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# A CHARACTERIZATION OF THE TOPOLOGICAL DIMENSION

BY

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ABSTRACT. This paper gives a new characterization of the dimension of a normal Hausdorff space, which joins together the Eilenberg–Otto characterization and the characterization by finite coverings. The link is furnished by the notion of a system of faces of a certain type $(N_1, \ldots, N_K)$, where $N_1, \ldots, N_K, K$ are natural numbers. It is shown that a space $X$ contains a system of faces of type $(N_1, \ldots, N_K)$ if and only if $\dim(X) \geq N_1 + \cdots + N_K$. The two limit cases of the theorem, namely $N_k = 1$ for $1 \leq k \leq K$ on the one hand, and $K = 1$ on the other hand, give the two known results mentioned above.

Let $X$ be a normal Hausdorff space. The dimension of $X$ is the largest natural number $n$ such that there exists an essential mapping from $X$ onto the $n$-dimensional simplex $\Delta_n$, with $n = 0$ or $n = \infty$ in the limit cases.

Several other properties of $X$ are known to be equivalent with $\dim X = n$, see for example [4]. In this paper, we describe a further characterization of the dimension number, which turns out to be a simultaneous generalization of two well known results, due to P. Alexandroff [1] and due to S. Eilenberg and E. Otto [2].

DEFINITION. Let $X$ be a topological space, and let $K, N_1, \ldots, N_K$ be natural numbers. A system $(A_n^k)$ $(1 \leq k \leq K$ and, for each $k$, $0 \leq n \leq N_K)$ of closed subsets of $X$ is called a *system of faces of type* $(N_1, \ldots, N_k)$, if

(i) $\bigcap_n A_n^k = \varnothing \qquad \forall k$,

(ii) If $(B_n^k)_{n,k}$ are closed subsets of $X$ with $A_n^k \subset B_n^k \ \forall n, k$ and $\bigcup_n B_n^k = X \ \forall k$, then $\bigcap_{n,k} B_n^k \neq \varnothing$.

EXAMPLES. (a) $X = \Delta_2$, $(A_0^1, A_1^1, A_2^1)$ are the three edges. (b) $X = \Delta_1 \times \Delta_1$, $(A_0^1, A_1^1)$ and $(A_0^2, A_1^2)$ are the two pairs of opposite edges. (c) $X = \Delta_2 \times \Delta_1$, $(A_0^1, A_1^1, A_2^1)$ are the three side faces, and $(A_0^2, A_1^2)$ are the bottom and the top face.

THEOREM. *A normal Hausdorff space $X$ possesses a system of faces of a given type $(N_1, \ldots, N_K)$ if and only if the topological dimension of $X$ is at least $N_1 + \cdots + N_K$.*

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The two special cases mentioned in the introduction are:

(1) $K=1$. This well known characterization of the dimension is usually formulated in terms of open coverings of the space. See [1] and [4].

(2) $N_k=1\ \forall 1\le k\le K$. This is essentially a characterization proved in [2] for separable metric spaces and in [3] for normal spaces. See also [4], p. 30.

**Proof.** Let the dimension of $X$ be at least $N_1+\cdots+N_K$. We have to construct a system of faces of type $(N_1,\ldots,N_K)$. First, we define such a system in the space

$$\Delta:=\Delta_{N_1}\times\cdots\times\Delta_{N_K}:$$

$$M_n^k=\{p\in\Delta \mid p_n^k=0\}\qquad(1\le k\le K, 0\le n\le N_k),$$

where $p_n^k$ is the barycentric coordinate of the component $p^k$ of $p$ in $\Delta_{N_k}$. (We don't need the fact that this is a system of faces in $\Delta$, which indeed will follow from the proof using the fact that the identity on $\Delta$ is essential.)

We choose an essential function

$$\Phi: X\to\Delta,$$

which exists by definition of the dimension, and since $\Delta$ is topologically the same as $\Delta_{N_1+\cdots+N_K}$. Now we transport the system $(M_n^k)$ to $X$:

$$A_n^k:=\Phi^{-1}(M_n^k)\ \forall n, k,$$

and verify that $(A_n^k)$ is a system of faces in $X$.

Property (i) of the definition is fulfilled, since $\bigcap_n M_n^k=\varnothing\ \forall k$. Let $(B_n^k)$ be closed subsets of $X$ with $A_n^k\subset B_n^k\ \forall n, k$ and $\bigcup_n B_n^k=X\ \forall k$. In order to verify (ii), we have to show that $\bigcap_{n,k} B_n^k\neq\varnothing$. Assume the contrary.

By the "shrinking lemma", choose open sets $U_n^k$ and $V_n^k$ in $X$ such that

$$A_n^k\subset U_n^k\ \forall n, k,\quad\text{and}\quad \bigcap_n U_n^k=\varnothing\ \forall k,$$

$$B_n^k\subset V_n^k\ \forall n, k\quad\text{and}\quad \bigcap_{n,k} V_n^k=\varnothing.$$

Let $u_n^k: X\to[0,1]$ be continuous such that

$$\begin{aligned} u_n^k&=0 &&\text{on } A_n^k,\\ u_n^k&>0 &&\text{on the complement of } U_n^k,\\ u_n^k&<1 &&\text{on } B_n^k, \text{ and}\\ u_n^k&=1 &&\text{on the complement of } V_n^k.\end{aligned}$$

Then $s^k:=\sum_n u_n^k$ is $>0$ on $X$, since $\bigcap_n U_n^k=\varnothing$. We define

$$\varphi: X\to\Delta,\qquad \varphi(x)_n^k=s^k(x)^{-1}u_n^k(x)\ \forall x\in X.$$

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Then $\varphi$ is continuous and $\varphi(A_n^k) \subset M_n^k \; \forall n, k$, since $u_n^k = 0$ on $A_n^k$. Let $p \in \Delta$ be the point with the coordinates $p_n^k = (N_k+1)^{-1} \; \forall n,k$. We show that $p \notin \varphi(X)$.

