This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to I. G. Connell, Department of Mathematics, McGill University, Montreal, P. Q.

ON THE BOUNDARY AND TENSOR PRODUCT OF FUNCTION ALGEBRAS

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Introduction. Let $\mathcal{F}$ be an arbitrary family of continuous complex-valued functions defined on a compact Hausdorff space X. A closed subset $B \subseteq X$ is called a boundary for $\mathcal{F}$ if every $f \in \mathcal{F}$ attains its maximum modulus at some point of B. A boundary, B, is said to be minimal if there exists no boundary for $\mathcal{F}$ properly contained in B. It can be shown that minimal boundaries exist regardless of the algebraic structure which $\mathcal{F}$ may possess. Under certain conditions on the family $\mathcal{F}$, it can be shown that a unique minimal boundary for $\mathcal{F}$ exists. In particular, this is the case if $\mathcal{F}$ is a subalgebra or subspace of C(X) where X is compact and Hausdorff (see for example [2]). This unique minimal boundary for an algebra $\mathcal{F}$ of functions is called the Šilov boundary of $\mathcal{F}$.

It is well known (see [4] for example), that in the special case of C(X), a boundary for a uniformly dense subalgebra $A \subseteq C(X)$ is a boundary for C(X). It is in both cases the space X itself. Section 1 of this note shows that the boundaries also coincide when C(X) is replaced by an arbitrary family $\mathcal{F}$,

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and A is a uniformly dense subfamily of $\mathcal{F}$. This result is used to show that if A and B are function algebras on X and Y respectively, the Šilov boundary of the tensor product, $A \otimes B$, of A and B is the product of the Šilov boundaries of A and B.

1. Uniformly dense subfamilies of $\mathcal{F}$. We shall without loss of generality assume that the functions under discussion are real valued and non-negative.

PROPOSITION 1. Let $\mathcal{F}$ be a family of continuous real valued functions on a compact Hausdorff space X and let B be a boundary for a uniformly dense subfamily A of $\mathcal{F}$. Then B is also a boundary for $\mathcal{F}$.

Proof. Let F be the uniform limit of a sequence (f_n) of functions, $f_n \in A$ for all n, and let $D \subseteq X$ be closed and such that all the functions f_n attain their maximum values on D. We show F attains its maximum on D. Let M denote $\max_{x \in X} F(x)$. Then it is clear that F attains its maximum on D if and only if for all $\varepsilon > 0$, $\{x / F(x) \geq M - \varepsilon\} \cap D \neq \emptyset$. We must therefore show that for $\varepsilon > 0$ there exists an $x \in D$ with $F(x) \geq M - \varepsilon$. Let F attain its maximum at $x_1 \in X$ and let f_n attain its maximum at $x_0 \in D$, where n is chosen so that $|f_n(x) - F(x)| < \varepsilon/2$ for all $x \in X$. Then we have $-\varepsilon/2 < f_n(x_o) - F(x_o) < \varepsilon/2$ and $-\varepsilon/2 < f_n(x_1) - F(x_1) < \varepsilon/2$. Consequently $F(x_o) > f_n(x_o) - \varepsilon/2$, and, since $f_n(x_o) \geq f_n(x_1)$, we have $F(x_o) > f_n(x_1) - \varepsilon/2$. Further, $F(x_1) - \varepsilon < f_n(x_1) - \varepsilon/2$. Hence we have $F(x_o) \geq F(x_1) - \varepsilon = M - \varepsilon$.

2. The tensor product of function algebras. In this section X and Y will always denote compact Hausdorff topological spaces. Let A and B be point separating uniformly closed algebras of continuous complex valued functions defined on X and Y respectively. In addition, let A and B both contain the constant function 1.

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Let $f \in A$, $g \in B$. For $(x,y) \in X \times Y$, define $f \otimes g(x,y)$ to be $f(x)g(y)$. Denote by L the linear span of $\{f \otimes g \,/\, f \in A,\ g \in B\}$. L is a linear subspace of $C(X \times Y)$ and is in fact closed under multiplication since $f_1 \times g_1$
$f_1 \otimes g_1 \circ f_2 \otimes g_2(x,y) = f_1(x)g_1(y)f_2(x)g_2(y) = f_1f_2 \otimes g_1g_2(x,y)$.
Hence L is a subalgebra of $C(X \times Y)$.

DEFINITION. The tensor product $A \otimes B$ of A and B is the uniform closure of L in $C(X \times Y)$.

By [2], the Šilov boundaries of A, B and $A \otimes B$ exist. Denote these by $\partial_A(X)$, $\partial_B(Y)$ and $\partial_{A \otimes B}(X \times Y)$ respectively.

THEOREM. Let A and B be given as above. Then

$$\partial_{A \otimes B}(X \times Y) = \partial_A(X) \times \partial_B(Y).$$

<u>Proof</u>. $\partial_L(X \times Y)$ exists by [2], and hence as a consequence of the proposition proved above it suffices to show that

$$\partial_L(X \times Y) = \partial_A(X) \times \partial_B(Y).$$

Let (x_o, y_o) be any point at which the function $\Sigma f_i \otimes g_i$ attains its maximum modulus. Then the function $\Sigma f_i(x_o)g_i$ attains its maximum modulus also at some $y_o' \in \partial_B(Y)$ and therefore $|\Sigma f_i(x_o)g_i(y_o')| = |\Sigma f_i(x_o)g_i(y_o)|$; similarly, there exists an $x_o' \in \partial_A(X)$ at which the function $\Sigma g_i(y_o')f_i$ attains its maximum modulus and hence $|\Sigma f_i(x_o')g_i(y_o')| = |\Sigma f_i(x_o)g_i(y_o)|$. Thus $\Sigma f_i \otimes g_i$ also takes on its maximum modulus at the point (x_o', y_o') in $\partial_A(X) \times \partial_B(Y)$ which shows that $\partial_L(X \times Y) \subseteq \partial_A(X) \times \partial_B(Y)$. On the other hand let $(x_o, y_o) \in \partial_A(X) \times \partial_B(Y)$, and let $U(x_o)$ and $V(y_o)$ be neighbourhoods of x_o and y_o respectively. By Theorem 2 in [3] we may choose an $f \in A$ and a $g \in B$ such that

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$\underset{(x,y)\in X\times Y}{\mathrm{Max}} |f(x)g(y)|$ is attained on $U(x_o) \times V(y_o)$ and not outside it. Hence by the same theorem $\partial_A(X) \times \partial_B(Y) \subseteq \partial_L(X \times Y)$ and therefore we have the desired result.

Example. Let $X = Y = \{ z \ |z| \leq 1\}$ and let $A = B$ be the disc algebra, that is the family of continuous functions on the closed unit disc analytic on the open unit disc. It can be shown that the tensor product $A \otimes A$ is the algebra of continuous functions on the polydisc $X \times Y$ which are analytic on its interior.

The Šilov boundary of $A \otimes A$ exists and by the theorem it is $\partial_A(X) \times \partial_A(X)$. Using the Maximum Modulus Principle, it is easily shown that $\partial_A(X) = \{ z \ / \ |z| = 1\}$ and hence we have

$$\partial_{A \otimes A}(X \times X) = S^1 \times S^1 .$$

This is the well known Bergmann Distinguished Boundary of the polydisc $X \times Y$.

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