On a Linear Refinement of the Prékopa-Leindler Inequality

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Abstract. If $f, g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are non-negative measurable functions, then the Prékopa-Leindler inequality asserts that the integral of the Asplund sum (provided that it is measurable) is greater than or equal to the 0-mean of the integrals of f and g. In this paper we prove that under the sole assumption that f and g have a common projection onto a hyperplane, the Prékopa–Leindler inequality admits a linear refinement. Moreover, the same inequality can be obtained when assuming that both projections (not necessarily equal as functions) have the same integral. An analogous approach may be also carried out for the so-called Borell-Brascamp-Lieb inequality.

1 Introduction

The Prékopa–Leindler inequality, originally proved in [17, 21], states that if $\lambda \in (0, 1)$ and $f, g, h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are non-negative measurable functions such that, for any $x, y \in \mathbb{R}^n$,

$$(1.1) \qquad h\big((1 - \lambda)x + \lambda y \big) \geq f(x)^{1-\lambda} g(y)^{\lambda},$$

then

$$\int_{\mathbb{R}^n} h \, dx \geq \Big(\int_{\mathbb{R}^n} f \, dx \Big)^{1-\lambda} \Big(\int_{\mathbb{R}^n} g \, dx \Big)^{\lambda}.$$

A more stringent version of this result can be obtained considering the smallest function h verifying condition (1.1), given f, g, and λ. Such a function is nothing but the so-called *Asplund sum* of f and g, defined as follows (see *e.g.*, [24, p. 517]).

Definition 1.1 Given two non-negative functions $f, g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\lambda \in (0, 1)$, the function $(1 - \lambda)f \star \lambda g : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is defined by

$$(1 - \lambda)f \star \lambda g \, (x) = \sup_{(1-\lambda)x_1 + \lambda x_2 = x} f(x_1)^{1-\lambda} g(x_2)^{\lambda}.$$

It is worth noting that the assumption that f and g are measurable is not sufficient to guarantee that $(1-\lambda)f \star \lambda g$ is measurable; see [9, Section 10]. The Prékopa–Leindler inequality can be now rephrased in the following way.

Theorem A (Prékopa–Leindler inequality)   *Let $\lambda \in (0,1)$ and let $f, g \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be non-negative measurable functions such that $(1-\lambda)f \star \lambda g$ is measurable as well. Then*

$$(1.2) \qquad \int_{\mathbb{R}^n} (1-\lambda)f \star \lambda g \, \mathrm{d}x \geq \Big(\int_{\mathbb{R}^n} f \, \mathrm{d}x \Big)^{1-\lambda} \Big(\int_{\mathbb{R}^n} g \, \mathrm{d}x \Big)^{\lambda}.$$

This functional inequality can be seen as the analytic counterpart of a geometric inequality, *i.e.*, the Brunn–Minkowski inequality. Let $\lambda \in [0,1]$ and let A and B be two nonempty (Lebesgue) measurable subsets of $\mathbb{R}^n$, such that their vector linear combination

$$(1-\lambda)A + \lambda B = \big\{ (1-\lambda)x + \lambda y : x \in A, y \in B \big\}$$

is also measurable. Then

$$(1.3) \qquad \mathrm{vol}\big((1-\lambda)A + \lambda B \big)^{1/n} \geq (1-\lambda)\mathrm{vol}(A)^{1/n} + \lambda\mathrm{vol}(B)^{1/n},$$

where by $\mathrm{vol}(\,\cdot\,)$ we denote the Lebesgue measure. The Brunn–Minkowski inequality admits an equivalent form, often referred to as its multiplicative or dimension-free version,

$$(1.4) \qquad \mathrm{vol}\big((1-\lambda)K + \lambda L \big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}.$$

Note that a straightforward proof of (1.4) can be obtained by applying (1.2) to characteristic functions. Indeed, for $A, B \subset \mathbb{R}^n$,

$$(1.5) \qquad (1-\lambda)\chi_A \star \lambda\chi_B = \chi_{(1-\lambda)A+\lambda B}$$

where χ denotes the characteristic function. On the other hand, the Prékopa–Leindler inequality can be proved by induction on the dimension n, and the initial case $n = 1$ follows easily from (1.3) (see, for instance, [20, p. 3] or [9, Section 7]).

Inequalities (1.2) and (1.3) have a strong link with convexity, as is shown by the description of the equality conditions. Indeed, equality may occur in (1.2) if and only if, roughly speaking, there exists a *log-concave* function F (*i.e.*, $F = e^{-u}$, where u is convex) such that f, g, and h coincide a.e. with F, up to translations and rescaling of the coordinates ([7]). As a consequence, equality holds in the Brunn–Minkowski inequality if and only if A and B are two homothetic compact convex sets, up to subsets negligible with respect to the Lebesgue measure.

The Brunn–Minkowski inequality is one of the most powerful results in convex geometry and, together with its analytic companion (1.2), has a wide range of applications in analysis, probability, information theory, and other areas of mathematics. We refer the reader to the updated monograph [24], entirely devoted to convex geometry, and to the extensive and detailed survey [9] concerning the Brunn–Minkowski inequality.

In [1, Section 50], linear refinements of the Brunn–Minkowski inequality were obtained for *convex bodies* (*i.e.*, nonempty compact convex sets) having a common orthogonal projection onto a hyperplane or more generally a projection with the same $(n-1)$-dimensional volume. To state these results we need some notation: $\mathcal{K}^n$ denotes the set of convex bodies in $\mathbb{R}^n$, $\mathcal{L}_{n-1}^n$ is the set of $(n-1)$-dimensional subspaces of $\mathbb{R}^n$ (*i.e.*, hyperplanes containing the origin) and, given $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_{n-1}^n$, $K|H$

is the orthogonal projection of K onto H (which is a convex body as well). Moreover, $\text{vol}_{n-1}(\cdot)$ denotes the $(n-1)$-dimensional Lebesgue measure in $\mathbb{R}^n$.

Theorem B ([1,10])　*Let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists $H \in \mathcal{L}_{n-1}^n$ with $K|H = L|H$. Then, for all $\lambda \in [0,1]$,*

$$\text{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)\text{vol}(K) + \lambda\text{vol}(L).$$

In other words, the volume itself is a concave function on the "segment" joining K and L in $\mathcal{K}^n$.

Theorem C ([1,10,19])　*Let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists $H \in \mathcal{L}_{n-1}^n$ with $\text{vol}_{n-1}(K|H) = \text{vol}_{n-1}(L|H)$. Then, for all $\lambda \in [0,1]$,*

$$(1.6) \qquad \text{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)\text{vol}(K) + \lambda\text{vol}(L).$$

These results have been extended to compact sets in [19] and more recently in [10, Subsection 1.2.4]; see also [13] for related topics.

We would like to point out that, contrary to Theorem B, Theorem C does not provide the concavity of the function $f(\lambda) = \text{vol}\big((1-\lambda)K + \lambda L\big)$ for $\lambda \in [0,1]$. More precisely, inequality (1.6) only yields the condition $f(\lambda) \geq (1-\lambda)f(0) + \lambda f(1)$. Nevertheless, when working with convex bodies K and L having a common projection onto a hyperplane, it is easy to check that the above condition implies, indeed, concavity of f (see *e.g.*, the proof of [13, Theorem 2.1.3]). On the other hand, Diskant [6] constructed an example where the above-mentioned function is not concave under the sole assumption of a common volume projection (the bodies used by Diskant are essentially a *cap body* of a ball and a half-ball). For further details about this topic we refer to Notes for Section 7.7 in [24] and the references therein.

At this point it is natural to wonder whether analogous results to Theorems B and C could be obtained for Prékopa–Leindler inequality. The aim of this paper is to provide an answer to this question. As a first step we notice that there is a rather natural way to define the "projection" of a function (see, for instance, [16]).

Definition 1.2　Given $f: \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $H \in \mathcal{L}_{n-1}^n$, the *projection* of f onto H is the (extended) function $\text{proj}_H(f): H \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined by

$$\text{proj}_H(f)(h) = \sup_{\alpha \in \mathbb{R}} f(h + \alpha v)$$

for $h \in H$, where v is a normal unit vector of H.

The geometric idea behind this definition is very simple: the hypograph of the projection of f onto H is the projection of the hypograph of f onto H. In particular, the projection of the characteristic function of a set A is just the characteristic function of the projection of A.

In this paper we show that under the equal projection assumption for the functions f and g, the Prékopa–Leindler inequality becomes linear in λ. This is the analytical counterpart of Theorem B. Indeed, taking $f = \chi_K$ and $g = \chi_L$, Theorem B may be obtained as a corollary.

Theorem 1.3 *Let $\lambda \in (0,1)$ and let $f, g \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be non-negative measurable functions such that $(1-\lambda)f \star \lambda g$ is measurable. If there exists $H \in \mathcal{L}_{n-1}^n$ such that $\operatorname{proj}_H(f) = \operatorname{proj}_H(g)$, then*

$$(1.7) \qquad \int_{\mathbb{R}^n}(1-\lambda)f \star \lambda g\, dx \geq (1-\lambda)\int_{\mathbb{R}^n} f\, dx + \lambda\int_{\mathbb{R}^n} g\, dx.$$

Notice that by means of the Arithmetic-Geometric mean inequality, the Prékopa–Leindler inequality (Theorem A) directly follows from the above result (and hence, indeed, (1.7) is a stronger inequality under the common projection assumption). As we already remarked, the Prékopa–Leindler inequality is naturally connected to log-concave functions, *i.e.*, functions of the form e^{-u} where $u \colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex. As a consequence of the above result we have the following statement.

