SEPARATION AND APPROXIMATION IN TOPOLOGICAL VECTOR LATTICES

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1. Introduction. Spectral theory in its lattice-theoretic setting proves abstractly that the indicators of measurable sets generate the space L of Lebesgue-integrable functions on an interval. We are concerned here with abstractions suggested by the fact that indicators of intervals suffice to generate L. Our results show that the approximation of arbitrary elements of a topological vector lattice rests upon the ability to separate disjoint elements f and g by an operation that behaves in the limit like a projection annihilating f and leaving g invariant.

The introduction of this concept of separation together with the notion of limit unit leads (via the Fundamental Lemma) to abstract generalizations of the Radon-Nikodym Theorem (Theorem 1) and the Stone-Weierstrass Theorem (Theorem 3). Even for lattices which have representations as function spaces our abstract approach has several advantages: (i) the domain plays no explicit role in the theory, (ii) we are not restricted to the topology of uniform convergence, and (iii) the functions under consideration need not be bounded, although they must be limits of bounded functions. Thus, Theorem 3 is actually stronger than Stone's theorem **(12)**. We do not assume conditional σ-completeness **(1)** in our lattices, so countable-additivity plays no role in the Boolean ring of Theorem 1.

The author is indebted to the referees for clarifying the general setting of the theory.

2. Positive operators on a vector lattice. Let $\mathfrak{L}$ be a vector lattice with real scalars. The following lattice-group properties will prove useful **(1, 4, 9)**:

$$f + g = f \vee g + f \wedge g \tag{2.1}$$

$$(f - f \wedge g) \wedge (g - f \wedge g) = 0 \tag{2.2}$$

$$|f \wedge h - g \wedge h| \leqslant |f - g| \tag{2.3}$$

$$|f \vee h - g \vee h| \leqslant |f - g|. \tag{2.4}$$

An *operator* on $\mathfrak{L}$ is a linear mapping of $\mathfrak{L}$ into itself. The operators on $\mathfrak{L}$ are partially ordered by defining $P \leqslant Q$ whenever $Pf \leqslant Qf$ for all $f \geqslant 0$ in $\mathfrak{L}$. Thus, positive operators are order-preserving:

$$\text{If } P \geqslant 0 \text{ and } f \leqslant g, \text{ then } Pf \leqslant Pg. \tag{2.5}$$

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A *contractor* is an operator P such that

$$0 \leqslant P \leqslant I \tag{2.6}$$

where I is the identity operator. We shall use the abbreviation P' for $I - P$. Thus, P is a contractor if, and only if, both P and P' are positive operators. Note that P' is a contractor whenever P is a contractor, and PQ is a contractor whenever P and Q are contractors.

Contractors interest us because they commute with the lattice operations:

$$P(f \wedge g) = Pf \wedge Pg \tag{2.7}$$

$$P(f \vee g) = Pf \vee Pg \tag{2.8}$$

and

$$P|f| = |Pf|. \tag{2.9}$$

To prove (2.7) let $h = Pf \wedge Pg$. Since $f \wedge g \leqslant f$ and $f \wedge g \leqslant g$, (2.5) gives $P(f \wedge g) \leqslant Pf$ and $P(f \wedge g) \leqslant Pg$. Hence $P(f \wedge g) \leqslant h$. To reverse this inequality we have $h \leqslant Pf$ and $P'(f \wedge g) \leqslant P'f$. Adding these gives $h + P'(f \wedge g) \leqslant f$. Similarly, $h + P'(f \wedge g) \leqslant g$. Hence $h + P'(f \wedge g) \leqslant f \wedge g$. Transposing the second term on the left gives $h \leqslant P(f \wedge g)$. Hence (2.7). The dual statement (2.8) follows from (2.7) and the identity (2.1). To obtain (2.9) set $g = -f$ in (2.8).

We call an idempotent contractor a *projector*. If A and B are projectors and $f \geqslant 0$, then

$$ABf = Af \wedge Bf. \tag{2.10}$$

To derive (2.10) let $g = Af \wedge Bf$. Now $ABf \leqslant Bf \leqslant f$ by (2.6). Applying A to the latter inequality gives $ABf \leqslant Af$. Hence $ABf \leqslant g$. To reverse this inequality note that $0 \leqslant g \leqslant Af$ and $0 \leqslant g \leqslant Bf$. Since $A^2 = A$, $A'A = 0$, so $A'g = 0$ by (2.5). Thus $Ag = g$ and similarly $Bg = g$. Hence $ABg = Ag = g$. Since $Af \leqslant f$ and $Bf \leqslant f$, $g \leqslant f$. So $ABg \leqslant ABf$. That is, $g \leqslant ABf$. Hence (2.10).

