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DISTRIBUTION OF UNITARILY k-FREE INTEGERS

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1. Introduction

Let k be a fixed integer $\geqq 2$. A positive integer n is called *unitarily k-free*, if the multiplicity of each prime divisor of n is not a multiple of k; or equivalently, if n is not divisible unitarily by the kth power of any integer > 1. By a unitary divisor, we mean as usual a divisor $d > 0$ of n such that $(d,(n/d)) = 1$. The integer 1 is also considered to be unitarily k-free. These integers were first defined by Cohen (1961; § 1). Let Q_k^* denote the set of unitarily k-free integers. When $k = 2$, the set Q_2^* coincides with the set Q^* of exponentially odd integers (that is, integers in whose canonical representation each exponent is odd) discussed by Cohen himself in an earlier paper (1961; § 1 and § 6). Let x denote a real variable $\geqq 1$ and let $Q_k^*(x)$ denote the number of unitarily k-free integers $\leqq x$. Cohen (1961; Theorem 3.2) established by purely elementary methods that

$$Q_k^*(x) = \alpha_k x + O(x^{1/k}\log x), \tag{1.1}$$

where

$$\alpha_k = \zeta(k) \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right), \tag{1.2}$$

the product being extended over all primes p and $\zeta(k)$ denotes the Riemann Zeta function. In the same paper Cohen (1961; Theorem 4.2) improved the order estimate of the error term in (1.1) to $O(x^{1/k})$, by making use of the properties of real Dirichlet series. Later, he (Cohen; 1964) proved the same result by purely elementary methods eliminating the use of Dirichlet series.

The object of the present paper is to further improve the order estimate of the error term in (1.1). In fact, we prove that

$$\Delta_k^*(x) = Q_k^*(x) - \alpha_k x = O(x^{1/k}\exp\{-A\log^{3/5}x(\log\log x)^{-1/5}\}),$$

wehere A is a positive constant. Further, on the assumption of the Riemann hypothesis, we prove that $\Delta_k^*(x) = O(x^{2/(2k+1)} \exp\{A \log x (\log\log x)^{-1}\})$, where A is a positive constant.

2. Preliminaries

In this section we introduce some notation and then prove some lemmas which are needed in our present discussion.

Let $\mu(n)$ and $\phi(n)$ denote respectively the Möbius function and the Euler totient function. Let $\phi(x, n)$ denote the number of positive integers $\leqq x$ which are prime to n. Let $\mu^*(n)$ denote the unitary analogue of the Möbius μ-function defined by $\mu^*(n) = (-1)^{\omega(n)}$ where $\omega(n)$ is the number of distinct prime factors of $n > 1$, $\omega(1) = 0$. Let $\sigma_s^*(n)$ denote the sum of all the sth powers of the square-free divisors of n. Let $a(n) = \prod_{i=1}^r (1 - \beta_i)$, if $n = \prod_{i=1}^r p_i^{\beta_i}$ is the canonical representation of $n > 1$ and $a(1) = 1$. Then we have the following:

LEMMA 2.1. (Cohen (1960; Lemma 3.4)). *For* $x \geqq 1$,

$$\phi(x, n) = x\frac{\phi(n)}{n} + O(\theta(n)), \tag{2.1}$$

uniformly in x and n, where $\theta(n) = \sigma_0(n)$, the number of square-free divisors of n.

REMARK 2.1. *It is sometimes convenient to replace $\theta(n)$ in (2.1) by $\tau(n)$, where $\tau(n)$ is the number of divisors of n. Clearly, $\theta(n) \leqq \tau(n)$.*

LEMMA 2.2. (Cohen (1961; Lemma 3.5)) $r \to \infty$,

$$\alpha_k = \sum_{n=1}^{\infty} \frac{\mu^*(n)\phi(n)}{n^{k+1}} = \zeta(k) \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right). \tag{2.2}$$

LEMMA 2.3. $\mu^*(n) = \sum_{d\delta = n} a(d)\mu(\delta)$.