Corollary 1.4 *Let $f, g \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be log-concave functions and let $\lambda \in (0,1)$. If there exists $H \in \mathcal{L}_{n-1}^n$ such that $\operatorname{proj}_H(f) = \operatorname{proj}_H(g)$, then*

$$\int_{\mathbb{R}^n}(1-\lambda)f \star \lambda g\, dx \geq (1-\lambda)\int_{\mathbb{R}^n} f\, dx + \lambda\int_{\mathbb{R}^n} g\, dx.$$

Proof The Asplund sum preserves log-concavity, as easily follows with the so-called infimal convolution (see Section 3). Hence it preserves measurability, because a log-concave function ϕ defined in $\mathbb{R}^n$ is, in fact, continuous in the interior of $\{x \in \mathbb{R}^n : \phi(x) > 0\}$. ∎

We prove that (1.7) can be obtained under the less restrictive hypothesis that the integral of the projections coincide, establishing a functional version of Theorem C. In the general case of measurable f and g (see Theorem 3.2 in Section 3), this result requires two mild (but technical) measurability assumptions. For simplicity, here we present this result for log-concave functions decaying to zero at infinity.

Theorem 1.5 *Let $f, g \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be log-concave functions such that*

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} g(x) = 0,$$

and let $\lambda \in (0,1)$. If there exists $H \in \mathcal{L}_{n-1}^n$ such that

$$\int_H \operatorname{proj}_H(f)(x)\, dx = \int_H \operatorname{proj}_H(g)(x)\, dx < \infty$$

then

$$\int_{\mathbb{R}^n}(1-\lambda)f \star \lambda g\, dx \geq (1-\lambda)\int_{\mathbb{R}^n} f\, dx + \lambda\int_{\mathbb{R}^n} g\, dx.$$

In the special case $n = 1$, Theorem 1.3 reduces to the following fact: if f and g are non-negative measurable functions defined on $\mathbb{R}$ such that $(1-\lambda)f \star \lambda g$ is measurable and $\sup_{\mathbb{R}} f = \sup_{\mathbb{R}} g$, then (1.7) holds. This result can be found in [3, Theorem 3.1].

The Prékopa–Leindler inequality has been generalized by introducing pth means (see Section 4 for detailed definitions and explanations) on both sides of (1.2); the resulting inequalities came to be called Borell-Brascamp-Lieb inequalities due to [2, 3].

We have been able to extend our approach to these inequalities by obtaining the suitable versions of Theorems 1.3 and 1.5. Again, for simplicity and in order to avoid technical measurability assumptions, we present here the result for the case of p-concave functions (see Section 4 for the definition).

Theorem 1.6 *Let $f, g \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be p-concave functions, where $-1/n \leq p \leq \infty$, and let $\lambda \in (0, 1)$. If there exists $H \in \mathcal{L}_{n-1}^n$ such that*

$$\int_H \mathrm{proj}_H(f)(x)\,\mathrm{d}x = \int_H \mathrm{proj}_H(g)(x)\,\mathrm{d}x,$$

then

$$\int_{\mathbb{R}^n} (1-\lambda)f \star_p \lambda g\,\mathrm{d}x \geq (1-\lambda)\int_{\mathbb{R}^n} f\,\mathrm{d}x + \lambda \int_{\mathbb{R}^n} g\,\mathrm{d}x.$$

The paper is organized as follows. Section 2 is devoted to collecting some definitions and preliminary constructions, whereas Theorems 1.3 and 1.5 (in fact, a more general version of the latter) will be proven in Section 3, as well as other related results. Finally in Section 4 we deal with the Borell-Brascamp-Lieb extensions, proving, among other results, Theorem 1.6.

2 Background Material and Auxiliary Results

Working in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 1$, let $\mathcal{K}^n$ denote the set of all convex bodies in $\mathbb{R}^n$. Given a subset A of $\mathbb{R}^n$, χ_A is the characteristic function of A.

With $\mathcal{L}_k^n$, $k \in \{0, 1, 2, \dots, n\}$, we will represent the set of all k-dimensional linear subspaces of $\mathbb{R}^n$. For $H \in \mathcal{L}_k^n$, $H^\perp \in \mathcal{L}_{n-k}^n$ denotes the orthogonal complement of H. Given $A \subset \mathbb{R}^n$ and $H \in \mathcal{L}_k^n$, the orthogonal projection of A onto H is denoted by $A|H$. For $k \in \{0, 1, \dots, n\}$ and $A \subset \mathbb{R}^n$, $\mathrm{vol}_k(A)$ denotes the k-dimensional Lebesgue measure of A (assuming that A is measurable with respect to this measure). We will often omit the index k when it is equal to the dimension n of the ambient space; in this case $\mathrm{vol}(\cdot) = \mathrm{vol}_n(\cdot)$ is just the (n-dimensional) Lebesgue measure.

Let $f \colon \mathbb{R}^n \to \mathbb{R}$; we define the *strict epigraph* of f by

$$\mathrm{epi}_s(f) = \{(x, t) : x \in \mathbb{R}^n, t \in \mathbb{R}, t > f(x)\} \subset \mathbb{R}^{n+1},$$

while its *strict hypograph* (or subgraph) will be denoted as

$$\mathrm{hyp}_s(f) = \{(x, t) : x \in \mathbb{R}^n, t \in \mathbb{R}, f(x) > t\} \subset \mathbb{R}^{n+1}.$$

The same definitions are valid for functions that take values on $\mathbb{R} \cup \{\infty\}$ (or $\mathbb{R} \cup \{\pm\infty\}$). In this case the strict epigraph of a function f is empty if and only if f is identically equal to infinity. Since we will work with non-negative functions, we also define

$$\mathrm{hyp}_s^+(f) = \{(x, t) : x \in \mathbb{R}^n, t \in \mathbb{R}_{\geq 0}, f(x) > t\} \subset \mathbb{R}^{n+1}.$$

Throughout this paper, given $H \in \mathcal{L}_{n-1}^n$, we set $\widetilde{H} = H \times \mathbb{R}$, *i.e.*, $\widetilde{H}$ is the n-dimensional subspace (in $\mathbb{R}^{n+1}$) *associated to H* when working with epigraphs and hypographs of functions.

The proof of our main result is based on symmetrization procedures; in fact, we will use two distinct types of symmetrization of functions that will be introduced in the rest of this section.

2.1 The Steiner symmetrization of a Function

To begin with, we briefly recall the Steiner symmetrization of sets: given a nonempty measurable set $A \subset \mathbb{R}^n$ and $H \in \mathcal{L}_{n-1}^n$, the *Steiner symmetral* of A with respect to H is given by

$$S_H(A) = \left\{ h + l \in \mathbb{R}^n : h \in A|H, l \in H^\perp, |l| \le \frac{1}{2}\mathrm{vol}_1\big(A \cap (h + H^\perp) \big) \right\}$$

(see *e.g.*, [11, p. 169] or [1, Section 9] for the compact convex case). Notice that $S_H(A)$ is well defined, since the sections of a measurable set are also measurable (for vol_{n-1} a.e. $h \in A|H$) and it is measurable (see, for instance, [8, p. 67]). To complete the picture, we define $S_H(\varnothing) = \varnothing$.

Next we define the first type of symmetrization of a (non-negative measurable) function f with respect to a hyperplane H; roughly speaking this is simply obtained by the Steiner symmetrization of the hypograph of f with respect to H. This technique is very well known in the theory of partial differential equations and calculus of variations; see, for instance [15].

Given a measurable $f \colon \mathbb{R}^n \to \mathbb{R}_{\ge 0} \cup \{\infty\}$, it will be convenient to write it in the form $f = e^{-u}$ where $u \colon \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is a measurable function, *i.e.*, $u(x) = -\log f(x)$ with the conventions that $\log 0 = -\infty$ and $\log \infty = \infty$. First we will consider the "symmetral" u_H of u with respect to $H \in \mathcal{L}_{n-1}^n$, which is given by

$$(2.1) \qquad \mathrm{epi}_s(u_H) = S_{\widetilde{H}}\big(\mathrm{epi}_s(u) \big).$$

Notice that (2.1) defines u_H completely. Indeed, as u is measurable, its strict epigraph is also measurable and hence $S_{\widetilde{H}}\big(\mathrm{epi}_s(u) \big)$ is well defined and measurable. Moreover, it is easy to see that if a point $(x, \bar{t}) \in S_{\widetilde{H}}\big(\mathrm{epi}_s(u) \big)$, with $x \in \mathbb{R}^n$ and $\bar{t} \in \mathbb{R}$, then the entire "vertical" half line $\{(x, t) : t \ge \bar{t}\}$ above it is contained in $S_{\widetilde{H}}\big(\mathrm{epi}_s(u) \big)$. Hence (2.1) is equivalent to the following explicit expression for $u_H \colon \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$:

$$u_H(x) = \inf\big\{ r \in \mathbb{R} : (x, r) \in S_{\widetilde{H}}\big(\mathrm{epi}_s(u) \big) \big\}.$$

Note also that the measurability of its epigraph implies the measurability of u_H. The next step is to define the symmetral of $f = e^{-u}$ through the symmetral of u, as follows.