From (2.10) it follows that projectors commute: $AB = BA$. Moreover, in terms of the operator ordering, (2.10) gives $A \cap B = AB$ and hence $A \cup B = A + B - AB$, which are easily seen to be projectors. Thus, the projectors on $\mathfrak{L}$ form a Boolean algebra with I as unit.

We remark that if $\mathfrak{L}$ is non-Archimedean, contractors need not commute.

3. Topological vector lattices. L is a *topological vector lattice* if it is a vector lattice with a topology making it a topological vector space possessing a local base of neighbourhoods $\mathfrak{N}$ of 0 such that

$$f \text{ is in } \mathfrak{N} \text{ whenever } |f| \leqslant |g| \text{ for some } g \text{ in } \mathfrak{N}. \tag{3.1}$$

(In **(10)** $\mathfrak{L}$ is called a locally-solid lattice-ordered linear topological space.) The lattice operations as well as the vector operations are continuous in $\mathfrak{L}$. Every Banach lattice **(1)** is clearly a topological vector lattice.

Given an arbitrary set $\mathfrak{U}$ of elements in a topological vector space $\mathfrak{V}$, we say $\mathfrak{U}$ *generates* $\mathfrak{W}$ if $\mathfrak{W}$ is the smallest closed linear subspace of $\mathfrak{V}$ which contains $\mathfrak{U}$.

A positive element u in a topological vector lattice is a *limit bound* of f if

$$(3.2) \qquad |f| \wedge nu \to |f| \qquad \text{as } n \to \infty.$$

f is *bounded* relative to u if $|f| \leqslant nu$ for some n. Now, u is a limit bound of f if, and only if, f is a limit of elements bounded relative to u. For, given (3.2) and $|h| \leqslant |f|$ we have, using (2.3), $0 \leqslant |f| \wedge nu - h \wedge nu \leqslant |f| - h$. Hence $0 \leqslant h - h \wedge nu \leqslant |f| - |f| \wedge nu$. From (3.2) and (3.1) we have $h \wedge nu \to h$. Taking first $h = f^+$ and then $h = f^-$ gives $f^+ \wedge nu - f^- \wedge nu \to f$ as $n \to \infty$. Conversely, given a net **(8)** of bounded elements converging to f, $f_t \to f$, we have $|f_t| = |f_t| \wedge nu$ for n sufficiently large. So using (2.3),

$$0 \leqslant |f| - |f| \wedge nu \leqslant \big||f| - |f_t|\big| + \big||f_t| - |f| \wedge nu\big| \leqslant 2\big||f| - |f_t|\big| \leqslant 2|f - f_t|.$$

Hence (3.2) follows from (3.1).

We say u is a *limit unit* in $\mathfrak{L}$ if u is a limit bound for every f in $\mathfrak{L}$, that is, if the bounded elements relative to u are dense in $\mathfrak{L}$. A limit unit is always a weak unit **(1)** if the topology in $\mathfrak{L}$ is T_1, that is, if finite sets are closed. To prove this let $f \wedge u = 0$. Then we have $1/n\ (f \wedge nu) \leqslant f$ and $1/n\ (f \wedge nu) \leqslant u$. So $f \wedge nu = 0$. Hence (3.2) implies $f = 0$. We remark that a weak unit need not be a limit unit.

A set $\mathfrak{C}$ of operators on a topological vector lattice $\mathfrak{L}$ is said to *separate f from g* if for every neighbourhood $\mathfrak{N}$ of 0 in $\mathfrak{L}$ there exists P in $\mathfrak{C}$ such that both $f - Pf$ and Pg are in $\mathfrak{N}$, that is, if there exists a net P_t in $\mathfrak{C}$ such that $P_tf \to f$ and $P_tg \to 0$. We say $\mathfrak{C}$ *separates f and g* if it separates f from g and g from f.