Assume $x \in X$ with $\varphi(x) = p$. Then $\varphi_n^k(x) = (N_k+1)^{-1}$ and $u_n^k(x) = s^k(x)(N_k+1)^{-1} \; \forall n, k$. For each $k$, there exists some $m$, $1 \le m \le N_k$, with $x \in B_m^k$, since $\bigcup_n B_n^k = X$. Then we have $u_m^k(x) < 1$ and hence $u_n^k(x) < 1 \; \forall n$. This implies $x \in V_n^k \; \forall n, k$, since $u_n^k$ is 1 on the complement of $V_n^k$. But this is not possible, since $\bigcap_{n,k} V_n^k = \varnothing$. Hence $p \notin \varphi(X)$.

We now use the function $\varphi$ to show that $\Phi$ is inessential. Consider

$$Y := \bigcup_{n,k} A_n^k.$$

Then $Y = \Phi^{-1}$ (boundary $\Delta$). Since $p \notin \varphi(X)$, there exists a function

$$\psi : X \to \text{boundary } \Delta, \quad \psi = \varphi \quad \text{on} \quad \varphi^{-1}(\text{boundary } \Delta) \supset Y.$$

Note that $\varphi$ and hence $\psi$ maps $A_n^k$ into $M_n^k \; \forall n, k$. Consider now the continuous extensions $\bar\psi, \bar\Phi : \beta X \to \Delta$ of $\psi$ and $\Phi$ on the Stone–Čech compactification $\beta X$ of $X$. Denoting the closure of a set $E \subset \beta X$ by $\tilde E$, we have

$$\bar\psi(\tilde A_n^k) \subset M_n^k, \quad \text{and} \quad \bar\Phi(\tilde A_n^k) \subset M_n^k,$$

since $M_n^k$ is compact. From the convexity of $M_n^k$ and from $\tilde Y = \bigcup \tilde A_n^k$ it now follows that the restrictions

$$\bar\psi' : \tilde Y \to \text{boundary } \Delta,$$
$$\bar\Phi' : \tilde Y \to \text{boundary } \Delta$$

are homotopic: consider $(1-t)\bar\psi' + t\bar\Phi'$, $0 \le t \le 1$. Thus we can apply Borsuk's homotopy extension theorem in order to obtain a function

$$\bar\Psi : \beta X \to \text{boundary } \Delta$$

with $\bar\Psi = \bar\Phi$ on $\tilde Y$. In particular, the restriction $\Psi$ of $\bar\Psi$ on $X$ fulfills $\Psi = \Phi$ on $Y$ and $\Psi(X) \subset$ boundary $\Delta$. Hence $\Phi$ is inessential, and we have a contradiction.

To prove the converse, assume now that $X$ possesses a system of faces $(A_n^k)$ of type $(N_1, \ldots, N_K)$. We have to show that the dimension of $X$ is at least $N_1 + \cdots + N_K$. Thus we have to construct an essential function

$$\varphi : X \to \Delta_{N_1+\cdots+N_K} = \Delta_{N_1} \times \cdots \times \Delta_{N_K} =: \Delta.$$

Since $\bigcap_n A_n^k = \varnothing \; \forall k$, there exist continuous functions

$$\varphi_n^k : X \to [0, 1], \qquad \varphi_n^k = 0 \quad \text{on} \quad A_n^k, \qquad \sum_n \varphi_n^k = 1.$$

We show that the function $\varphi : X \to \Delta$, $\varphi(x)_n^k = \varphi_n^k(x) \; \forall x \in X$, is essential. Assume the contrary. Then there exists a continuous

$$\psi : X \to \text{boundary } \Delta, \qquad \psi = \varphi \quad \text{on} \quad \varphi^{-1}(\text{boundary } \Delta).$$

For all $n$, $k$ define

$$B_n^k=\{x\in X \mid \psi_n^k(x)\le (N_k+1)^{-1}\}.$$

Then $B_n^k\subset X$ is closed, $A_n^k\subset B_n^k\ \forall n, k$, and $\bigcup_n B_n^k=X\ \forall k$, since it is not possible that $\psi_n^k(x)>(N_k+1)^{-1}\ \forall n$ for some $x\in X:\sum_n \psi_n^k(x)=1\ \forall x\in X$. On the other hand, we have

$$\begin{aligned}\bigcap_{n,k} B_n^k&=\{x\in X \mid \psi_n^k(x)\le (N_k+1)^{-1}\ \forall n, k\}\\&=\{x\in X \mid \psi_n^k(x)=(N_k+1)^{-1}\ \forall n, k\}\\&=\varnothing,\end{aligned}$$

since $\psi(X)\subset$ boundary $\Delta$. This is a contradiction to property (ii) of a system of faces, hence $\varphi$ must be essential, and the dimension of $X$ is at least $N_1+\cdots+N_K$.

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