Definition 2.1 Let $u \colon \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function and let $H \in \mathcal{L}_{n-1}^n$. Then the *Steiner symmetral* of $f = e^{-u}$ is $S_H(f) = e^{-u_H}$, where u_H is given by (2.1).

As u_H is measurable, $S_H(f)$ is measurable as well. Moreover, since $t \mapsto e^{-t}$ is a decreasing bijection between $\mathbb{R} \cup \{\pm\infty\}$ and $\mathbb{R}_{\ge 0} \cup \{\infty\}$, we also have

$$\mathrm{hyp}_s\big(S_H(f) \big) = S_{\widetilde{H}}\big(\mathrm{hyp}_s(f) \big),$$

which would have allowed us to define $S_H(f)$ directly. The above equality still holds if hypographs are replaced by positive hypographs or even if we consider sections $(h + H^\perp) \times \mathbb{R}$ for any $h \in H$ (see Figure 1), *i.e.*, we replace f by any of its restrictions to a line perpendicular to H.

From the definition of $S_{\widetilde{H}}(\cdot)$ we clearly have

$$\mathrm{epi}_s(u)|\widetilde{H} = S_{\widetilde{H}}(\mathrm{epi}_s(u))|\widetilde{H} = S_{\widetilde{H}}(\mathrm{epi}_s(u)) \cap \widetilde{H},$$
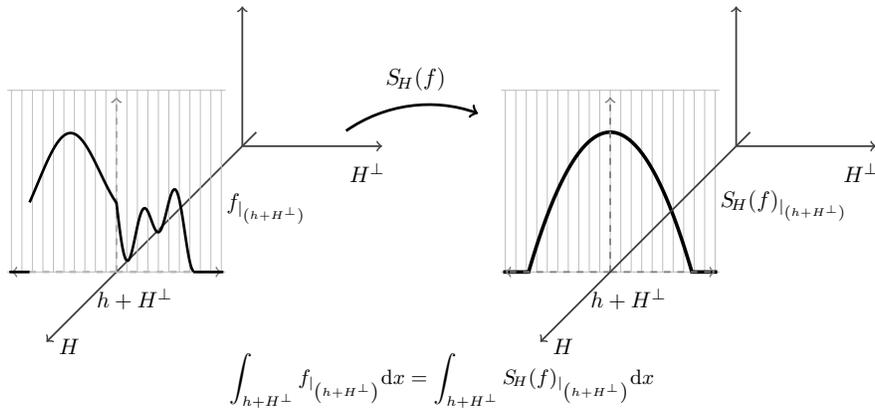
Figure 1

which is equivalent to

$$\mathrm{hyp}_s(f)|\widetilde{H} = \mathrm{hyp}_s\big(S_H(f)\big)|\widetilde{H} = \mathrm{hyp}_s\big(S_H(f)\big) \cap \widetilde{H}.$$

Hence, it follows that

$$\mathrm{proj}_H(f) = \mathrm{proj}_H(S_H(f)) = S_H(f)_{|H},$$

where we have used the following notation: for a function g defined in $\mathbb{R}^n$ and $H \in \mathcal{L}^n_{n-1}$, $g_{|H}$ is the restriction of g to H.

On the other hand, by construction of $S_H(\cdot)$, it is clear that (for any fixed $h \in H$)

$$\mathrm{vol}_1(x \in h + H^\perp : f(x) > t) = \mathrm{vol}_1(x \in h + H^\perp : S_H(f)(x) > t)$$

and hence

$$\begin{aligned}
\int_{\mathbb{R}^n} f \, \mathrm{d}x &= \int_H \int_0^{\mathrm{proj}_H(f)(h)} \mathrm{vol}_1(x \in h + H^\perp : f(x) > t) \, \mathrm{d}t \, \mathrm{d}h \\
&= \int_H \int_0^{\mathrm{proj}_H(S_H(f))(h)} \mathrm{vol}_1(x \in h + H^\perp : S_H(f)(x) > t) \, \mathrm{d}t \, \mathrm{d}h \\
&= \int_{\mathbb{R}^n} S_H(f) \, \mathrm{d}x.
\end{aligned}$$

Therefore, we have shown the following result.

Proposition 2.2 *Let $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a non-negative measurable function and let $H \in \mathcal{L}^n_{n-1}$ be a hyperplane. Then*

(i)   *$S_H(f) : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a non-negative measurable function.*

(ii) $\int_{\mathbb{R}^n} f \, \mathrm{d}x = \int_{\mathbb{R}^n} S_H(f) \, \mathrm{d}x.$

(iii) $\mathrm{proj}_H(f) = \mathrm{proj}_H(S_H(f)) = S_H(f)_{|H}.$

It will be important to relate the Steiner symmetral of the Asplund sum of two nonnegative functions with the Asplund sum of their symmetrals. For this we will need the following inclusion involving the symmetrals of the (nonempty) measurable sets A, B, and $(1-\lambda)A + \lambda B$ respectively. The proof can be carried out following the ideas of the proof for convex bodies (see *e.g.*, [11, Proposition 9.1]); we include it here for completeness.

Proposition 2.3 *Let $A, B \subset \mathbb{R}^n$ be nonempty measurable sets such that $(1-\lambda)A + \lambda B$ is measurable for a given $\lambda \in (0,1)$ and let $H \in \mathcal{L}_{n-1}^n$ be a hyperplane. Then we have*

$$(2.2) \qquad S_H\big((1-\lambda)A + \lambda B\big) \supset (1-\lambda)S_H(A) + \lambda S_H(B).$$

Proof   Let $x \in S_H(A)$, $y \in S_H(B)$, or, equivalently, $x = h_x + l_x$, $y = h_y + l_y$, where $h_x, h_y \in H$ and $l_x, l_y \in H^\perp$ are such that $|l_x| \leq \frac{1}{2}\mathrm{vol}_1\big(A \cap (x + H^\perp)\big)$ and $|l_y| \leq \frac{1}{2}\mathrm{vol}_1\big(B \cap (y + H^\perp)\big)$. Then $(1-\lambda)x + \lambda y = \big((1-\lambda)h_x + \lambda h_y\big) + \big((1-\lambda)l_x + \lambda l_y\big)$, with $(1-\lambda)h_x + \lambda h_y \in H$ and $(1-\lambda)l_x + \lambda l_y \in H^\perp$. By means of the (1-dimensional) Brunn–Minkowski inequality, we have

$$
\begin{aligned}
\big|(1-\lambda)l_x + \lambda l_y\big| &\leq (1-\lambda)\,|l_x| + \lambda\,|l_y| \\
&\leq \frac{1}{2}\big((1-\lambda)\mathrm{vol}_1\big(A \cap (x + H^\perp)\big) + \lambda\mathrm{vol}_1\big(B \cap (y + H^\perp)\big)\big) \\
&\leq \frac{1}{2}\mathrm{vol}_1\big((1-\lambda)\big(A \cap (x + H^\perp)\big) + \lambda\big(B \cap (y + H^\perp)\big)\big) \\
&\leq \frac{1}{2}\mathrm{vol}_1\big(\big((1-\lambda)A + \lambda B\big) \cap \big((1-\lambda)x + \lambda y + H^\perp\big)\big).
\end{aligned}
$$

Hence $(1-\lambda)x + \lambda y \in S_H\big((1-\lambda)A + \lambda B\big)$. ∎

2.2 Schwarz-type Symmetrization of a Function

The second symmetrization of functions which we will use is defined as follows.

Definition 2.4   Given $H \in \mathcal{L}_{n-1}^n$ and a non-negative measurable function $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, the *symmetrization of f with respect to $H^\perp$* is the function $S_{H^\perp}(f) : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ given by

$$S_{H^\perp}(f)(h + \alpha v) = d_\alpha \chi_{B_{n-1}}(h),$$

for $h \in H$ and $\alpha \in \mathbb{R}$, where

$$d_\alpha = \int_{\alpha v + H} f_{|(\alpha v + H)}\, \mathrm{d}x,$$

v is a normal unit vector of H, and $B_{n-1} \in \mathcal{K}^{n-1}$ is the Euclidean ball of volume 1 (lying in H).

See Figure 2, where, for a clearer representation, we have made a change of axes in relation to those of Figure 1.
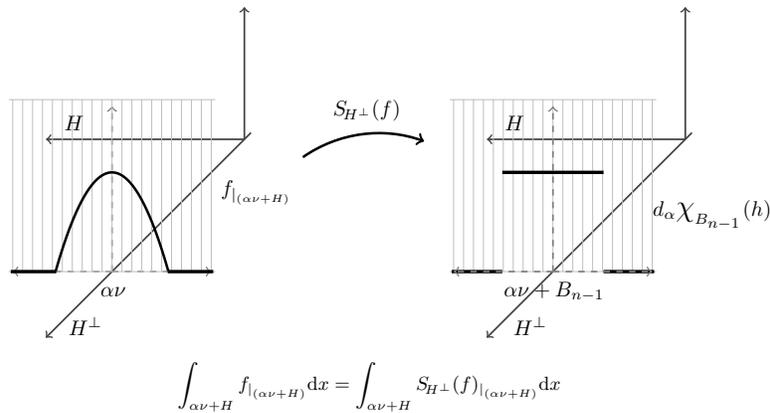
$$\int_{\alpha\nu+H} f_{|(\alpha\nu+H)}\,\mathrm{d}x = \int_{\alpha\nu+H} S_{H^\perp}(f)_{|(\alpha\nu+H)}\,\mathrm{d}x$$

Figure 2

The choice of the unit volume ball B_{n-1} in the previous definition is not relevant: any other (fixed) convex body (for instance, a cube of edge length 1) could have replaced it with no essential change. The behavior of the function $S_{H^\perp}(f)$ is basically described by the function $\alpha \to d_\alpha$, depending on the real variable α.

