mathrm{Fil}^m_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}})$$

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give rise to an exact triangle

$$\bigoplus_{q+j=\ell+1, i>0, j>0} \mathrm{R}\Gamma(\mathcal{X}, \mathrm{Fil}^q_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}})/p^n \hat{\otimes}_{W_n} \big(\mathcal{I}^{[j]}/p^n\big)$$
$$\to \bigoplus_{q+j=\ell} \mathrm{R}\Gamma(\mathcal{X}, \mathrm{Fil}^q_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}})/p^n \hat{\otimes}_{W_n} \big(\mathcal{I}^{[j]}/p^n\big) \to \mathrm{R}\Gamma(\mathcal{X}, \mathrm{Fil}^n_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}})/p^n$$

for any $\ell \in \mathbb{Z}$ and any $n \in \mathbb{Z} \cup \{\infty\}$.

Proof. The filtration comparison (Proposition 5.22(2)) shows the right-hand side is given by the ℓth Day convolution filtration on $\mathrm{R}\Gamma(\mathcal{X}, \mathrm{dR}^\wedge_{-/\mathfrak{S}})/p^n \hat{\otimes}_{W_n} \mathrm{dR}^\wedge_{W/\mathfrak{S}}/p^n$. Here the filtered ring $\mathrm{dR}^\wedge_{W/\mathfrak{S}}/p^n$ is given by $(S/p^n, \mathcal{I}^{[\bullet]})/p^n$ (see Remark 5.23). Finally, we apply Lemma 5.24 to conclude the proof. □

Theorem 5.26 (cf. [Bre98, p. 559 Remarques.(2)]). *For any $j, \ell \in \mathbb{Z}$ and any $n \in \mathbb{Z} \cup \{\infty\}$, use*

$$\mathrm{Im}\big(\mathrm{H}^j(\mathcal{X}, \mathrm{Fil}^\ell_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}}/p^n) \to \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^\wedge_{-/\mathfrak{S}})/p^n\big) =: \mathrm{Fil}^\ell \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^\wedge_{-/\mathfrak{S}}/p^n)$$

to filter $\mathrm{H}^j(\mathcal{X}, \mathrm{dR}^\wedge_{-/\mathfrak{S}}/p^n)$, and similarly filter $\mathrm{H}^j(\mathcal{X}, \mathrm{dR}^\wedge_{-/W}/p^n)$. Then we have a filtered isomorphism

$$\mathrm{H}^j(\mathcal{X}, \mathrm{dR}^\wedge_{-/W}/p^n) \hat{\otimes}_{W_n} \big(S/p^n\big) \xrightarrow{\cong} \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^\wedge_{-/\mathfrak{S}}/p^n).$$

Moreover, it is compatible with the divided Frobenii φ_m on the mth filtration of both sides for all $m \leq p-1$.

Here again the ring S/p^n is equipped with the divided power ideal filtration. Concretely, we have

$$\mathrm{Fil}^\ell \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^\wedge_{-/\mathfrak{S}}/p^n) = \sum_{r+s=\ell} \mathrm{Fil}^r \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^\wedge_{-/W}/p^n) \hat{\otimes}_{W_n} \big(\mathcal{I}^{[s]}/p^n\big)$$

as sub-W_n-modules inside $\mathrm{H}^j(\mathcal{X}, \mathrm{dR}^\wedge_{-/\mathfrak{S}}/p^n) \hat{\otimes}_{W_n} \big(S/p^n\big) \xrightarrow{\cong} \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^\wedge_{-/\mathfrak{S}}/p^n)$.

Proof. By Corollary 5.25, it suffices to show that the exact triangle obtained induces a short exact sequence after applying H^q. To that end, it suffices to show the map

$$\mathrm{H}^q(\mathcal{X}, \mathrm{Fil}^\ell_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}}/p^n) \hat{\otimes}_{W_n} \big(\mathcal{I}^{[j+1]}/p^n\big) \to \mathrm{H}^q(\mathcal{X}, \mathrm{Fil}^\ell_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}}/p^n) \hat{\otimes}_{W_n} \big(\mathcal{I}^{[j]}/p^n\big)$$

is injective for any q, j, ℓ, n. But this follows from the fact that $\big(\mathcal{I}^{[j]}/p^n\big)/\big(\mathcal{I}^{[j+1]}/p^n\big) \simeq W_n \cdot \gamma_j(u-p)$ is p-completely flat over W_n. The compatibility with divided Frobenii was checked in Proposition 5.22(3). □

We arrive at the following result, already proved by Fontaine and Messing [FM87, Cor. 2.7] and Kato [Kato87, II.Proposition 2.5]. In fact they did not need the existence of a lift all the way to $\mathrm{Spf}(W)$.

Corollary 5.27. *Let $\mathcal{X}$ be a proper smooth p-adic formal scheme over W. Let $j \leq p-1$ and $n \in \mathbb{N}$. Then the natural map $\mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n, \mathcal{I}^{[i]}_{\mathrm{crys}}) \to \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n)$ is injective, and the triple*

$$\big(\mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n), \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n, \mathcal{I}^{[i]}_{\mathrm{crys}}), \varphi_i : \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n, \mathcal{I}^{[i]}_{\mathrm{crys}}) \to \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n)\big)$$

is an object in $\mathrm{FM}_{W(k)}$.

Proof. The injectivity follows from Theorem 4.18(1). The triple tensored up to S is identified with

$$(\mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n), \mathrm{H}^j(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{H}}\,\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n), \varphi_j)$$

by Theorem 5.26. We have shown that the map $\mathrm{H}^j(\mathcal{X}, \mathrm{Fil}^{\ell}_{\mathrm{H}}\,\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n) \to \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n)$ is injective, and the divided Frobenius φ_j generates the image: for $j \le p-2$, this was the main result in our previous paper [LL20, Theorem 7.22 and Corollary 7.25]; and for $j = p-1$, use Lemma 5.3 and Corollary 5.8. Using the 'if' part of Lemma 2.16, we see that $\big(\mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n), \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n, \mathcal{I}^{[j]}_{\mathrm{crys}}), \varphi_i : \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n, \mathcal{I}^{[i]}_{\mathrm{crys}}) \to \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n)\big)$ is an object in $\mathrm{FM}_{W(k)}$. □

5.5 Comparison to étale cohomology

In this section, we study how the crystalline cohomology $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}/S_n)$ compares to the étale cohomology $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z})$ in the boundary case $e \cdot i = p-1$. We shall freely use the notation and terminology from § 2.

We first treat the case when $e = 1$ and $p-1$, in which case Corollary 5.27 shows that

$$M := \big(\mathrm{H}^{p-1}_{\mathrm{crys}}(\mathcal{X}_n/W_n), \mathrm{H}^{p-1}_{\mathrm{crys}}(\mathcal{X}_n/W_n, \mathcal{I}^{[p-1]}_{\mathrm{crys}}), \varphi_{p-1}\big)$$

is an object in $\mathrm{FM}_{W(k)}$.

Theorem 5.28. *With notation as above, there exists a natural map* $\eta\colon \mathrm{H}^{p-1}_{\text{ét}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z})(p-1) \to T_{\mathrm{FM}}(M)$ *of G_K-representations such that:*

(1) $\ker(\eta)$ *is an unramified representation of G_K killed by p;*
(2) $\mathrm{coker}(\eta)$ *sits in a natural exact sequence* $0 \to W \to \mathrm{coker}(\eta) \to W'$, *where* $W \cong \ker(\eta)$ *and* $W' \cong \ker(\mathrm{Sp}^{p-1}_n)$ *is given by the kernel of specialization map in degree $p-1$.*

Note that by our Corollary 4.15(3), $\ker(\mathrm{Sp}^{p-1}_n)$ is also an unramified G_K-representation killed by p. The $T_{\mathrm{FM}}(M)$ in the above theorem is what we meant by $\rho^{p-1}_{n,\mathrm{FL}}$ in Theorem 1.9.