PROOF. Since $a(n)$ and $\mu(n)$ are both multiplicative, it follows that $\sum_{d\delta = n} a(d)\mu(\delta)$ is multiplicative. Also, $\mu^*(n)$ is multiplicative. Hence it is enough if we verify the identity for $n = p^\alpha$, a prime power. Now,

$$\begin{aligned}\sum_{d\delta = p^\alpha} a(d)\mu(\delta) &= a(p^\alpha)\mu(1) + a(p^{\alpha-1})\mu(p) \\ &= (1-\alpha) - (2-\alpha) = -1 = \mu^*(p^\alpha).\end{aligned}$$

Hence the lemma follows.

LEMMA 2.4. $\sum_{n \leqq x} |a(n)| = O(x^{1/2})$.

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PROOF. By Euler's identity (Hardy and Wright (1965; Theorem 280)), we have

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

for $s > 1$. Hence

$$\frac{\zeta(2s)}{\zeta(4s)} \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \left\{\prod_p \left(1 + \frac{1}{p^{2s}}\right)\right\} \left\{\prod_p \left(1 - \frac{1}{p^{2s}} - \frac{2}{p^{3s}} - \frac{3}{p^{4s}} - \cdots\right)\right\}$$

(2.3)
$$\begin{cases} = \prod_p \left(1 - \dfrac{2}{p^{3s}} - \dfrac{4}{p^{4s}} - \dfrac{6}{p^{5s}} - \cdots\right) \\ = \sum_{n=1}^{\infty} \dfrac{b(n)}{n^s}, \text{ say.} \end{cases}$$

Since

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} \quad \textit{(Hardy and Wright (1965; Theorem 300))},$$

where $\lambda(n)$ denotes Liouville's function, we have by (2.3) for $s > 1$,

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^{2s}} \sum_{n=1}^{\infty} \frac{b(n)}{n^s}.$$

Hence

(2.4)
$$a(n) = \sum_{d^2\delta = n} \lambda(d) b(\delta).$$

By (2.3), we see that the abscissa of absolute convergence α of

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} \text{ is given by } \alpha = \frac{1}{3}.$$

Hence, we have

$$\sum_{n \leq x} \frac{|b(n)|}{n^{1/3+\varepsilon}} = O(1) \quad \text{for every } \varepsilon > 0.$$

Now, by partial summation, we have

$$\begin{aligned} \sum_{n \leq x} |b(n)| &= \sum_{n \leq x} \frac{|b(n)|}{n^{1/3+\varepsilon}} \cdot n^{1/3+\varepsilon} \\ &= O(x^{1/3+\varepsilon}) + O\left(\int_1^x t^{-2/3+\varepsilon} dt\right) \\ &= O(x^{1/3+\varepsilon}) + O(x^{1/3+\varepsilon}) \\ &= O(x^{1/3+\varepsilon}). \end{aligned}$$

In particular, taking $\varepsilon = 1/15$, we have

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$$\sum_{n \leqq x} |b(n)| = O(x^{2/5}). \tag{2.5}$$

Hence by (2.4) and (2.5), we have

$$\begin{aligned}
\sum_{n \leqq x} |a(n)| &= \sum_{n \leqq x} \sum_{d^2\delta = n} |b(\delta)| = \sum_{d^2\delta \leqq x} |b(\delta)| \\
&= \sum_{d \leqq \sqrt{x}} \sum_{\delta \leqq x/d^2} |b(\delta)| = O\left(\sum_{d \leqq \sqrt{x}} \left(\frac{x}{d^2}\right)^{2/5} \right) \\
&= O(x^{2/5} \sum_{d \leqq \sqrt{x}} d^{-4/5}) = O(x^{2/5}(x^{1/2})^{1/5}) \\
&= O(x^{1/2}).
\end{aligned}$$

LEMMA 2.5. *For every* $0 < \eta < 1/2$, *we have*

$$\sum_{n \leqq x} \frac{|a(n)|}{n^{1-\eta}} = O(1), \qquad x \to \infty \tag{2.6}$$

and

$$\sum_{n \leqq x} \frac{|a(n)|}{n^{1/2}} = O(\log x), \qquad x \to \infty. \tag{2.7}$$

PROOF. By partial summation and Lemma 2.4, we have

$$\begin{aligned}
\sum_{n \leqq x} \frac{|a(n)|}{n^{1-\eta}} &= O\left(\frac{x^{1/2}}{x^{1-\eta}}\right) + O\left(\int_1^x \frac{t^{1/2}}{t^{2-\eta}} dt\right) \\
&= O(x^{-1/2+\eta}) + O\left(\int_1^x t^{-3/2+\eta} dt\right).
\end{aligned}$$

Now, (2.6) follows since $0 < \eta < 1/2$.