By means of Fubini's theorem (together with the fact that the cartesian product of measurable sets is also measurable) it is clear that $S_{H^\perp}(f)$ is also a non-negative measurable function. Moreover, we have

(2.3)
$$\int_{\mathbb{R}^n} S_{H^\perp}(f)\,\mathrm{d}x = \int_{\mathbb{R}^n} f\,\mathrm{d}x.$$

Notice also that the symmetrization $S_{H^\perp}(\cdot)$ is increasing in the sense that if $f \le g$, then $S_{H^\perp}(f) \le S_{H^\perp}(g)$.

In next section (Propositions 3.3 and 3.4) we will show that the behavior of these symmetrizations, $S_H(\cdot), S_{H^\perp}(\cdot)$, with respect to the operation $\star$ is "good" (which may be seen as the analytic counterpart of Proposition 2.3). Roughly speaking we will show that, for both symmetrizations, the symmetral of the Asplund sum is pointwise larger than the Asplund sum of the symmetrals, which will allow us to obtain the inequality of Theorem 3.2.

3   Proof of Main Results

In this section we prove Theorem 1.3 and a more general version of Theorem 1.5 (see Theorem 3.2 below). We start by showing a result needed for the proof of Theorem 1.3.

Proposition 3.1   *Let $f, g\colon \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ be non-negative functions such that*

$$\mathrm{proj}_H(f), \mathrm{proj}_H(g)\colon H \to \mathbb{R}_{\ge 0}$$

for an $H \in \mathcal{L}_{n-1}^n$ and let $\lambda \in (0,1)$. Then

$$(3.1) \qquad \mathrm{proj}_H\big((1-\lambda)f \star \lambda g\big) \geq (1-\lambda)\,\mathrm{proj}_H(f) \star \lambda\,\mathrm{proj}_H(g).$$

Proof　Let v be a normal unit vector of H. Then for $h, h_1, h_2 \in H$ such that $(1-\lambda)h_1 + \lambda h_2 = h$ and any $\alpha_1, \alpha_2 \in \mathbb{R}$, we clearly have

$$\mathrm{proj}_H((1-\lambda)f \star \lambda g)(h) \geq (1-\lambda)f \star \lambda g\big((1-\lambda)(h_1 + \alpha_1 v) + \lambda(h_2 + \alpha_2 v)\big)$$
$$\geq f(h_1 + \alpha_1 v)^{1-\lambda} g(h_2 + \alpha_2 v)^\lambda.$$

Hence, by taking suprema over $\alpha_1, \alpha_2 \in \mathbb{R}$, it follows

$$\mathrm{proj}_H((1-\lambda)f \star \lambda g)(h) \geq \mathrm{proj}_H(f)(h_1)^{1-\lambda}\mathrm{proj}_H(g)(h_2)^\lambda$$

for all $h_1, h_2 \in H$ with $(1-\lambda)h_1 + \lambda h_2 = h$. ∎

Proof of Theorem 1.3　Let $U \colon H \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be the (extended) function given by $U(h) = \mathrm{proj}_H(f)(h) = \mathrm{proj}_H(g)(h)$. We claim that

$$(3.2) \qquad \mathrm{proj}_H\big((1-\lambda)f \star \lambda g\big) \geq U.$$

Indeed, if $U \colon H \to \mathbb{R}_{\geq 0}$, then

$$\mathrm{proj}_H((1-\lambda)f \star \lambda g) \geq (1-\lambda)\,U \star \lambda\,U \geq U \quad (cf.\ (3.1));$$

otherwise following the proof of Proposition 3.1 (taking $h_1 = h_2 = h$) we may assert that (3.2) holds. Moreover,

$$(3.3) \qquad \big\{x \in h + H^\perp : f(x) \geq t\big\}, \big\{y \in h + H^\perp : g(y) \geq t\big\} \neq \varnothing,$$

for all $0 \leq t < U(h)$. By the definition of $\star$ together with (3.3), it is clear that

$$\big\{z \in h + H^\perp : \big((1-\lambda)f \star \lambda g\big)(z) \geq t\big\}$$
$$\supset (1-\lambda)\big\{x \in h + H^\perp : f(x) \geq t\big\} + \lambda\big\{y \in h + H^\perp : g(y) \geq t\big\}$$

(for all $0 \leq t < U(h)$) and hence, by the Brunn–Minkowski inequality, we obtain

$$\mathrm{vol}_1\big(\big\{z \in h + H^\perp : \big((1-\lambda)f \star \lambda g\big)(z) \geq t\big\}\big)$$
$$\geq (1-\lambda)\mathrm{vol}_1\big(\big\{x \in h + H^\perp : f(x) \geq t\big\}\big) + \lambda\mathrm{vol}_1\big(\big\{y \in h + H^\perp : g(y) \geq t\big\}\big).$$

From the above inequality, and using Fubini's theorem together with (3.2) we get

$$
\int_{\mathbb{R}^n} (1-\lambda) f \star \lambda g \, \mathrm{d}x
$$

$$
= \int_H \int_{h+H^\perp} \big((1-\lambda) f \star \lambda g \big)_{|(h+H^\perp)} \, \mathrm{d}x \, \mathrm{d}h
$$

$$
= \int_H \int_0^{\mathrm{proj}_H((1-\lambda)f\star\lambda g)(h)} \mathrm{vol}_1 \big(\big\{ x \in h + H^\perp : \big((1-\lambda) f \star \lambda g \big)(x) \geq t \big\} \big) \, \mathrm{d}t \, \mathrm{d}h
$$

$$
\geq \int_H \int_0^{U(h)} \mathrm{vol}_1 \big(\big\{ x \in h + H^\perp : \big((1-\lambda) f \star \lambda g \big)(x) \geq t \big\} \big) \, \mathrm{d}t \, \mathrm{d}h
$$

$$
\geq \int_H \int_0^{U(h)} (1-\lambda) \mathrm{vol}_1 \big(\big\{ x \in h + H^\perp : f(x) \geq t \big\} \big) \, \mathrm{d}t \, \mathrm{d}h
$$

$$
+ \int_H \int_0^{U(h)} \lambda \mathrm{vol}_1 \big(\big\{ x \in h + H^\perp : g(x) \geq t \big\} \big) \, \mathrm{d}t \, \mathrm{d}h
$$

$$
= (1-\lambda) \int_{\mathbb{R}^n} f \, \mathrm{d}x + \lambda \int_{\mathbb{R}^n} g \, \mathrm{d}x,
$$

as desired.   ∎

Given a non-negative (extended) function $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, we will denote by $\overline{f} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ the function given by

$$
\overline{f}(x) = \begin{cases} 0 & \text{if } f(x) = \infty, \\ f(x) & \text{otherwise.} \end{cases}
$$

Note that if f is measurable, then $\overline{f}$ is measurable as well. Indeed $\overline{f} = f \cdot \chi_F$ where $F = \{ x \in \mathbb{R}^n : f(x) < \infty \}$; the measurability of f implies that F is a measurable set, i.e., χ_F is measurable. Hence $\overline{f}$ is measurable.

Theorem 3.2   *Let $f, g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be non-negative measurable functions such that $(1-\lambda) f \star \lambda g$ is measurable for $\lambda \in (0,1)$ fixed. If there exists $H \in \mathcal{L}_{n-1}^n$ such that*

$$
(1-\lambda)\overline{S_H(f)} \star \lambda \overline{S_H(g)}, \quad (1-\lambda) S_{H^\perp}\big(\overline{S_H(f)} \big) \star \lambda S_{H^\perp}\big(\overline{S_H(g)} \big)
$$

are measurable functions and

$$
(3.4) \qquad \int_H \mathrm{proj}_H(f)(x) \, \mathrm{d}x = \int_H \mathrm{proj}_H(g)(x) \, \mathrm{d}x < \infty,
$$

then

$$
\int_{\mathbb{R}^n} (1-\lambda) f \star \lambda g \, \mathrm{d}x \geq (1-\lambda) \int_{\mathbb{R}^n} f \, \mathrm{d}x + \lambda \int_{\mathbb{R}^n} g \, \mathrm{d}x.
$$

We first see how this result implies, in particular, Theorem 1.5.

Proof of Theorem 1.5   If $\phi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a log-concave function satisfying the condition

$$
\lim_{|x| \to \infty} \phi(x) = 0,
$$

then there exist constants $K, a > 0$ such that

$$
\phi(x) \leq K e^{-a\|x\|}
$$

for every $x \in \mathbb{R}^n$ (see [4, Lemma 2.5]). In particular ϕ is bounded. It is an easy exercise to check that $S_H(\phi)$ is log-concave as well and, by the boundedness of ϕ, is also finite, since it is also bounded. Consequently $\overline{S_H(\phi)} = S_H(\phi)$. Moreover, by the log-concavity of $S_H(\phi)$ and [3, Corollary 3.5], the function $\alpha \to d_\alpha$ in the definition of $S_{H^\perp}\big(\overline{S_H(\phi)}\big)$ is log-concave and then $S_{H^\perp}\big(\overline{S_H(\phi)}\big)$ is still log-concave, as a product of log-concave functions.