Proof. Let $\mathfrak{M} := \mathrm{H}^{p-1}_{\Delta}(\mathcal{X}/\mathfrak{S}_n)$ (note that here we do not have a Frobenius twist) and $\mathcal{M} = \mathrm{H}^{p-1}_{\mathrm{crys}}(\mathcal{X}/S_n)$. We have shown that the natural exact sequence (⚀) induces a natural exact sequence in $\mathrm{Mod}^{\varphi,p-1,\nabla}_{S,\mathrm{tor}}$:

$$0 \longrightarrow \underline{\mathcal{M}}(\mathfrak{M}) \longrightarrow \mathcal{M} \longrightarrow \overline{M} \longrightarrow 0$$

(see Propositions 5.6, 5.7, 5.13 and 5.19 and Corollaries 5.8 and 5.18 for descriptions of the filtrations, Frobenii action, and connections). Furthermore, our Theorem 5.26 says that $\mathcal{M} = \underline{\mathcal{M}}_{\mathrm{FM}}(M)$. Therefore, by left exactness of T_S, we have a natural sequence of G_K-representations:

$$0 \to T_S(\underline{\mathcal{M}}(\mathfrak{M})) \hookrightarrow T_S(\mathcal{M}) = T_{\mathrm{FM}}(M) \to T_S(\overline{M}).$$

On the other hand, we also have natural maps of G_K-representations:

$$\eta : \mathrm{H}^{p-1}_{\text{ét}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z})(p-1) \xrightarrow{\sim} T_{\mathfrak{S}}(\mathfrak{M})(p-1) \xrightarrow[\alpha]{\sim} T^{p-1}_{\mathfrak{S}}(\mathfrak{M}) \xrightarrow{\iota} T_S(\underline{\mathcal{M}}(\mathfrak{M}))\,.$$

The first isomorphism is proved by [LL20, Cor. 7.4, Rem. 7.5]. As explained before Lemma 2.18, the map $\iota \circ \alpha$ is a map compatible with G_K-actions if the natural map $f : \mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \to \mathcal{M}(\mathfrak{M}) \otimes_S A_{\mathrm{crys}}$ is compatible with G_K-actions on the both sides, where the G_K-action on $\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ given by $\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \simeq \mathrm{H}^{p-1}_{\Delta}(\mathcal{X}_{\mathcal{O}_{\mathbf{C}}}/A_{\mathrm{inf}})$ and the G_K-action on $\underline{\mathcal{M}}(\mathfrak{M}) \otimes_S A_{\mathrm{crys}}$ are defined by formula (2.17). To prove that f is compatible with G_K-actions, note that the natural map $f' : \mathrm{H}^{p-1}_{\mathfrak{S}}(\mathcal{X}_{\mathcal{O}_\mathbf{C}}/A_{\mathrm{inf}}) \to \mathrm{H}^{p-1}_{\mathrm{crys}}(\mathcal{X}_{\mathcal{O}_\mathbf{C}}/A_{\mathrm{crys}})$, which is compatible with G_K-actions, factors through $f : \mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \to \underline{\mathcal{M}}(\mathfrak{M}) \otimes_S A_{\mathrm{crys}}$ by using the inclusion $\underline{\mathcal{M}}(\mathfrak{M}) \subset \mathcal{M}$ and the isomorphism $\beta : \mathcal{M} \otimes_S A_{\mathrm{crys}} \simeq \mathrm{H}^{p-1}_{\mathrm{crys}}(\mathcal{X}_{\mathcal{O}_\mathbf{C}}/A_{\mathrm{crys}})$. So it suffices to check that $\underline{\mathcal{M}}(\mathfrak{M}) \otimes_S A_{\mathrm{crys}} \to \mathcal{M} \otimes_S A_{\mathrm{crys}} \simeq \mathrm{H}^{p-1}_{\mathrm{crys}}(\mathcal{X}_{\mathcal{O}_\mathbf{C}}/A_{\mathrm{crys}})$ are compatible with G_K-actions. The compatibility of first map is due to the fact that $\underline{\mathcal{M}}(\mathfrak{M}) \subset \mathcal{M}$ is stable under ∇ on $\mathcal{M}$ by Corollary 5.18, and the compatibility of the second isomorphism is proved in [LL20, § 5.3]. In summary, we obtain a natural map $\eta : \mathrm{H}^{p-1}_{\mathrm{ét}}(\mathcal{X}_\mathbf{C}, \mathbb{Z}/p^n\mathbb{Z})(p-1) \to T_{\mathrm{FM}}(M)$ of G_K-representations.

Now we shall justify the two extra statements concerning kernel and cokernel of η. Since T_S is left exact, $\ker(\eta) \simeq \ker(\iota)$ which is unramified and killed by p, thanks to Corollary 2.20.

An easy diagram chase gives us a natural exact sequence:

$$0 \to \mathrm{coker}(\iota) \to \mathrm{coker}(\eta) \to T_S(\overline{M}).$$

By Corollary 2.20 we have $\mathrm{coker}(\iota) \cong \ker(\iota)$. The fact that $T_S(\overline{M}) \cong \ker(\mathrm{Sp}^{p-1}_n)$ follows from Corollary 5.20 and Theorem 4.14. □

Remark 5.29. (1) From the proof, we see that the appearance of $\ker(\eta)$ and V is due to the defect of a key functor in integral p-adic Hodge theory, and the potential u-torsion in the degree p (mod p^n) prismatic cohomology of $\mathcal{X}$ is to be blamed for the appearance of V'.

(2) It is unclear to us if the whole $\mathrm{coker}(\eta)$ is unramified and/or killed by p. It could even very well be the case that the sequence $0 \to W \to \mathrm{coker}(\eta) \to W'$ is split exact (in particular, right exact) as G_K-representations. One would need extra input from integral p-adic Hodge theory, especially a further study of Breuil and Fontaine–Laffaille modules in the boundary degree case, in order to obtain such refinements.

We now discuss the case where $e > 1$ but $h \leq p-2$. We first recall that for $i \leq p-1$, in [LL20, § 5.2] we have shown that $\mathcal{M}^i_n := (\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}/S_n), \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}/S_n, \mathcal{I}^{[i]}), \varphi_i)$ is an object in ${}^{\sim}\mathrm{Mod}^{\varphi,i}_S$. By the discussion before equation (7.24) in [LL20], we get the following exact sequence for $i \leq h \leq p-2$:

$$\cdots \mathrm{H}^{i-1}_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}) \to \mathrm{H}^i_{\mathrm{ét}}(\mathcal{X}_\mathbf{C}, \mathbb{Z}/p^n\mathbb{Z}(h)) \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[h]}_{\mathrm{crys}}) \xrightarrow{\varphi_h - 1} \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}). \tag{5.30}$$

Note that the crucial input is [AMMN22, Theorem F]. Thanks to $A_{\mathrm{crys},n}$ being flat over S_n, we have

$$\begin{aligned}\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[h]}_{\mathrm{crys}}) \cong \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{I}^{[h]}_{\mathrm{crys}}) \otimes_S A_{\mathrm{crys}} \text{ and } \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}) \\ \cong \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n) \otimes_S A_{\mathrm{crys}}.\end{aligned}$$

In this case, we can still define

$$T_S(\mathcal{M}^i_n) := \mathrm{Fil}^i(\mathcal{M}^i_n \otimes_S A_{\mathrm{crys}})^{\varphi_i = 1} = \ker\{\varphi_i - 1 : \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[i]}_{\mathrm{crys}}) \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})\}.$$

The only difference is that the natural map $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}}) \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ is not expected to be injective without the condition $e \cdot i < p-1$.