Also, by partial summation, we see that

$$\begin{aligned}
\sum_{n \leqq x} \frac{|a(n)|}{n^{1/2}} &= O(1) + O\left(\int_1^x \frac{dt}{t}\right) \\
&= O(1) + O(\log x),
\end{aligned}$$

so that (2.7) follows.

LEMMA 2.6. *For* $s > 0$ *and* $0 \leqq u < 1$,

$$\sum_{n \leqq x} \frac{\sigma^*_{-s}(n)}{n^u} = O(x^{1-u}), \qquad x \to \infty. \tag{2.8}$$

PROOF. We have

$$\sum_{n \leqq x} \frac{\sigma^*_{-s}(n)}{n^u} = \sum_{n \leqq x} \frac{1}{n^u} \sum_{d\delta = n} \frac{\mu^2(d)}{d^s} = \sum_{d\delta \leqq x} \frac{\mu^2(d)}{d^{u+s}\delta^u}$$
$$= \sum_{d \leqq x} \frac{\mu^2(d)}{d^{u+s}} \sum_{\delta \leqq (x/d)} \frac{1}{\delta^u} = O\left(\sum_{d \leqq x} \frac{1}{d^{u+s}} \left(\frac{x}{d}\right)^{1-u}\right)$$
$$= O\left(x^{1-u} \sum_{d \leqq x} \frac{1}{d^{1+s}}\right) = O(x^{1-u}).$$

LEMMA 2.7. (Suryanarayana and Siva Rama Prasad (1973; Lemma 3.5)). *For $x \geqq 3$ and for every $\varepsilon > 0$,*

$$M_n(x) \equiv \sum_{\substack{m \leqq x \\ (m,n)=1}} \mu(m) = O(\sigma^*_{-1+\varepsilon}(n) x \delta(x)), \tag{2.9}$$

uniformly in x and n, where

$$\delta(x) = \exp\{-A \log^{3/5} x (\log\log x)^{-1/5}\}, \tag{2.10}$$

A being a positive constant.

LEMMA 2.8. *For $x \geqq 3$ and for every $\varepsilon > 0$,*

$$M^*_n(x) \equiv \sum_{\substack{m \leqq x \\ (m,n)=1}} \mu^*(m) = O(\sigma^*_{-1+\varepsilon}(n) x \delta(x)), \tag{2.11}$$

uniformly in x and n, where $\delta(x)$ is given by (2.10).

PROOF. We have by Lemma 2.3,

$$M^*_n(x) = \sum_{\substack{m \leqq x \\ (m,n)=1}} \sum_{d\delta = m} a(d)\mu(\delta) = \sum_{\substack{d\delta \leqq x \\ (d,n)=(\delta,n)=1}} a(d)\mu(\delta)$$
$$= \sum_{\substack{d \leqq x \\ (d,n)=1}} a(d) \sum_{\substack{\delta \leqq x/d \\ (\delta,n)=1}} \mu(\delta) = \sum_{\substack{m \leqq x \\ (m,n)=1}} a(m) M_n\left(\frac{x}{m}\right). \tag{2.12}$$

Hence by Lemma 2.7,

$$M^*_n(x) = O\left(\sum_{\substack{m \leqq x \\ (m,n)=1}} |a(m)|\, \sigma^*_{-1+\varepsilon}(n) \frac{x}{m} \delta\left(\frac{x}{m}\right)\right)$$
$$= O\left(\sigma^*_{-1+\varepsilon}(n) \sum_{m \leqq x} |a(m)| \left(\frac{x}{m}\right)^{1-\eta} \left(\frac{x}{m}\right)^{\eta} \delta\left(\frac{x}{m}\right)\right),$$

where $0 < \eta < 1/2$. Since $x^\eta \delta(x)$ is monotonic increasing for large x, we have by (2.6),

$$M^*_n(x) = O\left(\sigma^*_{-1+\varepsilon}(n) x \delta(x) \sum_{m \leqq x} \frac{|a(m)|}{m^{1-\eta}}\right)$$
$$= O(\sigma^*_{-1+\varepsilon}(n) x \delta(x)).$$

LEMMA 2.9. *For $x \geqq 3$ and for every $\varepsilon > 0$,*

(2.13)
$$\sum_{\substack{m \leq x \\ (m,n)=1}} \mu^*(m)m = O(\sigma^*_{-1+\varepsilon}(n)x^2\delta(x)),$$

uniformly, where $\delta(x)$ *is given by* (2.10).