If we apply these considerations to the functions f and g in the statement of the present theorem, we get that $\overline{S_H(f)}$, $\overline{S_H(g)}$, $S_{H^\perp}\big(\overline{S_H(f)}\big)$, and $S_{H^\perp}\big(\overline{S_H(g)}\big)$ are log-concave functions. On the other hand, the Asplund sum preserves log-concavity, so that

$$(1-\lambda)\overline{S_H(f)} \star \lambda\overline{S_H(g)}, \quad (1-\lambda)S_{H^\perp}\big(\overline{S_H(f)}\big) \star \lambda S_{H^\perp}\big(\overline{S_H(g)}\big)$$

are measurable functions. The proof is concluded applying Theorem 3.2. ∎

Remark In the statement of the above theorem, we can exchange the condition of decaying to zero at infinity (for both functions f and g) for that of boundedness, and we would still obtain the same inequality.

In order to establish Theorem 3.2 we need to study the interaction between the symmetrizations $S_H(\cdot)$, and $S_{H^\perp}(\cdot)$ (for a given $H \in \mathcal{L}_{n-1}^n$) and the Asplund sum; this is done in Propositions 3.3 and 3.4.

Proposition 3.3 *Let $f, g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be non-negative measurable functions such that $(1-\lambda)f \star \lambda g$ is measurable for $\lambda \in (0,1)$ fixed and let $H \in \mathcal{L}_{n-1}^n$ be a hyperplane. Then*

$$(3.5) \qquad S_H\big((1-\lambda)f \star \lambda g\big) \geq (1-\lambda)\overline{S_H(f)} \star \lambda\overline{S_H(g)}.$$

Proof Writing $f = e^{-u}$, and $g = e^{-v}$, we have that $(1-\lambda)f \star \lambda g = e^{-w}$ with $w = (1-\lambda)u \oplus \lambda v$, where $\oplus$ denotes the infimal convolution of u, v (see [22, p. 34, 38], [23, Section 1.H]) given by

$$\big((1-\lambda)u \oplus \lambda v\big)(t) = \inf\big\{(1-\lambda)u(x) + \lambda v(y) : (1-\lambda)x + \lambda y = t\big\}.$$

The strict epigraph of the infimal convolution satisfies (see [23, p. 25])

$$(3.6) \qquad \text{epi}_s\big((1-\lambda)u \oplus \lambda v\big) = (1-\lambda)\text{epi}_s(u) + \lambda\text{epi}_s(v).$$

Write $\overline{S_H(f)} = e^{-\tilde{u}_H}$, $\overline{S_H(g)} = e^{-\tilde{v}_H}$, *i.e.*, $\tilde{u}_H = -\log(\overline{S_H(f)})$ and $\tilde{v}_H = -\log(\overline{S_H(g)})$. We may assume without loss of generality that both f and g are not identically zero so that $\text{epi}_s(\tilde{u}_H)$, $\text{epi}_s(\tilde{v}_H) \neq \varnothing$. By (3.6), (2.2), and the definition of $\tilde{u}_H, \tilde{v}_H$, we have

$$(3.7) \qquad \begin{aligned} \text{epi}_s(w_H) &= S_{\widetilde{H}}\big(\text{epi}_s(w)\big) \\ &= S_{\widetilde{H}}\big((1-\lambda)\text{epi}_s(u) + \lambda\text{epi}_s(v)\big) \\ &\supset (1-\lambda)S_{\widetilde{H}}\big(\text{epi}_s(u)\big) + \lambda S_{\widetilde{H}}\big(\text{epi}_s(v)\big) \\ &= (1-\lambda)\text{epi}_s(u_H) + \lambda\text{epi}_s(v_H) \\ &\supset (1-\lambda)\text{epi}_s(\tilde{u}_H) + \lambda\text{epi}_s(\tilde{v}_H) = \text{epi}_s((1-\lambda)\tilde{u}_H \oplus \lambda\tilde{v}_H). \end{aligned}$$

This is equivalent to $w_H \leq (1-\lambda)\tilde{u}_H \oplus \lambda\tilde{v}_H$; therefore inequality (3.5) holds. ∎

Remark Notice that it was necessary to introduce $\bar{\cdot}$, as the Asplund sum of two non-negative functions f and g that may attain ∞ is in general not well defined. We also point out that the "$< \infty$" assumption in (3.4) has arisen in order to avoid some ambiguities of the type "$\infty \cdot 0$" (see the proofs of Proposition 3.4 and Theorem 3.2). All these conflicts will disappear in Section 4 when working with the p-th mean (for $p < 0$) instead of the 0-th mean.

On the other hand, for real-valued functions, for instance when working with bounded functions, in the penultimate line of (3.7) we would have

$$(1-\lambda)\mathrm{epi}_s(u_H) + \lambda\mathrm{epi}_s(v_H) = \mathrm{epi}_s\big((1-\lambda)u_H \oplus \lambda v_H\big),$$

obtaining $S_H((1-\lambda)f \star \lambda g) \geq (1-\lambda)S_H(f) \star \lambda S_H(g)$.

Proposition 3.4 *Let $f, g\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be non-negative measurable functions such that $(1-\lambda)f \star \lambda g$ is measurable for some fixed $\lambda \in (0,1)$. Assume also that there exists $H \in \mathcal{L}^n_{n-1}$ such that*

$$\int_{\alpha v+H} f_{|(\alpha v+H)} \, \mathrm{d}x, \ \int_{\alpha v+H} g_{|(\alpha v+H)} \, \mathrm{d}x < \infty \quad \text{for all } \alpha \in \mathbb{R}.$$

Then $S_{H^\perp}((1-\lambda)f \star \lambda g) \geq (1-\lambda)S_{H^\perp}(f) \star \lambda S_{H^\perp}(g)$.

Proof We use the Prékopa–Leindler inequality (Theorem A). Indeed, given $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha = (1-\lambda)\alpha_1 + \lambda\alpha_2$, we clearly have

$$\int_{\alpha v+H}\big((1-\lambda)f \star \lambda g\big)_{|(\alpha v+H)} \, \mathrm{d}x$$
$$\geq \Big(\int_{\alpha_1 v+H} f_{|(\alpha_1 v+H)} \, \mathrm{d}x\Big)^{1-\lambda} \Big(\int_{\alpha_2 v+H} g_{|(\alpha_2 v+H)} \, \mathrm{d}x\Big)^{\lambda},$$

which allows us to assert that for given $h_1, h_2 \in H$

$$S_{H^\perp}\big((1-\lambda)f \star \lambda g\big)\big(((1-\lambda)h_1 + \lambda h_2) + ((1-\lambda)\alpha_1 + \lambda\alpha_2)v\big)$$
$$\geq S_{H^\perp}(f)(h_1 + \alpha_1 v)^{1-\lambda}S_{H^\perp}(g)(h_2 + \alpha_2 v)^{\lambda}. \quad \blacksquare$$

As it occurs for the (linear improvements of) Brunn–Minkowski inequality, Theorems B and C, the same inequality can be obtained when a condition on the integral of the projection is assumed.

Proof of Theorem 3.2 Hypothesis (3.4) together with the definition of S_H, $S_{H^\perp}$, and $\bar{\cdot}$ imply that (see Proposition 2.2 (iii))

$$(3.8) \qquad \mathrm{proj}_H\big(S_{H^\perp}\big(\overline{S_H(f)}\big)\big) = \mathrm{proj}_H\big(S_{H^\perp}\big(S_H(f)\big)\big)$$
$$= \mathrm{proj}_H\big(S_{H^\perp}\big(S_H(g)\big)\big) = \mathrm{proj}_H\big(S_{H^\perp}\big(\overline{S_H(g)}\big)\big).$$

Moreover, since the projections onto H of f and g are integrable, by means of Fubini's theorem we have (see Proposition 2.2 (ii) and (2.3))

$$(3.9) \qquad \int_{\mathbb{R}^n} S_{H^\perp}\big(\overline{S_H(f)}\big) \, \mathrm{d}x = \int_{\mathbb{R}^n} f \, \mathrm{d}x, \quad \int_{\mathbb{R}^n} S_{H^\perp}\big(\overline{S_H(g)}\big) \, \mathrm{d}x = \int_{\mathbb{R}^n} g \, \mathrm{d}x.$$

On the other hand, Proposition 3.3 together with the monotonicity of $S_{H^\perp}(\cdot)$ and Proposition 3.4 imply

$$(3.10) \qquad S_{H^\perp}\big(S_H((1-\lambda)f \star \lambda g)\big) \geq S_{H^\perp}\big((1-\lambda)\overline{S_H(f)} \star \lambda\overline{S_H(g)}\big)$$
$$\geq (1-\lambda)\, S_{H^\perp}\big(\,\overline{S_H(f)}\,\big) \star \lambda S_{H^\perp}\big(\,\overline{S_H(g)}\,\big).$$

Therefore, applying Proposition 2.2 (ii) together with (2.3), (3.10), Theorem 1.3 (taking into account (3.8)), and (3.9), respectively, we get

$$\int_{\mathbb{R}^n} (1-\lambda)f \star \lambda g \, dx = \int_{\mathbb{R}^n} S_{H^\perp}\big(S_H((1-\lambda)f \star \lambda g)\big) \, dx$$
$$\geq \int_{\mathbb{R}^n} (1-\lambda)S_{H^\perp}\big(\,\overline{S_H(f)}\,\big) \star \lambda\, S_{H^\perp}\big(\,\overline{S_H(g)}\,\big) \, dx$$
$$\geq (1-\lambda) \int_{\mathbb{R}^n} S_{H^\perp}\big(\,\overline{S_H(f)}\,\big) \, dx + \lambda \int_{\mathbb{R}^n} S_{H^\perp}\big(\,\overline{S_H(g)}\,\big) \, dx$$
$$= (1-\lambda) \int_{\mathbb{R}^n} f \, dx + \lambda \int_{\mathbb{R}^n} g \, dx,$$

as desired. ∎

As a consequence of the above theorem, we may immediately obtain the following refinement of the Brunn–Minkowski inequality (*cf.* Theorems B and C) for the more general case of measurable sets.