Proposition 5.31. *With notation as above, we have a functorial isomorphism* $T_S(\mathcal{M}^i_n) \cong \mathrm{H}^i_{\mathrm{ét}}(\mathcal{X}_\mathbf{C}, \mathbb{Z}/p^n\mathbb{Z}(i))$.

Proof. By (5.30), it suffices to show that $\varphi_i - 1 : \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[i]}_{\mathrm{crys}}) \longrightarrow \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$ is surjective for $i < p-2$. Choose an m large enough such that $\varphi_i(\mathrm{Fil}^m S_n) = 0$. So clearly $\varphi_i - 1$

https://doi.org/10.1112/S0010437X23007261 Published online by Cambridge University Press

restricted to the image of $\mathrm{Fil}^m S \otimes \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$ is bijective.[9] Hence, it suffices to show that

$$\begin{aligned}\varphi_i - 1 : \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[i]}_{\mathrm{crys}})/\,\mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})\\ \longrightarrow \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})/\,\mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})\end{aligned}$$

is surjective. Now we claim that both sides are finitely generated $W_n(\mathcal{O}^\flat_{\mathbf{C}})$-modules. Then the surjectivity of $\varphi_i - 1$ follows Lemma 5.34 below.

To check that both

$$\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[i]}_{\mathrm{crys}})/\,\mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$$

and

$$\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})/\,\mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$$

are finitely generated over $W_n(\mathcal{O}^\flat_{\mathbf{C}})$, it suffices to check that

$$\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})/\,\mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$$

and

$$\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)/\,\mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$$

are finitely generated $\mathfrak{S}_n$-modules. This is clear for $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)/\,\mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$: it is known that $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ is a finitely generated S_n-module (see [LL20, Proposition 7.19]). For $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})/\,\mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$, consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^{i-1}_{\mathrm{qSyn}}(\mathcal{X}_n, \mathbb{\Delta}^{(1)}/\,\mathrm{Fil}^i_{\mathrm{N}}\,\mathbb{\Delta}^{(1)}) \arrow[r, "\alpha"] \arrow[d, "\wr"] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}_n, \mathrm{Fil}^i_{\mathrm{N}}\,\mathbb{\Delta}^{(1)}) \arrow[r, "\beta"] \arrow[d] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}_n, \mathbb{\Delta}^{(1)}_{-/\mathfrak{S}}) \arrow[r] \arrow[d, "\iota"] & \cdots \\
\mathrm{H}^{i-1}_{\mathrm{qSyn}}(\mathcal{X}_n, \mathrm{dR}^\wedge_{-/\mathfrak{S}}\,/\,\mathrm{Fil}^i_{\mathrm{H}}\,\mathrm{dR}^\wedge_{-/\mathfrak{S}}) \arrow[r, "\alpha'"] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}_n, \mathrm{Fil}^i_{\mathrm{H}}\,\mathrm{dR}^\wedge_{-/\mathfrak{S}}) \arrow[r, "\beta'"] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}_n, \mathrm{dR}^\wedge_{-/\mathfrak{S}}) \arrow[r] & \cdots
\end{tikzcd}$$

Since $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}_n, \mathrm{dR}^\wedge_{\mathcal{X}_n/\mathfrak{S}_n})$ is finitely generated over S_n, the image of $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})/\,\mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ inside $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)/\,\mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ is also finitely $\mathfrak{S}_n$-generated. Here we have used the fact that $S_n/\,\mathrm{Fil}^m S_n$ is finitely generated over $\mathfrak{S}_n$. Note that $\ker(\beta') = \mathrm{Im}(\alpha')$ is also finitely generated over $\mathfrak{S}_n$. So $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})/\,\mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ is finitely $\mathfrak{S}_n$-generated. ☐

Lemma 5.32. *Let $C^\flat$ be a characteristic p algebraically closed complete non-Archimedean field. Denote its ring of integers by $\mathcal{O}^\flat_C$ with maximal ideal $m^\flat$ and residue field $k^\flat$. Let M and N be two finitely generated $\mathcal{O}^\flat_C$ modules, let $F\colon M \to N$ be a Frobenius semi-linear map, and let $G\colon M \to N$ be a linear map. The following assertions are equivalent.*

(1) *The map $F - G\colon M \to N$ is surjective.*
(2) *The cokernel of $F - G\colon M \to N$ is finite.*
(3) *The induced map $\overline{F - G}\colon M/m^\flat M \to N/m^\flat N$ is surjective.*
(4) *The cokernel of $\overline{F - G}\colon M/m^\flat M \to N/m^\flat N$ is finite.*

Proof. It is clear that (1) $\implies$ (2), (3) $\implies$ (4). Below we shall show (4) $\implies$ (1).

Without loss of generality we may assume that both M and N are finite free over $\mathcal{O}^\flat_C$. Indeed, let us choose maps from finite free modules, say P and Q, to M and N such that these maps are

[9] From now on, we abusively denote this image by $\mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$.

isomorphisms after modulo $m^\flat$. By Nakayama's lemma we see that these maps are surjective. Lift the two maps F and G, to get the diagram

$$\begin{tikzcd} P \arrow[r, "\widetilde{F}-\widetilde{G}"] \arrow[d] & Q \arrow[d] \\ M \arrow[r, "F-G"] & N \end{tikzcd}$$

By our choice of P and Q, condition (4) still holds for the top arrow. Since vertical arrows are surjective, it suffices to show that the top arrow is surjective. Therefore, we may and do assume M and N are finite free.

Let us name the reduction of M and N by V and W which are finite dimensional $k^\flat$-vector spaces, and denote the reduction of F and G by f and g. We claim there are exhaustive increasing filtrations Fil_i with $0 \le i \le \ell$ on V and W respectively such that:

- the maps f and g respect these two filtrations;
- the induced $f\colon \mathrm{Fil}_0 V \to \mathrm{Fil}_0 W$ is surjective;
- the induced $f\colon \mathrm{gr}_i V \to \mathrm{gr}_i W$ is 0 for all $1 \le i \le \ell$; and
- the induced $g\colon \mathrm{gr}_i V \to \mathrm{gr}_i W$ is an isomorphism for all $1 \le i \le \ell$.

To see the existence of such filtrations, we consider the following process. Notice the image of $f\colon V \to W$ is a $k^\flat$ subspace, and consider the map $g\colon V \to W/\mathrm{Im}(f)$. By the assumption of $\mathrm{Coker}(f-g)$ being finite, this map must be surjective, and finally we let

$$\mathrm{Fil}^0 V = \mathrm{Ker}\big(g\colon V \to W/\mathrm{Im}(f)\big),\ \ \mathrm{Fil}^0 W = \mathrm{Im}(f).$$

Replace V and W with $\mathrm{Fil}^0 V$ and $\mathrm{Fil}^0 W$ and repeat the above steps. This process terminates when we arrive at $\mathrm{Im}(f) = W$, and it will terminate as each time the dimension of W will drop. This way we get a decreasing filtration, after reversing the indexing order we arrive at the desired increasing filtration.

Choosing a subvector space $V_0 \subset \mathrm{Fil}_0(M/m^\flat M)$ on which f is an isomorphism, and lifting the basis of $\mathrm{gr}_i(M/m^\flat M)$ for $1 \le i \le \ell$ and the basis of V_0 all the way to elements in M, we generate a finite free submodule $\widetilde{M}$. Now we consider the map $\widetilde{M} \to N$.