4. Approximation by contractors on a limit unit.

Our approximation theorems all depend upon the following lemma:

FUNDAMENTAL LEMMA. *Let u be a limit unit in a topological vector lattice $\mathfrak{L}$ and $\mathfrak{C}$ a set of contractors on $\mathfrak{L}$ such that $\mathfrak{C}$ separates every pair f and g in $\mathfrak{L}$ for which $f \wedge g = 0$. Then the set of all $PQ'u$ with P and Q in $\mathfrak{C}$ generates $\mathfrak{L}$.*

Proof. Since u is a limit unit we need only show that for $|f| \leqslant \lambda u$ and $\mathfrak{N}$ any neighbourhood of 0 satisfying (3.1) there exists g of the form $\sum \lambda_k P_k Q_k' u$ with P_k and Q_k in $\mathfrak{C}$ such that $f - g$ is in $\mathfrak{N}$.

Consider an arbitrary $\epsilon > 0$. We may assume ϵ is small enough to ensure that ϵu is interior to $\mathfrak{N}$, using the continuity of scalar multiplication. Choose $\lambda_0, \lambda_1, \ldots, \lambda_N$ with $\lambda_k - \lambda_{k-1} = \epsilon$ for $k = 1, \ldots, N$ and $\lambda_0 u \leqslant f \leqslant \lambda_N u$. For notational simplicity let $f_k = f - \lambda_k u$. By the hypothesis of separation there exists for each k a net $P_k(t)$ in $\mathfrak{C}$ such that

$$(4.1) \qquad P_k f_k^+ \to 0 \qquad \text{and} \qquad P_k' f_k^- \to 0,$$

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the limits being taken with respect to t. (We hereafter abbreviate $P(t)$ to P.) Since $f_0^- = 0$ we may assume $P_0 = 0$. Also, since $f_N^+ = 0$ we may take the net P_N such that

$$P_N u \to u \tag{4.2}$$

applying the separation hypothesis to u and 0. Now,

$$\begin{aligned} 0 \leqslant \epsilon P_{k-1}P_k'u &= P_{k-1}P_k'(f_{k-1} - f_k) \leqslant P_{k-1}P_k'(|f_{k-1}| + |f_k|) \\ &\leqslant P_{k-1}f_{k-1}^+ + P_k'f_{k-1}^- + P_{k-1}f_k^+ + P_k'f_k^- \leqslant 2P_{k-1}f_{k-1}^+ + 2P_k'f_k^- \end{aligned} \tag{4.3}$$

since $f_{k-1}^- \leqslant f_k^-$ and $f_k^+ \leqslant f_{k-1}^+$. Since the right side of (4.3) converges to 0 by (4.1), we have via (3.1)

$$P_{k-1}P_k'u \to 0. \tag{4.4}$$

Since $P_k P_{k-1}' = (P_k - P_{k-1}) + P_{k-1}P_k'$ and $P_0 = 0$,

$$\sum P_k P_{k-1}' = P_N + \sum P_{k-1}P_k' \tag{4.5}$$

with summation over $k = 1, \ldots, N$. Applying (4.5) to u and taking limits with respect to t, we obtain via (4.4) and (4.2)

$$\sum P_k P_{k-1}'u \to u. \tag{4.6}$$

Recalling that $|f| \leqslant \lambda u$ and $P_{k-1}P_k'$ is a contractor, we have

$$|P_{k-1}P_k'f| \leqslant \lambda P_{k-1}P_k'u$$

by (2.5) and (2.9). Hence (4.4) gives

$$P_{k-1}P_k'f \to 0. \tag{4.7}$$

Similarly, since $P_N'u \to 0$ by (4.2), $P_N'f \to 0$. So (4.5) and (4.7) give

$$\sum P_k P_{k-1}'f \to f. \tag{4.8}$$

Now since $f_k^- \leqslant f_{k-1}^- + \epsilon u$,

$$|P_k P_{k-1}'f_k| \leqslant P_k f_k^+ + P_k P_{k-1}'f_k^- \leqslant P_k f_k^+ + P_{k-1}'f_{k-1}^- + \epsilon P_k P_{k-1}'u. \tag{4.9}$$

Thus,

$$\begin{aligned} \left|f - \sum \lambda_k P_k P_{k-1}'u\right| &\leqslant \left|f - \sum P_k P_{k-1}'f\right| + \left|\sum P_k P_{k-1}'f_k\right| \\ &\leqslant \left|f - \sum P_k P_{k-1}'f\right| + \sum P_k f_k^+ \\ &\quad + \sum P_{k-1}'f_{k-1}^- + \epsilon \sum P_k P_{k-1}'u. \end{aligned} \tag{4.10}$$

By (4.8), (4.1), and (4.6) the right side of (4.10) converges to ϵu, which is interior to $\mathfrak{N}$. Hence, the right side of (4.10) is eventually in $\mathfrak{N}$. By (3.1), the left side of (4.10) is likewise eventually in $\mathfrak{N}$, which proves the lemma.