PROOF. By partial summation and Lemma 2.8, we have

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \mu^*(m)m = M^*_n(x)x - \int_n^x M^*_n(t)dt$$

$$= O(\sigma^*_{-1+\varepsilon}(n)x^2\delta(x)) + O\left(\int_1^x \sigma^*_{-1+\varepsilon}(n)t\delta(t)dt\right).$$

Since $x\delta(x)$ is monotonic increasing for large x, we have

$$\int_1^x t\delta(t)dt \leqq x\delta(x) \int_1^x dt = O(x^2\delta(x)).$$

Hence the lemma follows.

LEMMA 2.10. (Estermann (1962; Theorem 41)). *If* $f(n)$ *is multiplicative and* $\prod_p \{\sum_{m=0}^{\infty} |f(p^m)|\}$ *converges, then* $\sum_{n=1}^{\infty} f(n)$ *converges absolutely and*

$$\sum_{n=1}^{\infty} f(n) = \prod_p \left\{\sum_{m=0}^{\infty} f(p^m)\right\}.$$

LEMMA 2.11. *For* $s > 1/2$,

(2.14)
$$\sum_{n \leq x} \frac{\theta(n)}{\gamma(n)n^s} = O(1),$$

where $\theta(n)$ *is as in Lemma* 2.1 *and* $\gamma(n)$ *is the core* (*the maximal square-free divisor*) *of* n.

PROOF. We have for $s > 1/2$,

$$\prod_p \left\{\sum_{m=0}^{\infty} \frac{\theta(p^m)}{\gamma(p^m)\,p^{ms}}\right\} = \prod_p \left(1 + \frac{2}{p^{s+1}} + \frac{2}{p^{2s+1}} + \cdots\right)$$

$$= \prod_p \left\{\left(1 + \frac{1}{p^{2s}} + \frac{1}{p^{4s}} + \cdots\right)\left(1 + \frac{2}{p^{s+1}} - \frac{1}{p^{2s}} + \frac{2}{p^{2s+1}}\right)\right\}$$

$$= \zeta(2s) \prod_p \left(1 + \frac{2}{p^{s+1}} - \frac{1}{p^{2s}} + \frac{2}{p^{2s+1}}\right),$$

where the infinite product on the right is convergent. Hence by Lemma 2.10, it follows that $\sum_{n=1}^{\infty} (\theta(n))/(\gamma(n)n^s)$ is absolutely convergent for $s > 1/2$.

Hence the lemma follows.

LEMMA 2.12.

$$\phi(n) = n \sum_{\substack{d\delta = n \\ (d,\delta)=1}} \frac{\mu^*(d)}{\gamma(d)}.$$

PROOF. Since $\mu^*(n)$ and $\gamma(n)$ are multiplicative, it follows (Cohen (1960; Theorem 61)), that

$$\sum_{\substack{d\delta = n \\ (d,\delta)=1}} \frac{\mu^*(d)}{\gamma(d)}$$

is multiplicative. Also, $\phi(n)$ is multiplicative. Hence, it is enough if we verify the identity for $n = p^\alpha$, a prime power. Now,

$$\begin{aligned} p^\alpha \sum_{\substack{d\delta = p^\alpha \\ (d,\delta)=1}} \frac{\mu^*(d)}{\gamma(d)} &= p^\alpha \left\{ \frac{\mu^*(1)}{\gamma(1)} + \frac{\mu^*(p^\alpha)}{\gamma(p^\alpha)} \right\} \\ &= p^\alpha \left(1 - \frac{1}{p}\right) = \phi(p^\alpha). \end{aligned}$$

LEMMA 2.13. For $x \geqq 3$,

$$N^*(x) \equiv \sum_{n \leqq x} \mu^*(n)\phi(n) = O(x^2\delta(x)), \tag{2.15}$$

where $\delta(x)$ is given by (2.10).