After choosing bases, we may regard both sides as $\mathcal{O}_C^\flat$ points of formal affine space over $\mathcal{O}_C^\flat$, and the map $F-G$ can be promoted to an algebraic map $h\colon \mathrm{Spf}(\mathcal{O}_C^\flat\langle \underline{X}\rangle) \to \mathrm{Spf}(\mathcal{O}_C^\flat\langle \underline{Y}\rangle)$. Note that by our choice of $\widetilde{M}$, these two formal affine spaces have the same dimension. Our choice of $\widetilde{M}$ guarantees that the reduction of h is finite, due to the next lemma. Therefore, the rigid generic fiber map h^{rig} is also finite by [BGR84, 6.3.5 Theorem 1], which implies it is flat by miracle flatness [Sta21, Tag 00R4], hence inducing a surjective map at the level of $C^\flat$-points.[10] ☐

The following lemma was used in the proof above. We thank Johan de Jong for providing an elegant proof.

Lemma 5.33. *Let k be a field, let $m > 1$ be an integer, and let (a_{ij}) be an $n \times n$ matrix with entries in k. Let $\overline{h}\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ be the morphism given by $\overline{h}^\sharp(y_i) = x_i^m + \sum_j a_{i,j}x_j$. Then $\overline{h}$ is a finite morphism.*

[10] Note that the $C^\flat$-points of the rigid generic fiber of an admissible formal scheme over $\mathcal{O}_C^\flat$ are the same as the $\mathcal{O}_C^\flat$-points of the formal scheme (see [Bos14, § 8.3]).

Proof. This map can be compactified to a morphism between $\mathbb{P}^n_k$ preserving the infinity hyperplane. When restricted to the infinity hyperplane, the map becomes $[x_1\colon \cdots : x_n] \mapsto [x_1^m\colon \cdots : x_n^m]$, which is non-constant. Finally, observe that any endomorphism of $\mathbb{P}^n_k$ is either finite or constant. $\square$

Here we have crucially used the algebraic closedness of $\mathcal{O}_C^\flat$. Below is an example suggested to us by Johan de Jong illustrating the failure of (4) $\implies$ (3) when one drops the algebraically closed assumption. Starting with the field $L_0 = \mathbb{F}_p(t^{1/p^\infty})$ and picking a basis of $\mathrm{H}^1_{\text{ét}}(L_0, \mathbb{F}_p)$, we may find an (enormous!) Galois pro-p infinite field extension L_1 such that the induced map on $\mathrm{H}^1_{\text{ét}}(-, \mathbb{F}_p)$ kills every basis vector except the first. Repeating this process, we arrive at a perfect field L such that $\mathrm{H}^1_{\text{ét}}(L, \mathbb{F}_p)$ is one -dimensional over $\mathbb{F}_p$.

From the above we immediately obtain the following lemma.

Lemma 5.34. *Let M and N be two finitely generated A_{inf} modules. Let $F\colon M \to N$ be a Frobenius linear map and $G\colon M \to N$ be a linear map. Then the cokernel of $F - G$ (which is a $\mathbb{Z}_p$-linear map) is finitely generated over $\mathbb{Z}_p$ if and only if it is* 0*.*

Proof. The 'if' part is trivial. For the 'only if' part, using right exactness of tensors and Lemma 5.32, we conclude that the cokernel is zero after modulo p. Now since a finitely generated $\mathbb{Z}_p$ module is 0 if and only if its reduction modulo p is so, we get that the cokernel is zero. $\square$

6. An example

Inspired by the example in [BMS18, § 2.1], let us work out a direct generalization of their example (as suggested in [BMS18, Remark 1.3]) in this subsection. This example answers a question of Breuil [Bre02, Question 4.1] negatively.

Fix a positive integer n.[11] Let $\mathcal{E}_0$ be an ordinary elliptic curve over an algebraically closed field k of characteristic $p > 0$. Let $\mathcal{E}$ over $\mathrm{Spec}(W(k))$ be a lift. By the theory of canonical subgroups (see [Katz73, § 3.4]), we have a closed immersion $\mu_{p^n} \subset \mathcal{E}[p^n]$ of finite flat group schemes over $\mathrm{Spec}(W(k))$. Let $\mathcal{O}_K := W(k)[\zeta_{p^n}]$ and choose $\zeta_{p^n} - 1$ to be the uniformizer in order to get $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$. To avoid confusion let us denote its Eisenstein polynomial by

$$E = d = \frac{(u+1)^{p^n} - 1}{(u+1)^{p^{n-1}} - 1} \in \mathfrak{S}.$$

On $\mathrm{Spec}(\mathcal{O}_K)$ we have the canonical group scheme homomorphism $\mathbb{Z}/p^n \to \mu_{p^n}$.

Construction 6.1. Let $\mathcal{X} := [\mathcal{E}/(\mathbb{Z}/p^n)]$, a Deligne–Mumford stack which is smooth proper over $\mathrm{Spec}(\mathcal{O}_K)$. Here the action of $\mathbb{Z}/p^n$ on $\mathcal{E}$ is via μ_{p^n}. The generic fiber of $\mathcal{X}$ is the elliptic curve $\mathcal{E}_K/\mu_{p^n}$ and the special fiber is $\mathcal{E}_0 \times B(\mathbb{Z}/p^n)$. We have a factorization

$$\mathcal{E} \to \mathcal{X} \to \mathcal{E}/\mu_{p^n} =: \mathcal{E}'.$$

We want to understand the various cohomology theories of $\mathcal{X}$, but first we need a remark on how these cohomology theories (originally defined for schemes or formal schemes over $\mathcal{O}_K$) can be made meaningful for $\mathcal{X}$.

Remark 6.2 (cf. [Li22, Remark 4.9]). We may regard $\mathcal{X}$ as a sheaf valued in groupoids on the big étale site of $\mathcal{O}_K$. Then for any étale sheaf $\mathcal{F}$, it makes sense to talk about $\mathcal{F}(\mathcal{X})$. In fact, following [ABM21, Construction 2.7], the above can be generalized to any syntomic sheaf

[11] We suggest that first-time readers simply take $n = 1$ in order to simplify the notation and formulas.

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(all sheaves considered in this paper are quasi-syntomic sheaves) and any syntomic stack (such as our $\mathcal{X}$ here). Just as in [ABM21] we can use the cover $\mathcal{E} \to \mathcal{X}$ to get the Leray spectral sequence, that is, the cohomologies of m-fold self-products of $\mathcal{E}$ over $\mathcal{X}$ converge to cohomologies of $\mathcal{X}$. Since the cover is a $\mathbb{Z}/p^n$-torsor, the above spectral sequence becomes the usual Hochschild–Serre spectral sequence, that is, we can use the group cohomology of $\mathbb{Z}/p^n$ with coefficients being cohomologies of $\mathcal{E}$ to compute cohomologies of $\mathcal{X}$. For an even more general treatment concerning higher stacks (see [KP21, § 2]).

Let us first record the structure of the prismatic cohomology of $\mathcal{E}_{\mathcal{O}_K}$ relative to $\mathfrak{S}$. We need the following lemma explicating the Frobenius operator on the (-1) Breuil–Kisin twist $\mathfrak{S}\{-1\}$ (see [BL22, § 2.2]).