5. Approximation by projectors on a limit unit.

THEOREM 1. *Let $\mathfrak{R}$ be a Boolean ring of projectors on a topological vector lattice $\mathfrak{L}$ and u be a limit unit in $\mathfrak{L}$. Then $\mathfrak{R}u$, the set of all Eu for E in $\mathfrak{R}$, generates $\mathfrak{L}$ if, and only if, $\mathfrak{R}$ separates every pair f and g in $\mathfrak{L}$ for which $f \wedge g = 0$.*

Proof. Let $\mathfrak{R}u$ generate $\mathfrak{L}$. Then, given $f \wedge g = 0$, there exists a net f_t converging to f and a corresponding net g_t converging to g of the form:

$$(5.1) \qquad f_t = \sum \alpha_k E_k u, \qquad g_t = \sum \beta_k E_k u$$

where E_k is in $\mathfrak{R}$ and $E_i E_j = 0$ for $i \neq j$. Since $f_t \to f$, $|f_t| \to |f|$ by (3.1). Moreover $f \geqslant 0$, so we may assume $f_t \geqslant 0$, and similarly $g_t \geqslant 0$. That is, $\alpha_k \geqslant 0$ and $\beta_k \geqslant 0$ in (5.1). Let A_t be the sum of those E_k in (5.1) for which $\alpha_k \leqslant \beta_k$. Since $f_t \wedge g_t = \sum \delta_k E_k u$ where δ_k is the smaller of α_k and β_k, we have $0 \leqslant A_t f_t \leqslant f_t \wedge g_t$ and $0 \leqslant A_t' g_t \leqslant f_t \wedge g_t$. Therefore

$$(5.2) \qquad A_t f \leqslant |A_t f - A_t f_t| + A_t f_t \leqslant |f - f_t| + f_t \wedge g_t.$$

Since $f \wedge g = 0$,

$$f_t \wedge g_t \leqslant |f \wedge g - f \wedge g_t| + |f \wedge g_t - f_t \wedge g_t| \leqslant |g - g_t| + |f - f_t|$$

by (2.3). Hence (5.2) gives $|A_t f| \leqslant |g - g_t| + 2|f - f_t|$. Since $f_t \to f$ and $g_t \to g$, $A_t f \to 0$ by (3.1). Similarly

$$|A_t' g| \leqslant |f - f_t| + 2|g - g_t|.$$

Hence, $A_t' g \to 0$.

The converse follows directly from the fundamental lemma, since PQ' is in $\mathfrak{R}$ for P and Q in $\mathfrak{R}$.

6. Topological lattice algebras. Let $\mathfrak{A}$ be a T_1 topological vector lattice in which an associative, distributive multiplication is defined making $\mathfrak{A}$ a topological algebra with a multiplicative unit 1 which is also a limit unit. Moreover, let $fg \geqslant 0$ whenever both $f \geqslant 0$ and $g \geqslant 0$. We call $\mathfrak{A}$ a *topological lattice algebra*. From **(2)** it follows that multiplication is commutative in $\mathfrak{A}$.

We shall apply the results of the preceding sections by viewing the elements of $\mathfrak{A}$ as operators on $\mathfrak{A}$ via multiplication. This is effective because the operator ordering for elements of $\mathfrak{A}$ is just the ordering in $\mathfrak{A}$. A few simple lemmas serve to establish the basic properties of $\mathfrak{A}$.

LEMMA 1. *If $f \wedge g = 0$, then $fg = 0$.*

Proof. Let $f_n = f \wedge n1$ and $g_n = g \wedge n1$. Since 1 is a limit unit $f_n \to f$ and $g_n \to g$. Since multiplication is continuous $f_n g_n \to fg$. Thus, it suffices to show $f_n g_n = 0$. Since $0 \leqslant f_n \leqslant f$ and $0 \leqslant g_n \leqslant g$ we have $0 \leqslant f_n \wedge g_n \leqslant f \wedge g$. So $f_n \wedge g_n = 0$, since $f \wedge g = 0$. Moreover, $0 \leqslant f_n \leqslant n1$ and since $g_n \geqslant 0$, $0 \leqslant f_n g_n \leqslant n g_n$. Similarly $f_n g_n \leqslant n f_n$. Hence

$$0 \leqslant \frac{1}{n} f_n g_n \leqslant f_n \wedge g_n,$$

and so $f_n g_n = 0$.