Corollary 3.5 Let $A, B \subset \mathbb{R}^n$ be nonempty measurable sets and let $\lambda \in (0,1)$ be such that $(1-\lambda)A + \lambda B$ is measurable. If there exists $H \in \mathcal{L}^n_{n-1}$ such that

(i) $\mathrm{vol}_{n-1}(A|H) = \mathrm{vol}_{n-1}(B|H) < \infty$,
(ii) $(1-\lambda)S_H(A) + \lambda S_H(B)$ is a measurable set,
(iii) $(1-\lambda)S_{H^\perp}\big(\chi_{S_H(A)}\big) \star \lambda S_{H^\perp}\big(\chi_{S_H(B)}\big)$ is a measurable function,

then $\mathrm{vol}\big((1-\lambda)A + \lambda B\big) \geq (1-\lambda)\mathrm{vol}(A) + \lambda\mathrm{vol}(B)$.

Proof It is enough to consider the functions $f = \chi_A$, $g = \chi_B$ and apply the theorem above (recall that in this case we have $\mathrm{proj}_H(f) = \chi_{A|H}$, $\mathrm{proj}_H(g) = \chi_{B|H}$ and $(1-\lambda)f \star \lambda g = \chi_{(1-\lambda)A+\lambda B}$ (*cf.* (1.5)). Notice also that for any measurable set M, we have $S_H(\chi_M) = \chi_{S_H(M)}$). ∎

Remark Although the measurability conditions (ii) and (iii) could appear a little bit stronger, they may be also easily fulfilled, for instance, when working with compact sets A and B. Indeed, since A and B are compact, then $S_H(A)$ and $S_H(B)$ are also compact. Thus condition (ii) holds. On the other hand, for a general compact set K and by the construction of $S_H(\cdot)$ (for sets), the sections of $S_H(K)$ satisfy the following decreasing volume behavior:

$$\mathrm{vol}_{n-1}\big(S_H(K) \cap (t_1 v + H)\big) \geq \mathrm{vol}_{n-1}\big(S_H(K) \cap (t_2 v + H)\big)$$

if $|t_1| \leq |t_2|$ (where v is a normal unit vector of H). This fact together with the compactness condition of $S_H(K)$ imply that $S_{H^\perp}\big(\chi_{S_H(K)}\big)$ is an upper semi-continuous function.

Now (iii) follows from the fact that for non-negative upper semi-continuous functions $\phi = e^{-u}$ and $\psi = e^{-v}$ so that $\{x \in \mathbb{R}^n : u(x) < \infty\}$, $\{x \in \mathbb{R}^n : v(x) < \infty\}$ are bounded, we have [23, p. 25] $\mathrm{epi}((1-\lambda)u \oplus \lambda v) = (1-\lambda)\mathrm{epi}(u) + \lambda\mathrm{epi}(v)$, where $\mathrm{epi}(\,\cdot\,)$ denotes the epigraph of a function.

In [1] (see also [10, Corollary 1.2.1]) a similar result to Theorem C was proved, involving sections instead of projections. The aim of the following result is to prove that the inequality in Theorems 1.3 and 3.2 can be obtained if we replace the projection hypothesis by a suitable section condition.

Theorem 3.6 *Let $f, g: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be non-negative measurable functions such that $(1-\lambda)f \star \lambda g$ is measurable for $\lambda \in (0,1)$ fixed. If there exists $H \in \mathcal{L}_{n-1}^n$ such that*

$$(3.11) \qquad \sup_{y \in H^\perp} \int_{y+H} f_{|(y+H)} \, \mathrm{d}x = \sup_{y \in H^\perp} \int_{y+H} g_{|(y+H)} \, \mathrm{d}x < \infty,$$

and $(1-\lambda)S_{H^\perp}(f) \star \lambda S_{H^\perp}(g)$ is a measurable function, then

$$\int_{\mathbb{R}^n} (1-\lambda)f \star \lambda g \, \mathrm{d}x \geq (1-\lambda) \int_{\mathbb{R}^n} f \, \mathrm{d}x + \lambda \int_{\mathbb{R}^n} g \, \mathrm{d}x.$$

Proof Notice that hypothesis (3.11) together with the definition of $S_{H^\perp}$ implies that

$$\int_H \mathrm{proj}_H\big(S_{H^\perp}(f)\big) \, \mathrm{d}x = \sup_{y \in H^\perp} \int_{y+H} f_{|(y+H)} \, \mathrm{d}x$$

$$= \sup_{y \in H^\perp} \int_{y+H} g_{|(y+H)} \, \mathrm{d}x = \int_H \mathrm{proj}_H\big(S_{H^\perp}(g)\big) \, \mathrm{d}x,$$

and, furthermore, we have $\mathrm{proj}_H\big(S_{H^\perp}(f)\big) = \mathrm{proj}_H\big(S_{H^\perp}(g)\big)$. Thus, applying (2.3), Proposition 3.4, and Theorem 1.3, respectively, we get

$$\int_{\mathbb{R}^n} (1-\lambda)f \star \lambda g \, \mathrm{d}x = \int_{\mathbb{R}^n} S_{H^\perp}\big((1-\lambda)f \star \lambda g\big) \, \mathrm{d}x$$

$$\geq \int_{\mathbb{R}^n} (1-\lambda)\,S_{H^\perp}(f) \star \lambda S_{H^\perp}(g) \, \mathrm{d}x$$

$$\geq (1-\lambda) \int_{\mathbb{R}^n} f \, \mathrm{d}x + \lambda \int_{\mathbb{R}^n} g \, \mathrm{d}x,$$

as desired. ∎

Remark We would like to point out that Theorem 3.6 can be obtained as a particular consequence of results contained in [5], where the authors provide a standard proof of this Borell–Brascamp–Lieb type inequality based on induction in the dimension (*cf.* [5, Theorem 3.2]). Indeed, they obtain a "more general range" for the parameter p proving that the inequality holds not just for $p \geq -1/n$ but for $p \geq -1/(n-1)$. In Section 4 we provide an alternative proof of the inequality in this slightly smaller range based on symmetrization procedures. The work [5] deals with integral inequalities providing simple proofs of certain known inequalities (such as the Prékopa–Leindler and the Borell–Brascamp–Lieb inequalities) as well as new ones. Despite the relevance of the proven inequalities, this work seems not to be so well known in the literature.

We also refer the interested reader to [12,18] for related topics involving inequalities for functions and measures.

We end this section by remarking that (under a mild assumption on measurability) the analogous result to Corollary 3.5 for measurable sets A and B with a common maximal volume section (through parallel hyperplanes to a given one H) can be obtained. This is the content of the following result.

Corollary 3.7   *Let $A, B \subset \mathbb{R}^n$ be nonempty measurable sets and $\lambda \in (0,1)$ such that $(1 - \lambda)A + \lambda B$ is measurable. If there exists $H \in \mathcal{L}_{n-1}^n$ such that*

$$\sup_{x \in H^\perp} \mathrm{vol}_{n-1}\big(A \cap (x + H) \big) = \sup_{x \in H^\perp} \mathrm{vol}_{n-1}\big(B \cap (x + H) \big) < \infty,$$

then (provided that $(1 - \lambda)S_{H^\perp}\big(\chi_A \big) \star \lambda S_{H^\perp}\big(\chi_B \big)$ is a measurable function)

$$\mathrm{vol}\big((1 - \lambda)A + \lambda B \big) \geq (1 - \lambda)\mathrm{vol}(A) + \lambda\mathrm{vol}(B).$$

4   Extension to Borell–Brascamp–Lieb Inequalities

In this section we present an extension of Theorems 1.3 and 1.5 to the Borell–Brascamp–Lieb ("BBL" for short) inequalities. In order to describe these inequalities and our contribution, we need to introduce some further notation. More precisely we recall the definition of the p-th mean of two non-negative numbers, where p is a parameter varying in $\mathbb{R} \cup \{\pm\infty\}$. For this definition we follow [3], regarding a general reference for p-th means of non-negative numbers; we refer also to the classic text of Hardy, Littlewood, and Pólya [14].