PROOF. We have by Lemmas 2.12 and 2.9,

$$\begin{aligned} \sum_{n \leqq x} \mu^*(n)\phi(n) &= \sum_{n \leqq x} \mu^*(n)n \sum_{\substack{d\delta = n \\ (d,\delta)=1}} \frac{\mu^*(d)}{\gamma(d)} = \sum_{\substack{d\delta \leqq x \\ (d,\delta)=1}} \frac{\mu^*(d)\mu^*(d\delta)d\delta}{\gamma(d)} \\ &= \sum_{\substack{d\delta \leqq x \\ (d,\delta)=1}} \frac{\mu^*(\delta)d\delta}{\gamma(d)} = \sum_{d \leqq x} \frac{d}{\gamma(d)} \sum_{\substack{\delta \leqq x/d \\ (\delta,d)=1}} \mu^*(\delta)\delta \\ &= O\left(\sum_{n \leqq x} \frac{n}{\gamma(n)} \sigma^*_{-1+\varepsilon}(n) \left(\frac{x}{n}\right)^2 \delta\left(\frac{x}{n}\right)\right) \\ &= O\left(x \sum_{n \leqq x} \frac{\sigma^*_{-1+\varepsilon}(n)}{\gamma(n)} \left(\frac{x}{n}\right)^{1-\eta} \left(\frac{x}{n}\right)^{\eta} \delta\left(\frac{x}{n}\right)\right), \end{aligned}$$

where $0 < \eta < 1/2$. Since $x^\eta\delta(x)$ is monotonic increasing for large x and $\sigma^*_{-1+\varepsilon}(n) \leqq \theta(n)$ for $0 < \varepsilon < 1$, we have by Lemma 2.11,

$$N^*(x) = O\left(x^2\delta(x) \sum_{n \leqq x} \frac{\theta(n)}{\gamma(n)n^{1-\eta}}\right) = O(x^2\delta(x)).$$

LEMMA 2.14. *For $x \geqq 3$ and $s > 2$,*

$$\sum_{n > x} \frac{\mu^*(n)\phi(n)}{n^s} = O\left(\frac{\delta(x)}{x^{s-2}}\right), \tag{2.16}$$

where $\delta(x)$ is given by (2.10).

PROOF. Putting $f(n) = (1/n^s)$, we see that $f(n+1) - f(n) = O(1/n^{s+1})$. Hence by partial summation and Lemma 2.13, we have

$$\begin{aligned}\sum_{n>x} \frac{\mu^*(n)\phi(n)}{n^s} &= \sum_{n>x} \mu^*(n)\phi(n)f(n)\\ &= -N^*(x)f([x]+1) - \sum_{n>x} N^*(n)\{f(n+1)-f(n)\}\\ &= O\left(x^2\delta(x)\cdot\frac{1}{x^s}\right) + O\left(\sum_{n>x} n^2\delta(n)\cdot\frac{1}{n^{s+1}}\right)\\ &= O\left(\frac{\delta(x)}{x^{s-2}}\right) + O\left(\delta(x)\sum_{n>x}\frac{1}{n^{s-1}}\right),\end{aligned}$$

Since $\delta(x)$ is monotonic decreasing. Also, since $s > 2$, we have

$$\sum_{n>x}\frac{1}{n^{s-1}} = O\left(\frac{1}{x^{s-2}}\right).$$

Hence the lemma follows.

LEMMA 2.15. (Suryanarayana and Siva Rama Prasad (1973; Lemma 5.2)). *If the Riemann hypothesis is true, then for $x \geqq 3$ and every $\varepsilon > 0$,*

$$M_n(x) \equiv \sum_{\substack{m\leqq x\\(m,n)=1}} \mu(m) = O(\sigma^*_{-1/2+\varepsilon}(n)x^{1/2}\omega(x)), \tag{2.17}$$

uniformly in x and n, where

$$\omega(x) = \exp\{A\log x(\log\log x)^{-1}\}, \tag{2.18}$$

A being a positive constant.