Lemma 6.3. *The Frobenius module $\mathfrak{S}\{-1\}$ has a generator x such that*

$$\varphi(x) = E(u) \cdot p/E(0) \cdot x.$$

Proof. We know that modulo u the Breuil–Kisin prism $\mathfrak{S}$ reduces to crystalline prism, whose (-1)-twist has a canonical generator $\overline{x}$ satisfying $\varphi(\overline{x}) = p \cdot \overline{x}$. Lifting this generator, we see that there is a generator x' of $\mathfrak{S}\{-1\}$ such that $\varphi(x') = a \cdot x'$ with $a \equiv p \bmod u$. On the other hand, we know a is necessarily $E(u) \cdot$ unit, due to [BL22, Construction 2.2.14]. Therefore, we see that $a = E(u) \cdot p/E(0) \cdot v'$ where $v' \in \mathfrak{S}^\times$ and reduces to $1 \bmod u$. It is a simple exercise to verify that v' is of the form $\varphi(v)/v$ for some unit $v \in \mathfrak{S}^\times$ satisfying $v \equiv 1 \bmod u$ as well. Finally, $x = x'/v$ is our desired generator. □

In our concrete situation, the Eisenstein polynomial d of $\zeta_{p^n} - 1$ has constant term p. Therefore, our $\mathfrak{S}\{-1\}$ has a generator x such that $\varphi(x) = d \cdot x$.

Proposition 6.4. *We have an isomorphism of Frobenius modules over $\mathfrak{S}$:*

(1) $\mathrm{H}^0_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \cong \mathfrak{S}$;
(2) $\mathrm{H}^2_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \cong \mathfrak{S}\{-1\}$; *and*
(3) $\mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \simeq \mathfrak{S} \cdot \{e_1, e_2\}$ *with its Frobenius action given by* $\varphi(e_1) = e_1$, *and* $\varphi(e_2) = a \cdot e_1 + d \cdot e_2$ *for some* $a \in \mathfrak{S}$.

Proof. It is well known that an elliptic curve has torsion-free crystalline cohomology. Therefore, by Remark 3.9, we know all these prismatic cohomology groups are finite free $\mathfrak{S}$-modules.

The map $\mathcal{X} \to \mathrm{Spf}(\mathfrak{S}/d)$ always induces an isomorphism on H^0_{Δ} by Hodge–Tate comparison; this proves the first identification.

The second identification is well known. For instance, the relative prismatic Chern class [BL22, § 7.5] of (the line bundle associated with) the origin $0 \in \mathcal{E}_{\mathcal{O}_K}(\mathcal{O}_K)$ gives a map $c\colon \mathfrak{S}\{-1\} \to \mathrm{H}^2_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})$. Reducing mod u, this reduces to the first Chern class map in the crystalline cohomology which is well known to be an isomorphism. Since both source and target are finite free $\mathfrak{S}$-module, the map c is an isomorphism.

The cup product gives rise to a map of finite free Frobenius $\mathfrak{S}$-modules: $\bigwedge^2_{\mathfrak{S}} \mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \to \mathrm{H}^2_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})$. Modulo u this map reduces to the analogous map in the crystalline cohomology which is again well known to be an isomorphism, hence it is an isomorphism before mod u. Therefore, it suffices to justify the existence of e_1. Since $\varphi(u) = u^p$, we see that

$$\left(\mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})\right)^{\varphi=1} \cong \left(\mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})/u\right)^{\varphi=1}.$$

Now we may use the crystalline comparison $\mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})/u \cong \mathrm{H}^1_{\mathrm{crys}}(\mathcal{E}_0/W)^{(-1)}$ and the fact that $\mathcal{E}_0$ is ordinary to conclude the existence of e_1. □

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Next let us compute the prismatic cohomology $\mathrm{H}^*_{\Delta}(\mathcal{X}/\mathfrak{S})$. We consider the Leray spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(\mathbb{Z}/p^n, \mathrm{H}^j_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})) \Longrightarrow \mathrm{H}^{i+j}_{\Delta}(\mathcal{X}/\mathfrak{S})$$

which is compatible with Frobenius actions. In order to understand the E_2 terms, we need the following lemma.

Lemma 6.5. *The action of $\mathbb{Z}/p^n$ on $\mathrm{H}^j_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})$ is trivial.*

Proof. Let us use the p-completely flat base change $\mathfrak{S} \hookrightarrow W(C^\flat)$. Since our prismatic cohomologies, as $\mathfrak{S}$-modules, are free, we get injections $\mathrm{H}^j_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \hookrightarrow \mathrm{H}^j_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \otimes_{\mathfrak{S}} W(C^\flat)$ compatible with the $\mathbb{Z}/p^n$-action. Using the étale comparison [BMS18, Theorem 1.8.(iv)], the target is canonically identified with $\mathrm{H}^j_{\text{ét}}(\mathcal{E}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(C^\flat)$. We conclude that the $\mathbb{Z}/p^n$-action on the target is trivial by comparison to the topological situation. □

Therefore, the second page, which is the starting page, of the above spectral sequence is as follows:

$$\begin{tikzcd}
\mathfrak{S}\{-1\} \arrow[drr] & 0 & \mathfrak{S}\{-1\}/p^n & \cdots \\
\mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \arrow[drr, "d_2"'] & 0 & \mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})/p^n & \cdots \\
\mathfrak{S} & 0 & \mathfrak{S}/p^n & \cdots
\end{tikzcd} \tag{⚃}$$

Of interest to us is the differential

$$d_2 \colon E_2^{0,1} = \mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \longrightarrow E_2^{2,0} \cong \mathfrak{S}/p^n.$$

Using the multiplicative structure of the spectral sequence, this arrow determines the rest of the arrows; by degree reason the spectral sequence degenerates on the third page $E_3^{i,j} = E_\infty^{i,j}$.

Lemma 6.6. *The differential d_2 is divisible by u. In other words, it is zero after reduction modulo u.*

Proof. Let us consider the reduction modulo u of the spectral sequence (⚃), which computes the crystalline cohomology of $\mathcal{X}/W$ by the crystalline comparison. Using the fact that $\mathrm{H}^2_{\mathrm{crys}}(B(\mathbb{Z}/p^n)/W) \cong W/p^n$ (see, for instance, [Mon21, Theorem 1.2]), we see that d_2 modulo u must be zero. □

Lemma 6.7. *We have $d_2(e_1) = 0$.*

Proof. This is because d_2 is Frobenius equivariant. Now Proposition 6.4(3) implies $e_1 \in \mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})$ is fixed by Frobenius, yet Lemma 6.6 says its image under d_2 is divisible by u. So its image is divisible by arbitrary powers of u, hence must be zero. □

Lemma 6.8. *After scaling e_2 by a unit in $\mathbb{Z}_p^\times$ we have $d_2(e_2) = (u+1)^{p^{n-1}} - 1$.*

Proof. Note that $\varphi(e_2) \equiv d \cdot e_2$ (modulo e_1) by Proposition 6.4(3), so Lemma 6.7 implies that $d_2(e_2) \in \mathfrak{S}/p^n$ must satisfy the same Frobenius eigenclass condition. The next lemma guarantees that $d_2(e_2) = b \cdot ((u+1)^{p^{n-1}} - 1)$ for some $b \in \mathbb{Z}/p^n$. Étale comparison for prismatic cohomology says that base-changing the spectral sequence (⚃) along $\mathfrak{S} \hookrightarrow W(C^\flat)$ gives a spectral sequence computing the étale cohomology of $\mathcal{X}_C$ (base-changed along $\mathbb{Z}_p \hookrightarrow W(C^\flat)$). Since $\mathcal{X}_C$ is an elliptic

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curve, its second étale cohomology has no torsion, hence the base-changed d_2 must be surjective. In particular, we see that $b \notin p \cdot \mathbb{Z}/p^n$, hence b must be a unit in $(\mathbb{Z}/p^n)^\times$. $\square$

In the proof above, we have utilized the following lemma.