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LEMMA 2. $f^2 = |f|^2$. *Hence*, $f^2 \geqslant 0$.

Proof. By Lemma 1, $f^+ f^- = 0$. So $f^2 = (f^+ - f^-)^2 = f^{+2} + f^{-2} = |f|^2$.

LEMMA 3. *If* $f^2 = 0$, *then* $f = 0$.

Proof. By Lemma 2 we may assume without loss of generality that $f \geqslant 0$. Consider any $\epsilon > 0$. Now $(f - \epsilon 1)^2 = - 2\epsilon f + \epsilon^2 1$, which is positive by Lemma 2. So $2\epsilon f \leqslant \epsilon^2 1$. Dividing by ϵ we get $0 \leqslant 2f \leqslant \epsilon 1$. Letting $\epsilon \to 0$ gives $f = 0$.

LEMMA 4. *If* $f \geqslant 0$, $g \geqslant 0$, *and* $fg = 0$, *then* $f \wedge g = 0$.

Proof. Let $h = f \wedge g$. Then $0 \leqslant h \leqslant f$ and $0 \leqslant h \leqslant g$. Therefore $0 \leqslant h^2 \leqslant fh \leqslant fg \leqslant 0$. So $h^2 = 0$. By Lemma 3, $h = 0$.

LEMMA 5. $|fg| = |f|\,|g|$.

Proof. $fg = (f^+ - f^-)(g^+ - g^-) = (f^+g^+ + f^-g^-) - (f^+g^- + f^-g^+)$, a difference of two positive terms. That the product of these two terms is 0 follows from Lemma 1, using the commutative, distributive, and associative laws. Hence, by Lemma 4, the two terms are disjoint. Thus,

$$(fg)^+ = f^+g^+ + f^-g^-$$

and

$$(fg)^- = f^+g^- + f^-g^+.$$

Therefore,

$$|fg| = (fg)^+ + (fg)^- = (f^+ + f^-)(g^+ + g^-) = |f|\,|g|.$$

LEMMA 6. $fg = 0$ *if, and only if,* $|f| \wedge |g| = 0$.

Proof. By Lemma 5, $fg = 0$ if, and only if, $|f|\,|g| = 0$. By Lemmas 1 and 4, $|f|\,|g| = 0$ if, and only if, $|f| \wedge |g| = 0$.

7. Projectors on a topological lattice algebra.

LEMMA 7. *The identity*

$$(Ef)g = f(Eg) = (Ef)(Eg) \tag{7.1}$$

holds for every projector E *on* $\mathfrak{A}$.

Proof. $(Ef)g - f(Eg) = (Ef)(E'g) - (Eg)(E'f)$, an identity which can be verified by setting $E' = I - E$ on the right and expanding. We shall show that each of the terms on the right side of this identity is 0, in order to derive the first equation in (7.1). Now by (2.9), (2.5), and (2.10),

$$|Ef| \wedge |E'g| = E|f| \wedge E'|g| \leqslant E(|f| + |g|) \wedge E'(|f| + |g|) = EE'(|f| + |g|) = 0.$$

Thus, by Lemma 6, $(Ef)(E'g) = 0$. Similarly $(Eg)(E'f) = 0$. The second equation in (7.1) follows if we replace f in the first equation by Ef.

LEMMA 8. *The projectors E on $\mathfrak{A}$ are isomorphic to the idempotent elements e of $\mathfrak{A}$ via the correspondence $E \sim e$ induced by*

$$E1 = e \tag{7.2}$$

and

$$ef = Ef. \tag{7.3}$$

Proof. Given any idempotent $e = e^2$ in $\mathfrak{A}$, Lemma 2 implies $e \geqslant 0$. Since $1 - e$ is also idempotent we have $0 \leqslant e \leqslant 1$. Thus E defined by (7.3) is a projector. Conversely, every projector E defines an idempotent e via (7.2) which, by Lemma 7, satisfies (7.3). Clearly, $I \sim 1$ and for $A \sim a$ and $B \sim b$, $AB \sim ab$.

The next theorem follows directly from Theorem 1 via Lemmas 6 and 8.

THEOREM 2. *Let $\mathfrak{R}$ be a Boolean ring of idempotents in a topological lattice algebra $\mathfrak{R}$. Then $\mathfrak{R}$ generates $\mathfrak{A}$ if, and only if, $\mathfrak{R}$ separates every pair f and g in $\mathfrak{A}$ for which $fg = 0$.*

8. Subalgebras dense in $\mathfrak{A}$. A *subalgebra* of $\mathfrak{A}$ is a linear subspace which is closed under multiplication.