LEMMA 2.16. *If the Riemann hypothesis is true, then for $x \geqq 3$ and every $\varepsilon > 0$,*

$$M_n^*(x) \equiv \sum_{\substack{m\leqq x\\(m,n)=1}} \mu^*(m) = O(\sigma^*_{-1/2+\varepsilon}(n)x^{1/2}\omega(x)\log x), \tag{2.19}$$

uniformly in x and n, where $\omega(x)$ is given by (2.18).

PROOF. By (2.12) and (2.17),

$$\begin{aligned}M_n^*(x) &= \sum_{\substack{m\leqq x\\(m,n)=1}} a(m)M_n\left(\frac{x}{m}\right)\\ &= O\left(\sum_{\substack{m\leqq x\\(m,n)=1}} |a(m)|\,\sigma^*_{-1/2+\varepsilon}(n)\left(\frac{x}{m}\right)^{1/2}\omega\left(\frac{x}{m}\right)\right)\\ &= O\left(\sigma^*_{-1/2+\varepsilon}(n)x^{1/2}\omega(x)\sum_{m\leqq x}\frac{|a(m)|}{m^{1/2}}\right),\end{aligned}$$

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since $\omega(x)$ is monotonic increasing. Now, the lemma follows by (2.7).

LEMMA 2.17. *If the Riemann hypothesis is true, then for* $x \geqq 3$ *and every* $\varepsilon > 0$,

$$\sum_{\substack{m \leqq x \\ (m,n)=1}} \mu^*(m)m = O(\sigma^*_{-1/2+\varepsilon}(n)x^{3/2}\,\omega(x)\log x), \tag{2.20}$$

where $\omega(x)$ *is given by* (2.18).

PROOF. Following the same procedure adopted in proving Lemma 2.9 and making use of Lemma 2.16 instead of Lemma 2.8, we get this lemma.

LEMMA 2.18. *If the Riemann hypothesis is true, then for* $x \geqq 3$ *and every* $\varepsilon > 0$,

$$N^*(x) \equiv \sum_{n \leqq x} \mu^*(n)\phi(n) = O(x^{3/2}\omega(x)\log x), \tag{2.21}$$

where $\omega(x)$ *is given by* (2.18).

PROOF. Following the same procedure adopted in proving Lemma 2.13 and making use of Lemma 2.17 instead of Lemma 2.9, we get this lemma.

LEMMA 2.19. *If the Riemann hypothesis is true, then for* $x \geqq 3$ *and* $s > 2$,

$$\sum_{n > x} \frac{\mu^*(n)\phi(n)}{n^s} = O\left(\frac{\omega(x)\log x}{x^{s-3/2}}\right). \tag{2.22}$$

where $\omega(x)$ *is given by* (2.18).

PROOF. Following the same procedure adopted in proving Lemma 2.14 and making use of Lemma 2.18 instead of Lemma 2.13, we get this lemma.

3. Main results

We are now in a position to prove the following:

THEOREM 3.1. *For* $x \geqq 3$,

$$Q_k^*(x) = \alpha_k x + O(x^{1/k}\delta(x)), \tag{3.1}$$

where α_k *is given by* (2.2) *and* $\delta(x)$ *is given by* (2.10).

PROOF. Let $q_k^*(n)$ denote the characteristic function of the set Q_k^* of unitarily k-free integers, that is, $q_k{}^*(n) = 1$ or 0 according as n is or is not a member of Q_k^*. We have (Cohen (1971), (3.7) and (3.1) as $r \to \infty$)

$$q_k^*(n) = \sum_{\substack{d^k\delta = n \\ (d\ \delta)=1}} \mu^*(d).$$

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Hence we have

$$Q_k^*(x) = \sum_{n \leqq x} q_k^*(n) = \sum_{\substack{d^k\delta \leqq x \\ (d,\delta)=1}} \mu^*(d).$$