Lemma 6.9. *For any $m \leq n$, we have an exact sequence*

$$0 \to \mathbb{Z}/p^m \cdot ((u+1)^{p^{n-1}} - 1) \to \mathfrak{S}/p^m \xrightarrow{\varphi - d} \mathfrak{S}/p^m.$$

Proof. First of all, let us check that $(u+1)^{p^{n-1}} - 1$ does satisfy the Frobenius action condition. Recalling that

$$d = \frac{(u+1)^{p^n} - 1}{(u+1)^{p^{n-1}} - 1},$$

it suffices to know that $(u+1)^{p^n} \equiv (u^p + 1)^{p^{n-1}}$ modulo p^n. When $n = 1$ this is well known; induction on n proves the statement.

Next we verify this exact sequence for $m = 1$. In that situation $\mathfrak{S}/p \cong k[\![u]\!]$, and the Frobenius condition becomes $f^p = u^{p^{n-1}(p-1)} \cdot f$. One immediately verifies that $f \in \mathbb{F}_p \cdot u^{p^{n-1}}$.

Finally, we finish the proof by induction on m and applying the snake lemma to the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{S}/p^m \arrow[r, "\cdot p"] \arrow[d, "\varphi - d"] & \mathfrak{S}/p^{m+1} \arrow[r] \arrow[d, "\varphi - d"] & \mathfrak{S}/p \arrow[r] \arrow[d, "\varphi - d"] & 0 \\
0 \arrow[r] & \mathfrak{S}/p^m \arrow[r, "\cdot p"] & \mathfrak{S}/p^{m+1} \arrow[r] & \mathfrak{S}/p \arrow[r] & 0
\end{tikzcd}$$

Notice that we have verified that the kernel of the middle vertical arrow surjects onto the kernel of the right vertical arrow, thanks to the previous two paragraphs. The snake lemma tells us that the kernel of the middle vertical arrow has length $m+1$, but we also know that $\mathbb{Z}/p^{m+1} \cdot \big((u+1)^{p^{n-1}} - 1\big)$ sits inside the kernel. $\square$

From now on let us scale e_2 by the p-adic unit such that $d_2(e_2) = (u+1)^{p^{n-1}} - 1$. Using the multiplicativity of the spectral sequence (⚃), we can compute the prismatic cohomology of $\mathcal{X}$. Let us record the result below.

Corollary 6.10. *The prismatic cohomology ring of $\mathcal{X}/\mathfrak{S}$ is*

$$\mathrm{H}^*_{\Delta}(\mathcal{X}/\mathfrak{S}) \cong \mathfrak{S}\langle e, f\rangle[g]/((u+1)^{p^{n-1}} - 1 \cdot g, p^n \cdot g, f \cdot g),$$

where e and f have degree 1 and are pulled back to e_1 and $p^n \cdot e_2$ respectively inside $\mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})$, and g has degree 2, being the generator of $E_3^{2,0} = E_\infty^{2,0}$. Moreover, the Frobenius action is given by

$$\varphi(e) = e, \varphi(f) = p^n a' \cdot e + d \cdot f, \text{ and } \varphi(g) = g.$$

In particular, we see that

$$\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty] \cong \mathfrak{S}/((u+1)^{p^{n-1}} - 1, p^n) \cdot g$$

and

$$\mathrm{H}^\ell_{\Delta}(\mathcal{X}/\mathfrak{S}) \simeq \mathfrak{S}/((u+1)^{p^{n-1}} - 1, p^n)$$

for all $\ell \geq 3$ generated by either $g^{\ell/2}$ or $e \cdot g^{(\ell-1)/2}$ depending on the parity of ℓ.

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Here a' is a p-adic unit (which we used to scale e_2) times the constant a from Proposition 6.4(3). We remark that g can be taken as a generator of the group cohomology $\mathrm{H}^2(\mathbb{Z}/p^n, \mathbb{Z}_p)$.

Later on we will produce a schematic example using approximations of $B(\mathbb{Z}/p^n)$, but before that let us observe that our stacky example matches some predictions made in the following remark.

Remark 6.11. The discussion in § 4.1 extends to smooth proper Deligne–Mumford stacks, such as our $\mathcal{X}$. Since the generic fiber of $\mathcal{X}$ is $\left(\mathcal{E}/\mu_{p^n}\right)_K$, the map $g\colon \mathcal{X} \to$ Néron model of $\mathcal{X}_K$ becomes the natural map $[\mathcal{E}/(\mathbb{Z}/p^n)] \to \mathcal{E}/\mu_{p^n}$. Taking the special fiber and factoring through $\mathrm{Alb}(\mathcal{X}_0) = \mathcal{E}_0$, we see the map f becomes the natural quotient map $\mathcal{E}_0 \to \mathcal{E}_0/\mu_{p^n}$ which has kernel μ_{p^n}. Note that when $n = 1$, we have $e = p - 1$, and our Corollary 4.6(3) indeed predicts that $\ker(f)$ can be at worst a form of several copies of μ_p.

Since $\mathcal{X}_0 \cong \mathcal{E}_0 \times B(\mathbb{Z}/p^n)$, we know its π_1 is abelian with torsion given by $\mathbb{Z}/p^n$. Consequently, the torsion part in $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_0, \mathbb{Z}_p)$ is also given by $\mathbb{Z}/p^n$. Since $\mathcal{X}_C$ is an elliptic curve, its étale cohomology is torsion-free. Hence, the specialization map in degree 2 for p-adic étale cohomology has kernel given by $\mathbb{Z}/p^n$. This matches up with what Theorem 4.14 predicts. Indeed, since $\varphi(g) = g$, we see that

$$\left(\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty]\right)^{\varphi=1} = \left(\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty]/u\right)^{\varphi=1} = \left(W/p^n \cdot g\right)^{\varphi=1} = \mathbb{Z}/p^n \cdot g.$$

Here in the second identification we have used the fact that u divides $(u+1)^{p^{n-1}} - 1$.

The above stacky example can be turned into a scheme example, by the procedure of approximation explained below.

Construction 6.12. Choose a representation V of $\mathbb{Z}/p^n$ over $\mathbb{Z}_p$, such that inside $\mathbb{P}(V)$ one can find a $\mathbb{Z}/p^n$-stable complete intersection 3-fold $\mathcal{Y}$ with no fixed point and smooth proper over $\mathbb{Z}_p$ (see [BMS18, 2.7–2.9]). We now form $\mathcal{Z} \coloneqq (\mathcal{E} \times_{\mathbb{Z}_p} \mathcal{Y})_{\mathcal{O}_K}/(\mathbb{Z}/p^n)$, which is a smooth proper relative 4-fold over $\mathcal{O}_K$. Here the action of $\mathbb{Z}/p^n$ is the diagonal action.

Let us show that the prismatic cohomology of $\mathcal{Z}/\mathfrak{S}$ approximates that of $\mathcal{X}/\mathfrak{S}$ in degrees ≤ 2 in a suitable sense.