THEOREM 3. *Let $\mathfrak{R}$ be a subalgebra of a topological lattice algebra $\mathfrak{A}$. Then $\mathfrak{R}$ is dense in $\mathfrak{A}$ if, and only if, $\mathfrak{R}$ separates every pair f and g in $\mathfrak{A}$ for which $fg = 0$.*

To prove this theorem we need another lemma.

LEMMA 9. *The following conditions are equivalent:*

(i) *$\mathfrak{R}$ separates f and g whenever $fg = 0$.*

(ii) *The set of all contractors in the closure of $\mathfrak{R}$ separates f and g whenever $f \wedge g = 0$.*

Proof. We first show that (i) implies that the closure of $\mathfrak{R}$ is a lattice and contains the unit 1. Now the trivial identity $f - g = (f - f \wedge g) - (g - f \wedge g)$ gives, in view of (2.2),

$$(f - g)^+ = f - f \wedge g. \tag{8.1}$$

Thus, to show that the closure of $\mathfrak{R}$ is a lattice we need only show that it contains f^+ whenever it contains f. Since $f^+f^- = 0$, (i) implies the existence of a net h_t in $\mathfrak{R}$ such that $h_t f^+ \to f^+$ and $h_t f^- \to 0$. Hence $h_t f \to f^+$. Since $h_t f$ is in the closure of $\mathfrak{R}$, so is f^+. That 1 is in the closure of $\mathfrak{R}$ follows from (i), since $\mathfrak{R}$ must separate 1 from 0.

Given $f \wedge g = 0$, (i) gives a net h_t in $\mathfrak{R}$ with $h_t f \to 0$ and $h_t g \to g$. Let $p_t = |h_t| \wedge 1$ which is in the closure of $\mathfrak{R}$ by the preceding arguments. Clearly, p_t is a net of contractors: $0 \leqslant p_t \leqslant 1$. Moreover, since $0 \leqslant p_t \leqslant |h_t|$, $0 \leqslant p_t f \leqslant |h_t f|$ using Lemma 5. So by (3.1), $p_t f \to 0$. From the identity (8.1) we

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have $1 - p_t = (1 - |h_t|)^+$. So $(1 - p_t)g \leqslant |(1 - |h_t|)g| \leqslant |g - h_t g|$. Hence, $p_t g \to g$. Thus (i) implies (ii).

Given (ii) and $fg = 0$, $|f| \wedge |g| = 0$ by Lemma 6. So there exists a net of contractors p_t in the closure of $\mathfrak{R}$ separating $|g|$ from $|f|$: $p_t|f| \to 0$ and $p_t|g| \to |g|$ with $0 \leqslant p_t \leqslant 1$. Using Lemma 5 we have $p_t f \to 0$ and $(1 - p_t)g \to 0$. Since p_t is in the closure of $\mathfrak{R}$ there exists h_t in $\mathfrak{R}$ such that $p_t - h_t \to 0$. Hence $|h_t f| \leqslant |h_t - p_t|\,|f| + p_t|f|$ and $|(1 - h_t)g| \leqslant (1 - p_t)|g| + |p_t - h_t|\,|g|$. So $h_t f \to 0$ and $h_t g \to g$, giving (i).

Proof of Theorem 3. Given (i) we have (ii) by Lemma 9. By the Fundamental Lemma, (ii) implies $\mathfrak{R}$ is dense in $\mathfrak{A}$. Conversely, we shall show that if the closure of $\mathfrak{R}$ is $\mathfrak{A}$, then (ii), and hence (i) holds.

Given $f \wedge g = 0$ let

$$p_n = n\left(g \wedge \frac{1}{n} 1\right).$$

We contend that p_n is a sequence of contractors separating g from f. Clearly, $0 \leqslant p_n \leqslant 1$. Since $0 \leqslant p_n \leqslant ng$, $0 \leqslant p_n f \leqslant nfg$. Now $fg = 0$ by Lemma 6, so $p_n f = 0$.

Noting that

$$1 - p_n = n\left(\frac{1}{n} 1 - g \wedge \frac{1}{n} 1\right),$$

apply (2.2) to $1/n\ 1$ and g to obtain, via Lemma 6,

$$(1 - p_n)\left(g - \frac{1}{n} p_n\right) = 0.$$

So

$$(1 - p_n)g = \frac{1}{n} p_n(1 - p_n).$$

Hence,

$$0 \leqslant (1 - p_n)g \leqslant \frac{1}{n} 1.$$

So $(1 - p_n)g \to 0$.