Consider first the case $p \in \mathbb{R}$ and $p \neq 0$. Given $a, b \geq 0$ such that $ab \neq 0$ and $\lambda \in [0,1]$, we set $M_p(a, b, \lambda) = ((1 - \lambda)a^p + \lambda b^p)^{1/p}$. For $p = 0$ we set

$$M_0(a, b, \lambda) = a^{1-\lambda}b^\lambda$$

and, to complete the picture, for $p = \pm\infty$ we define $M_\infty(a, b, \lambda) = \max\{a, b\}$ and $M_{-\infty}(a, b, \lambda) = \min\{a, b\}$. Finally, if $ab = 0$, we will define $M_p(a, b, \lambda) = 0$ for all $p \in \mathbb{R} \cup \{\pm\infty\}$. Note that $M_p(a, b, \lambda) = 0$ if $ab = 0$ is redundant for all $p \leq 0$, however it is relevant for $p > 0$ (as we will briefly comment later on). Furthermore, for $p \neq 0$, we will allow that a, b take the value ∞ and in that case, as usual, $M_p(a, b, \lambda)$ will be the value that is obtained "by continuity".

The next step is to define a family of functional operations based on these means, including the Asplund sum for the special case $p = 0$. Given non-negative functions $f, g: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $p \in \mathbb{R} \cup \{\pm\infty\}$, and $\lambda \in [0,1]$, we define

$$(4.1) \qquad (1-\lambda)f \star_p \lambda g(x) = \sup_{(1-\lambda)x_1 + \lambda x_2 = x} M_p\big(f(x_1), g(x_2), \lambda \big).$$

In this way, the Asplund sum is obtained for $p = 0$. Note further that

$$(4.2) \qquad (1 - \lambda)f \star_p \lambda g \leq (1 - \lambda)f \star_q \lambda g$$

if $p \leq q$.

The following theorem contains the Borell–Brascamp–Lieb inequality (see [2,3,9] for a detailed presentation).

Theorem D (Borell–Brascamp–Lieb inequality) *Let $\lambda \in (0,1)$, $-1/n \leq p \leq \infty$ and let $f, g: R^n \to \mathbb{R}_{\geq 0}$ be non-negative measurable functions such that $(1-\lambda)f \star_p \lambda g$ is measurable as well. Then*

$$(4.3) \qquad \int_{\mathbb{R}^n} (1-\lambda)f \star_p \lambda g \, dx \geq M_{p/(np+1)}\Big(\int_{\mathbb{R}^n} f \, dx, \int_{\mathbb{R}^n} g \, dx, \lambda \Big).$$

Note that the fact that $M_p(a, b, \lambda) = 0$ if $ab = 0$ prevents us from obtaining trivial inequalities when $p > 0$. Furthermore, the above theorem is a generalization of both the classical Brunn–Minkowski inequality (1.3) ($p = \infty$ and taking f and g characteristic functions) and Prékopa–Leindler inequality, Theorem A ($p = 0$).

Regarding the functions which are naturally connected to the above theorem, we give the following definition: a non-negative function $f: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is p-concave, $p \in \mathbb{R} \cup \{\pm\infty\}$, if

$$(4.4) \qquad f\big((1-\lambda)x_1 + \lambda x_2\big) \geq M_p\big(f(x_1), f(x_2), \lambda\big)$$

for all $x_1, x_2 \in \mathbb{R}^n$ and all $\lambda \in (0,1)$. This definition has the following meanings.

(i)   For $p = \infty$, f is ∞-concave if and only if f is constant on a convex set and 0 otherwise.

(ii) For $0 < p < \infty$, f is p-concave if and only if f^p is concave on a convex set and 0 elsewhere.

(iii) For $p = 0$, f is 0-concave if and only if f is log-concave.

(iv) For $-\infty < p < 0$, f is p-concave if and only if f^p is convex.

(v)   For $p = -\infty$, f is $(-\infty)$-concave if and only if its level sets $\{x \in \mathbb{R}^n : f(x) > t\}$ are convex (for all $t \in \mathbb{R}$).

Furthermore, for any $p \in \mathbb{R} \cup \{\pm\infty\}$, f is p-concave if and only if

$$(4.5) \qquad (1-\lambda)f \star_p \lambda f = f.$$

In the following, for $p \neq 0$, we will work with (non-negative) extended measurable functions $f, g: \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ for which we will define the functional operation $\star_p$ as in (4.1). Notice that since f and g can be approximated from below by bounded functions (in such a way that the integrals converge), we are allowed to extend Theorem D for such f and g (functions which may take ∞ as a value and provided that $p \neq 0$).

In the same way, we will say that a non-negative extended function

$$f: \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$$

is p-concave, $p \neq 0$, if and only if the equivalent conditions (4.4), (4.5) hold.

4.1 BBL Inequality Under an Equal Projection Assumption

Let $H \in \mathcal{L}^n_{n-1}$ and let v be a normal unit vector of H. Given $f, g: \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, for $h, h_1, h_2 \in H$ such that $(1-\lambda)h_1 + \lambda h_2 = h$ and any $\alpha_1, \alpha_2 \in \mathbb{R}$, we clearly have

$$\text{proj}_H((1-\lambda)f \star_{-\infty} \lambda g)(h)$$
$$\geq \big((1-\lambda)f \star_{-\infty} \lambda g\big)\big((1-\lambda)(h_1 + \alpha_1 v) + \lambda(h_2 + \alpha_2 v)\big)$$
$$\geq \min\big(f(h_1 + \alpha_1 v), g(h_2 + \alpha_2 v)\big).$$

Thus, by taking suprema over $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$\mathrm{proj}_H((1-\lambda)f \star_{-\infty} \lambda g)(h) \geq \min\big(\mathrm{proj}_H(f)(h_1), \mathrm{proj}_H(f)(h_2)\big)$$

for all $h_1, h_2 \in H$ with $(1-\lambda)h_1 + \lambda h_2 = h$. This implies that

$$\mathrm{proj}_H((1-\lambda)f \star_{-\infty} \lambda g) \geq (1-\lambda)\mathrm{proj}_H(f) \star_{-\infty} \lambda\mathrm{proj}_H(g).$$

In particular, if $\mathrm{proj}_H(f) = \mathrm{proj}_H(g)$ and we set $U\colon H \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ given by $U(h) = \mathrm{proj}_H(f)(h) = \mathrm{proj}_H(g)(h)$, then

$$(4.6) \qquad\qquad \mathrm{proj}_H((1-\lambda)f \star_{-\infty} \lambda g) \geq U.$$

On the other hand, it is clear that

$$(4.7) \qquad \big\{x \in h + H^\perp : f(x) \geq t\big\}, \big\{y \in h + H^\perp : g(y) \geq t\big\} \neq \varnothing,$$

for all $0 \leq t < U(h)$ and, by means of the definition of $\star_{-\infty}$ together with (4.7), we have

$$(4.8) \quad \big\{z \in h + H^\perp : \big((1-\lambda)f \star_{-\infty} \lambda g\big)(z) \geq t\big\}$$
$$\supset (1-\lambda)\big\{x \in h + H^\perp : f(x) \geq t\big\} + \lambda\big\{y \in h + H^\perp : g(y) \geq t\big\}$$

for all $0 \leq t < U(h)$. Now, using the same approach as in the proof of Theorem 1.3 and taking into account (4.6) and (4.8) together with (4.2), we may assert the following.

Theorem 4.1 *Let $f, g\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be non-negative measurable functions such that $(1-\lambda)f \star_p \lambda g$ is measurable for $p \in \mathbb{R} \cup \{\pm\infty\}$ and $\lambda \in (0,1)$ fixed. If there exists $H \in \mathcal{L}_{n-1}^n$ such that $\mathrm{proj}_H(f) = \mathrm{proj}_H(g)$, then*

$$\int_{\mathbb{R}^n} (1-\lambda)f \star_p \lambda g \, dx \geq (1-\lambda) \int_{\mathbb{R}^n} f \, dx + \lambda \int_{\mathbb{R}^n} g \, dx.$$

In the one-dimensional case this theorem was proved by Brascamp and Lieb (see [3, Theorem 3.1]). We notice that Theorem 1.3 is obtained when $p = 0$. Note further that since $p/(np+1) \in [-\infty, 1/n]$, the above inequality is stronger than (4.3) (*cf.* (4.2)).

4.2 BBL Inequality Under the Same Integral of a Projection

Given $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ measurable and $-\infty < p < 0$, it will be convenient to write it in the form $f = u^{1/p}$ where $u\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a measurable function, *i.e.*, $u(x) = f(x)^p$ with the conventions that $0^p = \infty$ and $\infty^p = 0$). Given an $H \in \mathcal{L}_{n-1}^n$, we may write the Steiner symmetral of $f = u^{1/p}$ in the form $S_H(f) = u_H^{1/p}$, *i.e.*, $u_H\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is the function given by $u_H(x) = \big(S_H(f)(x)\big)^p$. Notice further that as $t \mapsto t^{1/p}$ is a decreasing bijection on $\mathbb{R}_{\geq 0} \cup \{\infty\}$ and $\mathrm{hyp}_s(S_H(f)) = S_{\widetilde{H}}\big(\mathrm{hyp}_s(f)\big)$, we also have $\mathrm{epi}_s(u_H) = S_{\widetilde{H}}\big(\mathrm{epi}_s(u)\big)$.