Proposition 6.13. *The natural $\mathbb{Z}/p^n$-equivariant projection $(\mathcal{E} \times_{\mathbb{Z}_p} \mathcal{Y})_{\mathcal{O}_K} \to \mathcal{E}_{\mathcal{O}_K}$ gives rise to a map $\mathcal{Z} \to \mathcal{X}$, which induces isomorphisms*

$$\mathrm{H}^0_{\Delta}(\mathcal{X}/\mathfrak{S}) \xrightarrow{\cong} \mathrm{H}^0_{\Delta}(\mathcal{Z}/\mathfrak{S}) \quad \text{and} \quad \mathrm{H}^1_{\Delta}(\mathcal{X}/\mathfrak{S}) \xrightarrow{\cong} \mathrm{H}^1_{\Delta}(\mathcal{Z}/\mathfrak{S}).$$

Together with the similarly defined map $\mathcal{Z} \to \mathcal{Y}_{\mathcal{O}_K}/(\mathbb{Z}/p^n)$, we have

$$\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S}) \oplus \mathfrak{S}\{-1\} \xrightarrow{\cong} \mathrm{H}^2_{\Delta}(\mathcal{Z}/\mathfrak{S}).$$

Proof. We want to apply the Leray spectral sequence to the finite étale cover $(\mathcal{E} \times_{\mathbb{Z}_p} \mathcal{Y})_{\mathcal{O}_K} \to \mathcal{Z}$.

First we claim the natural embedding $\mathcal{Y}_{\mathcal{O}_K} \to \mathbb{P}(V)_{\mathcal{O}_K}$ induces an isomorphism of prismatic cohomology in degrees ≤ 2. It suffices to show the same for the Hodge–Tate cohomology, which in turn reduces us to showing it for the Hodge cohomology. This follows from $\mathcal{Y}$ being a smooth complete intersection inside $\mathbb{P}(V)$ (see [ABM21, Proposition 5.3]). Finally, it is well known that $\mathrm{H}^2_{\Delta}(\mathbb{P}(V)_{\mathcal{O}_K}/\mathfrak{S}) \cong \mathfrak{S}\{-1\}$ (see, for instance, [BL22, 10.1.6]).

Since $\mathrm{H}^1_{\Delta}(\mathcal{Y}_{\mathcal{O}_K}/\mathfrak{S}) = 0$, the Leray spectral sequence in degrees ≤ 2 is the direct sum of the spectral sequences for $\mathcal{X}$ and $\mathcal{Y}/(\mathbb{Z}/p^n)$, respectively. This gives the statement for cohomological degrees ≤ 1. Looking at the shape of the Leray spectral sequence for $\mathcal{Y}/(\mathbb{Z}/p^n)$, one easily sees

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that the $E_2^{0,2}$ term

$$\left(\mathrm{H}^2_{\Delta}(\mathcal{Y}_{\mathcal{O}_K}/\mathfrak{S})\right)^{\mathbb{Z}/p^n} \cong \mathrm{H}^2_{\Delta}(\mathcal{Y}_{\mathcal{O}_K}/\mathfrak{S}) \cong \mathfrak{S}\{-1\}$$

survives, hence proving the statement in cohomological degree 2. □

Remark 6.14. (1) Since $\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S}) \oplus \mathfrak{S}\{-1\} \cong \mathrm{H}^2_{\Delta}(\mathcal{Z}/\mathfrak{S})$ we know that $\mathrm{H}^2_{\Delta}(\mathcal{Z}/\mathfrak{S})[u^\infty] \cong \mathfrak{S}/((u+1)^{p^{n-1}}-1, p^n)$. In particular, its annihilator ideal is $((u+1)^{p^{n-1}}-1, p^n) \in \mathfrak{S}$, congruent to $(u^{p^{n-1}})$ modulo (p). The ramification index is $p^{n-1}(p-1)$, hence these examples demonstrate that the bound in Corollary 3.4 is sharp.

(2) Now assume $p \geq 3$; then $p-2+1 \geq 2$. Our previous result [LL20, Theorem 7.22] together with the fact that $\mathrm{H}^2_{\Delta}(\mathcal{Z}/\mathfrak{S})$ contains u-torsion implies that Breuil's first crystalline cohomology of $\mathcal{Z}$, with mod p^m coefficient for any m, together with Frobenius action and filtration is *not* a Breuil module. When $n=1$, we have $e=p-1$, which shows that our result [LL20, Corollary 7.25] is sharp. Below we shall see that the first crystalline cohomology cannot even support a strongly divisible lattice structure because it is torsion-free but not free.

(3) The same reasoning as in Remark 6.11 shows that the map $f\colon \mathrm{Alb}(\mathcal{Z}_0) \to \mathrm{Alb}(\mathcal{Z})_0$ is given by the quotient map $\mathcal{E}_0 \to \mathcal{E}_0/\mu_{p^n}$.

(4) The special fiber $\mathcal{Z}_0 = \mathcal{E}_0 \times (\mathcal{Y}_0/(\mathbb{Z}/p^n))$ has abelian π_1, with its torsion part being $\mathbb{Z}/p^n$. Here we used the fact that complete intersections of dimension ≥ 3 are simply connected (see [Sta21, Tag 0ELE]). On the other hand, the same argument as in [BMS18, proof of Proposition 2.2(i)] shows that $\pi_1(\mathcal{Z}_C) \cong \widehat{\mathbb{Z}}^{\oplus 2}$. Hence, we see again the specialization map $\mathrm{H}^2_{\text{ét}}(\mathcal{Z}_0) \to \mathrm{H}^2_{\text{ét}}(\mathcal{Z}_C)$ has kernel given by $\mathbb{Z}/p^n$ (cf. [BMS18, Remarks 2.3–2.4]).

In fact it was the desire to find examples with non-trivial kernel under specialization, together with inspiring discussions with Bhatt and Petrov separately, that lead us to analyze and generalize the example in [BMS18, § 2.1]. The Enriques surface used there turns out to be something of a red herring; the actual purpose it serves is just as an approximation of the classifying stack of $\mathbb{Z}/2$, like our $(\mathcal{Y}/(\mathbb{Z}/p^n))$ here.

Finally, let us explain why our example negates a prediction of Breuil [Bre02, Question 4.1]. Let S denote the p-adic PD envelope of $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$.

Proposition 6.15. *There is an exact sequence*

$$0 \to \mathrm{H}^1_{\mathrm{crys}}(\mathcal{Z}/S) \hookrightarrow S \cdot \{e_1, e_2\} \xrightarrow{d_2} S/p^n,$$

where $d_2(e_1) = 0$ and $d_2(e_2) = (u+1)^{p^n} - 1$. In particular, $\mathrm{H}^1_{\mathrm{crys}}(\mathcal{Z}/S)$ is torsion-free of rank 2 but not *free unless $(n,p) = (1,2)$.*

Proof. In Proposition 6.13 we see that the map $\mathcal{Z} \to \mathcal{X}$ induces isomorphism in the degree 1 prismatic cohomology and u^∞-torsion in the degree 2 prismatic cohomology. The comparison between prismatic and crystalline cohomology [LL20, Theorem 1.5] (see also [BS22, Theorem 5.2]) tells us that $\mathrm{H}^1_{\mathrm{crys}}(\mathcal{X}/S) \xrightarrow{\cong} \mathrm{H}^1_{\mathrm{crys}}(\mathcal{Z}/S)$. The same comparison result implies that after applying $-\otimes_{\mathfrak{S}} \varphi_* S$ to the spectral sequence (⚃), one calculates the crystalline cohomology of $\mathcal{Z}/S$. Therefore, the first statement follows from Lemmas 6.7 and 6.8. Note that $\varphi\big((u+1)^{p^{n-1}} - 1\big) \equiv (u+1)^{p^n} - 1$ modulo p^n.