9. Absolutely continuous set functions. Let u be a bounded, non-negative, finitely additive measure on a Boolean algebra $\mathfrak{B}$ with unit I. The Banach lattice $\mathfrak{B}$ dealt with in **(3)** and **(6)** consists of all finitely additive, real valued functions f on $\mathfrak{B}$ which are absolutely continuous with respect to u:

(9.1) $$f(E) \to 0 \quad \text{as} \quad u(E) \to 0.$$

The norm in $\mathfrak{B}$ is defined by

(9.2) $$||f|| = \sup f(E) - f(E')$$

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where the supremum is taken over all E in $\mathfrak{B}$. The partial ordering is induced by defining $f \geqslant 0$ whenever $f(E) \geqslant 0$ for all E in $\mathfrak{B}$. With this ordering

$$f \wedge g(A) = \inf f(EA) + g(E'A) \tag{9.3}$$

and

$$f \vee g(A) = \sup f(EA) + g(E'A) \tag{9.4}$$

taken over all E in $\mathfrak{B}$ **(1, 3, 4, 6)**. Since $|f| = f \vee -f$, (9.2) and (9.4) give

$$||f|| = |f|\ (I). \tag{9.5}$$

Every E in $\mathfrak{B}$ defines a projector E given by

$$Ef(A) = f(EA) \tag{9.6}$$

for all A in $\mathfrak{B}$. Thus $\mathfrak{B}$, modulo the ideal of all E with $u(E) = 0$, is isomorphic to a subalgebra of the Boolean algebra of all projectors on $\mathfrak{B}$.

Now (9.1) implies that u is a limit unit. To prove this let $f \geqslant 0$ and $f_n = f \wedge nu$. The sequence $(f - f_n)(I)$ is decreasing, hence converges to some limit λ. In view of (9.5) we need only show $\lambda = 0$. By (9.3), $f_n(I) = \inf f(E') + nu(E)$. Hence we may choose a sequence E_n such that

$$f_n(I) \leqslant f(E_n') + n\,u(E_n) \leqslant f_n(I) + \frac{1}{n}.$$

Multiplying by -1 and adding $f(I)$ we obtain

$$(f - f_n)(I) - \frac{1}{n} \leqslant f(E_n) - n\,u(E_n) \leqslant (f - f_n)(I).$$

Hence $f(E_n) - n\,u(E_n)$ converges to λ. Now $0 \leqslant f(E_n) \leqslant f(I)$ and $0 \leqslant \lambda \leqslant f(I)$ while n increases without bound. Hence $u(E_n)$ must converge to 0. By (9.1), $f(E_n)$ does likewise. So $\lambda = -\lim n\,u(E_n)$. Thus $\lambda \leqslant 0$. But $\lambda \geqslant 0$. So $\lambda = 0$.

Given $f \wedge g = 0$ there exists, via (9.3) with $A = I$, a sequence E_n in $\mathfrak{B}$ such that

$$f(E_n) + g(E_n') \to 0. \tag{9.7}$$

By (9.6) and (9.5), $||E_n f|| = f(E_n)$ and $||E_n' g|| = g(E_n')$. So (9.7) implies that $\mathfrak{B}$ separates f and g. By Theorem 1, $\mathfrak{B}u$ generates $\mathfrak{B}$. That is, the "step functions" are dense in $\mathfrak{B}$. (See **(3)** and **(6)**.) As was pointed out by Bochner **(3)**, this gives the Radon-Nikodym theorem **(11)**.

10. The finitely additive integral. Let $\mathfrak{B}$ be a Boolean algebra of subsets E of a set I with I as unit. Let u be a bounded, non-negative, finitely additive measure on $\mathfrak{B}$. A *partition* Δ is a finite class of disjoint sets in $\mathfrak{B}$ whose union is I. The partitions are ordered by defining $\Delta' \geqslant \Delta$ whenever Δ' is a refinement of Δ. For $f(x)$ real-valued on the domain I and $\Delta = \{E_1, \ldots, E_n\}$ any partition, let

$$s(\Delta) = \sum f(x_k) u(E_k) \tag{10.1}$$

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where x_k is any point in E_k and k ranges through $1, \ldots, n$. In general, $s(\Delta)$ is a many-valued function of Δ, a particular value depending on the choice of x_k in E_k. If $\lim s(\Delta)$ exists (in the Moore-Smith sense (**8**)) uniformly for all such choices, then f is said to be *integrable.*