Now writing $f = u^{1/p}$ and $g = v^{1/p}$, it is easy to check that $(1-\lambda)f \star_p \lambda g = w^{1/p}$ where $w = (1-\lambda)u \oplus \lambda v$. Therefore $S_H(f) = u_H^{1/p}$, $S_H(g) = v_H^{1/p}$ whereas

$$S_H\big((1-\lambda)f \star_p \lambda g\big) = w_H^{1/p}$$

and, without loss of generality, we may also assume that both f and g are not identically zero (which implies that $\text{epi}_s(u_H)$, $\text{epi}_s(v_H) \neq \varnothing$). Thus, using a similar argument to that at the end of Proposition 3.3, we have

$$\text{epi}_s(w_H) = S_{\widetilde{H}}\big(\text{epi}_s(w)\big) \supset \text{epi}_s((1-\lambda)u_H \oplus \lambda v_H).$$

This is equivalent to $w_H \leq (1-\lambda)u_H \oplus \lambda v_H$ and hence

$$S_H\big((1-\lambda)f \star_p \lambda g\big) = w_H^{1/p} \geq \big((1-\lambda)u_H \oplus \lambda v_H\big)^{1/p} = (1-\lambda)S_H(f) \star_p \lambda S_H(g).$$

Furthermore, notice that for $p = -\infty$, we have

$$\begin{aligned}
\big\{ z \in &\mathbb{R}^n : S_H((1-\lambda)f \star_{-\infty} \lambda g)(z) > t \big\} \\
&= S_H\big(\big\{ z \in \mathbb{R}^n : \big((1-\lambda)f \star_{-\infty} \lambda g\big)(z) > t \big\}\big) \\
&\supset (1-\lambda)S_H\big(\big\{ x \in \mathbb{R}^n : f(x) > t \big\}\big) + \lambda S_H\big(\big\{ y \in \mathbb{R}^n : g(y) > t \big\}\big) \\
&= (1-\lambda)\big\{ x \in \mathbb{R}^n : S_H(f)(x) > t \big\} + \lambda\big\{ y \in \mathbb{R}^n : S_H(g)(y) > t \big\} \\
&= \big\{ z \in \mathbb{R}^n : \big((1-\lambda)S_H(f) \star_{-\infty} \lambda S_H(g)\big)(z) > t \big\},
\end{aligned}$$

and thus $S_H((1-\lambda)f \star_{-\infty} \lambda g) \geq (1-\lambda)S_H(f) \star_{-\infty} \lambda S_H(g)$.

On the other hand, given $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha = (1-\lambda)\alpha_1 + \lambda\alpha_2$, and by means of the Borell–Brascamp–Lieb inequality (Theorem D), we have

$$\begin{aligned}
\int_{\alpha v+H} &\big((1-\lambda)f \star_p \lambda g\big)_{|(\alpha v+H)} \, dx \\
&\geq M_{p/(np+1)}\Big(\int_{\alpha_1 v+H} f_{|(\alpha_1 v+H)} \, dx, \int_{\alpha_2 v+H} g_{|(\alpha_2 v+H)} \, dx, \lambda \Big),
\end{aligned}$$

for $-1/n \leq p < 0$. This ensures that, for $h_1, h_2 \in H$,

$$\begin{aligned}
S_{H^\perp}\big((1-\lambda)f &\star_p \lambda g\big)\big(((1-\lambda)h_1 + \lambda h_2) + ((1-\lambda)\alpha_1 + \lambda\alpha_2)v\big) \\
&\geq M_{p/(np+1)}\big(S_{H^\perp}(f)(h_1 + \alpha_1 v), S_{H^\perp}(g)(h_2 + \alpha_2 v), \lambda \big).
\end{aligned}$$

Therefore, we have shown the following result.

Proposition 4.2 *Let $f, g \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be non-negative measurable functions such that $(1-\lambda)f \star_p \lambda g$ is measurable for $\lambda \in (0,1)$ fixed and $-\infty \leq p < 0$. Then given $H \in \mathcal{L}_{n-1}^n$,*

(4.9) $$S_H((1-\lambda)f \star_p \lambda g) \geq (1-\lambda)S_H(f) \star_p \lambda S_H(g).$$

Moreover, if $-1/n \leq p < 0$,

(4.10) $$S_{H^\perp}((1-\lambda)f \star_p \lambda g) \geq (1-\lambda)S_{H^\perp}(f) \star_q \lambda S_{H^\perp}(g),$$

where $q = p/(np+1)$.

Now as in Theorem 3.2, we extend the above theorem for the case of two functions with the same integral of a projection onto a hyperplane.

Theorem 4.3 *Let* $-1/n \le p < 0$ *and let* $f, g: \mathbb{R}^n \to \mathbb{R}_{\ge 0} \cup \{\infty\}$ *be non-negative measurable functions such that* $(1-\lambda) f \star_p \lambda g$ *is measurable for* $\lambda \in (0,1)$ *fixed. If there exists* $H \in \mathcal{L}_{n-1}^n$ *such that*

$$(1-\lambda) S_H(f) \star_p \lambda S_H(g) \quad and \quad (1-\lambda) S_{H^\perp}(S_H(f)) \star_{-\infty} \lambda S_{H^\perp}(S_H(g))$$

are measurable functions and

(4.11) $$\int_H \mathrm{proj}_H(f)(x) \, \mathrm{d}x = \int_H \mathrm{proj}_H(g)(x) \, \mathrm{d}x,$$

then $\int_{\mathbb{R}^n} (1-\lambda) f \star_p \lambda g \, \mathrm{d}x \ge (1-\lambda) \int_{\mathbb{R}^n} f \, \mathrm{d}x + \lambda \int_{\mathbb{R}^n} g \, \mathrm{d}x.$

Proof Hypothesis (4.11) together with the definition of S_H, $S_{H^\perp}$ implies

$$\mathrm{proj}_H\big(S_{H^\perp}(S_H(f))\big) = \mathrm{proj}_H\big(S_{H^\perp}(S_H(g))\big).$$

On the other hand, Proposition 4.2 (together with the monotonicity of $S_{H^\perp}(\cdot)$ and (4.2)) implies

$$\begin{aligned} S_{H^\perp}\big(S_H((1-\lambda) f \star_p \lambda g)\big) &\ge S_{H^\perp}\big((1-\lambda) S_H(f) \star_p \lambda S_H(g)\big) \\ &\ge (1-\lambda) S_{H^\perp}\big(S_H(f)\big) \star_q \lambda S_{H^\perp}\big(S_H(g)\big) \\ &\ge (1-\lambda) S_{H^\perp}\big(S_H(f)\big) \star_{-\infty} \lambda S_{H^\perp}\big(S_H(g)\big), \end{aligned}$$

where $q = p/(np+1)$. The proof is now concluded by following similar steps to Theorem 3.2. ∎

Remark Notice that if in the above theorem $f, g: \mathbb{R}^n \to \mathbb{R}_{\ge 0} \cup \{\infty\}$ are such that $(1-\lambda) f \star_{p'} \lambda g$ is measurable for $\lambda \in (0,1)$ and $p' \ge 0$, then (*cf.* (4.2))

$$\int_{\mathbb{R}^n} (1-\lambda) f \star_{p'} \lambda g \, \mathrm{d}x \ge (1-\lambda) \int_{\mathbb{R}^n} f \, \mathrm{d}x + \lambda \int_{\mathbb{R}^n} g \, \mathrm{d}x.$$

Proof of Theorem 1.6 Without loss of generality we may assume that $p < 0$ (*cf.* (4.2)). It is an easy exercise to check that if a function ϕ is p-concave, then $S_H(\phi)$ and $S_{H^\perp}(\phi)$ are, respectively, p-concave and q-concave ($q = p/(np+1)$) functions. In fact, this may be quickly obtained from (4.5), (4.9), and (4.10).

On the other hand, since q-concave functions are also $(-\infty)$-concave and the t-Asplund sum $\star_t$ preserves t-concavity, we can assert that

$$(1-\lambda) S_H(f) \star_p \lambda S_H(g), \quad (1-\lambda) S_{H^\perp}(S_H(f)) \star_{-\infty} \lambda S_{H^\perp}(S_H(g))$$

are measurable functions. The proof is concluded by applying Theorem 4.3. ∎

To end this paper, we establish here the analogous result to Theorem 3.6. It can be shown following the steps of the proof of the above-mentioned theorem.

Theorem 4.4 *Let* $-1/n \le p < 0$ *and let* $f, g: \mathbb{R}^n \to \mathbb{R}_{\ge 0} \cup \{\infty\}$ *be non-negative measurable functions such that* $(1-\lambda) f \star_p \lambda g$ *is measurable for* $\lambda \in (0,1)$ *fixed. If there exists* $H \in \mathcal{L}_{n-1}^n$ *such that*

$$\sup_{y \in H^\perp} \int_{y+H} f_{|(y+H)} \, \mathrm{d}x = \sup_{y \in H^\perp} \int_{y+H} g_{|(y+H)} \, \mathrm{d}x,$$

and $(1 - \lambda)S_{H^\perp}(f) \star_{-\infty} \lambda S_{H^\perp}(g)$ is a measurable function, then

$$\int_{\mathbb{R}^n} (1 - \lambda)f \star_p \lambda g \, dx \geq (1 - \lambda) \int_{\mathbb{R}^n} f \, dx + \lambda \int_{\mathbb{R}^n} g \, dx.$$

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