To see the second assertion, note that $\mathrm{H}^1_{\mathrm{crys}}(\mathcal{Z}/S) \cong S \cdot e_1 \oplus J \cdot e_2$ where J is the ideal

$$\{x \in S \mid p^n \text{ divides } x \cdot ((u+1)^{p^n} - 1)\}.$$

If J were free, then it would be generated by a particular such element, denoted below by g. Let $g = \sum_{i=0}^{\infty} a_i (u^i/e(i)!)$ with $a_i \in W(k)$ approaching 0 and $e(i) = \lfloor i/e \rfloor$ where

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$e = p^{n-1} \cdot (p-1)$; note that every element in S can be uniquely expressed in this form. Since p^n trivially lies in J, it must also be divisible by this g. Therefore, there exists $h_1 \in S$ such that $gh_1 = p^n$. Write $q_n = (u+1)^{p^n} - 1$.

Claim 6.16. *a_0 is non-zero and divisible by p.*

Proof. The fact that a_0 is non-zero follows from $gh_1 = p^n$. If a_0 is a unit in $W(k)$ then $g \in S^\times$ is a unit, which implies $q_n \in p^n S$. But this is equivalent to $n = 1$ and $p = 2$. □

So now we can assume that $a_0 = pa_0'$ with $a_0' \neq 0$. Pick $u^m/e(m)!$ such that $u^m/e(m)! q_n \in p^n S$ (select $m = p^n e - 1$, for example). Then we have $gh_2 = u^m/e(m)!$ for some $h_2 \in S$. The above equation implies that $h_2 = \sum_{i=m}^{\infty} b_i(u^i/e(i)!)$. But comparing u^m terms on both sides, we have $a_0 b_m = 1$ which contradicts $a_0 = pa_0'$. This finishes the proof. □

Remark 6.17. In Breuil's terminology, this shows that the first crystalline cohomologies of our examples are *not* strongly divisible lattices [Bre02, Definition 2.2.1]. This contradicts the claimed [Bre02, Theorem 4.2(2)], in the proof of which one is led to Faltings's paper [Fal99]. However, Faltings was treating the case of p-divisible groups, hence Breuil's theorem/proof should only be applied to abelian schemes. Now it is tempting to say that smooth proper schemes over $\mathcal{O}_K$ and their Albanese should share the same H^1 for whatever cohomology theory.[12] But our example clearly negates this philosophy: the stacky example is squeezed between two abelian schemes and neither should really be the 'mixed characteristic 1-motive' of our stack (even though sometimes these two abelian schemes are abstractly isomorphic). Indeed, the sequence $\mathcal{E}_{\mathcal{O}_K} \to \mathcal{X} \to \mathcal{E}'_{\mathcal{O}_K}$ has the property that the first map only induces an isomorphism of the first crystalline cohomology of the special fiber (relative to W) and the second map only induces an isomorphism of the first étale cohomology of the (geometric) generic fiber.

6.1 Raynaud's theorem on prolongations

Finally, we give a geometric proof of Raynaud's theorem [Ray74, Théorème 3.3.3] on the uniqueness of prolongations of finite flat commutative group schemes over a mildly ramified $\mathcal{O}_K$.

Let G_K be a finite flat commutative group scheme over K. A prolongation of G_K is a finite flat commutative group scheme G over $\mathcal{O}_K$ together with an isomorphism of its generic fiber with G_K (as finite flat commutative group schemes). Once G_K is fixed, its prolongations form a category with homomorphisms given by maps of group schemes compatible with the isomorphisms of their generic fiber.

Recall [Ray74, Corollaire 2.2.3] that the (possibly empty) category of prolongations of a finite flat group scheme G over K has an initial $G_{\min}$ and a terminal object $G_{\max}$. Moreover, these two are interchanged under Cartier duality.

Theorem 6.18 (cf. [Ray74, Théorème 3.3.3]). *Assume G_K is a finite flat commutative group scheme which has a prolongation over $\mathcal{O}_K$.*

(1) *If $e < p-1$, then the prolongation is unique.*
(2) *If $e < 2(p-1)$, then the reduction of the canonical map $G_{\min} \to G_{\max}$ has kernel and cokernel annihilated by p.*
(3) *If $e = p-1$, then the reduction of the above map has étale kernel and multiplicative type cokernel.*

[12] To quote Sir Humphrey Appleby: 'It is not for a humble mortal such as I to speculate on the complex and elevated deliberations of the mighty.' But we suspect this is what Breuil had in mind when he claimed that his conjecture holds for H^1.

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Proof. To ease the notation, let us denote $G_1 \coloneqq G_{\min}$ and $G_2 \coloneqq G_{\max}$. Denote the canonical map by $\rho\colon G_1 \to G_2$. Choose a group scheme embedding $G_2 \to \mathcal{A}$ of G_2 into an abelian scheme $\mathcal{A}$ over $\mathcal{O}_K$, which is guaranteed by yet another theorem of Raynaud (see [BBM82, Théorème 3.1.1]).

We shall consider the quotient stack $[\mathcal{A}/G_1]$, which is a smooth proper Artin stack. Similar to Construction 6.12, let us pick a smooth complete intersection $\mathcal{Y}$ with a fixed-point-free action by G_1, let $\mathcal{Z} \coloneqq (\mathcal{A} \times_{\mathcal{O}_K} \mathcal{Y})/G_1$, which is a smooth projective scheme over $\mathcal{O}_K$, thanks to the second factor. Moreover, $\mathcal{Z}$ is pointed because it admits a map from $\mathcal{A}$, which has a canonical point given by the identity section.

Let H be the image group scheme of the map $\rho_k\colon G_{1,k} \to G_{2,k}$. Applying the same reasoning as in Remark 6.11 shows us that the canonical map $f\colon \mathrm{Alb}(\mathcal{Z}_0) \to \mathrm{Alb}(\mathcal{Z})_0$ is identified with $\mathcal{A}_0/H \to \mathcal{A}_0/G_{2,k}$, whose kernel group scheme is given by $G_{2,k}/H$, which is none other than the cokernel of ρ_0. Now our statements on $\mathrm{coker}(\rho_0)$ follow directly from applying Corollary 4.6 to our $\mathcal{Z}$. The statements on $\ker(\rho_0)$ follows from Cartier duality. □

Remark 6.19. Note that Raynaud first proved his theorem on prolongations, then used it to prove statements concerning Thu Picard scheme of a p-adic integral scheme, which is directly related to statements concerning natural map between Albanese of reduction and reduction of Albanese (see Remark 4.8). Our roadmap is the exact opposite.

One can reduce Raynaud's theorem to a question about Breuil–Kisin modules more directly, for instance by using Kisin's result [Kis06, Theorem 0.5]. Our proof is rather a geometrization of the same approach.

Finally, we remark that the estimate of s such that p^s kills the corkernel of $G_{\min} \to G_{\max}$ has been studied before (see, for example, [VZ12] and [Bon06] and the references therein). Note that an affirmative answer to our Question 3.10 for $i = 2$, when specialized to the construction made in the above proof, agrees with Bondako's sharp estimate.

Acknowledgements

We are very grateful to Bhargav Bhatt for so many helpful conversations and for sharing his excitement. During the preparation of this paper, we benefited from discussions with the following mathematicians: Christophe Breuil, Johan de Jong, Luc Illusie, Dmitry Kubrak, Yu Min, Shubhodip Mondal, Sasha Petrov, and Ziquan Yang. We would also like to thank the anonymous referee for their comments and suggestions on the earlier version, helping us improving this paper.

This work is partially supported by the National Natural Science Foundation of China (No. 12288201). The first named author is grateful for the support of AMS-Simons Travel Grant 2021–2023. He is also very grateful to the department of Mathematics at the University of Michigan for the support and environment they provide.

Conflicts of Interest

None.

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Shizhang Li lishizhang@amss.ac.cn
Morningside Center of Mathematics and Hua Loo-Keng Key Laboratory of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China

Tong Liu tongliu@purdue.edu
Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907, USA

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