Introducing the upper and lower Darboux sums

$$\bar{s}(\Delta) = \sum \sup f(x_k)u(E_k) \tag{10.2}$$

and

$$\underline{s}(\Delta) = \sum \inf f(x_k)u(E_k),$$

let $S(\Delta, f) = \bar{s}(\Delta) - \underline{s}(\Delta)$. In (10.2) we assume $\infty \,.\, 0 = 0$. Since $\lim\sup s(\Delta) = \lim \bar{s}(\Delta)$ and $\lim\inf s(\Delta) = \lim \underline{s}(\Delta)$, f is integrable if, and only if, $\lim S(\Delta, f) = 0$. Note that for any f, $S(\Delta, f)$ is a decreasing function of Δ. Since $S(\Delta, \alpha f + \beta g) \leqslant |\alpha|\, S(\Delta, f) + |\beta|\, S(\Delta, g)$ the integrable functions form a vector space. Since $S(\Delta, 1) = 0$ the constant functions are integrable. That products of integrable functions are integrable follows from the inequality $S(\Delta, fg) \leqslant M(f)\, S(\Delta, g) + M(g)\, S(\Delta, f)$ where $M(f)$ is the supremum of $|f(x)|$ for x restricted to those sets in Δ which are not of measure zero. That $|f|$ is integrable whenever f is integrable follows from the inequality $S(\Delta, |f|) \leqslant S(\Delta, f)$. Given $|f(x) - g(x)| < \epsilon$ for all x we have $S(\Delta, f) \leqslant S(\Delta, g) + S(\Delta, f - g) \leqslant S(\Delta, g) + 2\epsilon u(I)$. So a uniform limit of integrable functions is integrable. Since an integrable function is bounded except on a set of measure zero, we shall consider only bounded integrable functions. These form a topological lattice algebra under uniform convergence with the usual ordering and algebraic operations. Using Theorem 2, we shall show that this algebra is generated by its idempotents. Thus, it suffices to show that for f any bounded integrable function, f^- can be separated from f^+ by integrable idempotents.

Consider any $\epsilon > 0$. Choose a sequence Δ_n of partitions such that $\Delta_{n+1} \geqslant \Delta_n$ and $S(\Delta_n, f) \to 0$, which is possible because f is integrable. Let C_n be the union of those sets E, belonging to the partition Δ_n, for which there exist x and y in E with $f^+(x) \geqslant \epsilon$ and $f^-(y) \geqslant \epsilon$. By induction, starting with $A_0 = B_0 = \phi$ and $C_0 = I$, let A_n be the union of A_{n-1} and those sets E in Δ_n which are contained in C_{n-1} and have $f^+(x) < \epsilon$ for all x in E. Let B_n be the union of B_{n-1} and those sets E in Δ_n which are contained in C_{n-1}, have $f^-(x) < \epsilon$ for all x in E, and have $f^+(y) \geqslant \epsilon$ for some y in E. Then A_{n-1} is a subset of A_n, B_{n-1} of B_n, and C_n of C_{n-1}. Since $2\epsilon u(C_n) \leqslant S(\Delta_n, f)$, we have $u(C_n) \to 0$. Let $A = \lim A_n$ and $C = \lim C_n$. Let E be the union of A with the set of all points x in C for which $f^+(x) = 0$. Let e be the indicator of E:

$$e(x) = \begin{cases} 1 \text{ for } x \text{ in } E \\ 0 \text{ for } x \text{ in } E'. \end{cases} \tag{10.3}$$

Since A_n is contained in E and B_n is contained in E', $e(x)$ equals 1 for x in A_n and 0 for x in B_n. Hence, $S(\Delta_n, e) \leqslant u(C_n)$ which converges to 0. So e

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is integrable. For x in E either x is in C with $f^+(x) = 0$ or x belongs to some A_n, implying $f^+(x) < \epsilon$. Clearly then $ef^+ < \epsilon 1$. For x in E', either x is in C with $f^+(x) > 0$, hence $f^-(x) = 0$, or x is in some B_n, implying $f^-(x) < \epsilon$. So $(1 - e)f^- < \epsilon 1$.

Thus, by Theorem 2, the algebra of bounded integrable functions is generated under uniform convergence by its idempotents.

A similar result can be obtained for the almost everywhere continuous functions on a closed interval, using Theorem 2. Combining these two results, we get Lebesgue's characterization of the Riemann integrable functions **(7)**.

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