Let $z = x^{1/k}$ and $0 < \rho = \rho(x) < 1$, where $\rho(x)$ will be suitably chosen later. If $d^k\delta \leqq x$, then both $d > \rho z$ and $\delta > \rho^{-k}$ can not simultaneously hold good, and so

$$Q_k^*(x) = \sum_{\substack{d^k\delta \leqq x \\ d \leqq \rho z \\ (d,\delta)=1}} \mu^*(d) + \sum_{\substack{d^k\delta \leqq x \\ d \leqq \rho^{-k} \\ (d,\delta)=1}} \mu^*(d) - \sum_{\substack{d \leqq \rho z \\ \delta \leqq \rho^{-k} \\ (d,\delta)=1}} \mu^*(d)$$

$$(3.2) \qquad = S_1 + S_2 - S_3, \text{ say.}$$

By Lemma 2.1 and Remark 2.1, we have

$$\begin{aligned} S_1 &= \sum_{d \leqq \rho z} \mu^*(d) \sum_{\substack{\delta \leqq (x/d^k) \\ (\delta,d)=1}} 1 = \sum_{d \leqq \rho z} \mu^*(d)\phi\left(\frac{x}{d^k}, d\right) \\ &= \sum_{d \leqq \rho z} \mu^*(d)\left\{\frac{x}{d^k}\frac{\phi(d)}{d} + O(\tau(d))\right\} \\ &= x \sum_{n \leqq \rho z} \frac{\mu^*(n)\phi(n)}{n^{k+1}} + O\left(\sum_{n \leqq \rho z} \tau(n)\right) \\ &= x \sum_{n=1}^{\infty} \frac{\mu^*(n)\phi(n)}{n^{k+1}} - x \sum_{n > \rho z} \frac{\mu^*(n)\phi(n)}{n^{k+1}} + O(\rho z \log(\rho z)), \end{aligned}$$

since $\sum_{n \leqq x} \tau(n) = O(x \log x)$ (Hardy and Wright (1965; Theorem 320)).

Hence by Lemmas 2.2 and 2.14, we have

$$\begin{aligned} S_1 &= \alpha_k x + O\left(\frac{z^k\delta(\rho z)}{(\rho z)^{k-1}}\right) + O(\rho z \log(\rho z\ ) \\ (3.3) \qquad &= \alpha_k x + O(\rho^{1-k} z\delta(\rho z)) + O(\rho z \log z). \end{aligned}$$

We have by Lemma 2.8,

$$S_2 = \sum_{\delta \leqq \rho^{-k}} \sum_{\substack{d \leqq \sqrt[k]{(x/\delta)} \\ (d,\delta)=1}} \mu^*(d) = \sum_{\delta \leqq \rho^{-k}} M_\delta^* \left(\sqrt[k]{\frac{x}{\delta}}\right) = \sum_{n \leqq \rho^{-k}} M_n^*\left(\sqrt[k]{\frac{x}{n}}\right)$$

$$= O\left(\sum_{n \leqq \rho^{-k}} \sigma^*_{-1+\varepsilon}(n)\left(\sqrt[k]{\frac{x}{n}}\right)\delta\left(\sqrt[k]{\frac{x}{n}}\right)\right).$$

Since $\delta(x)$ is monotonic decreasing and $\sqrt[k]{x/n} \geqq \rho z$, we have $\delta(\sqrt[k]{x/n}) \leqq \delta(\rho z)$. Also, by Lemma 2.6,

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$$\sum_{n \leqq \rho^{-k}} \frac{\sigma^*_{-1+\varepsilon}(n)}{n^{1/k}} = O((\rho^{-k})^{1-1/k}) = O(\rho^{1-k}).$$

Hence

$$(3.4) \qquad S_2 = O(\rho^{1-k} z\delta(\rho z)).$$

Also, by Lemmas 2.8 and 2.6,

$$S_3 = \sum_{\delta \leqq \rho^{-k}} \sum_{\substack{d \leqq \rho z \\ (d,\delta)=1}} \mu^*(d) = \sum_{\delta \leqq \rho^{-k}} M^*_\delta(\rho z) = \sum_{n \leqq \rho^{-k}} M^*_n(\rho z)$$

$$(3.5) \quad = O\left(\sum_{n \leqq \rho^{-k}} \sigma^*_{-1+\varepsilon}(n)\rho z\delta(\rho z)\right) = O(\rho^{1-k} z\delta(\rho z)).$$

Hence, by (3.2), (3.3), (3.4) and (3.5), we have

$$(3.6) \qquad Q^*_k(x) = \alpha_k x + O(\rho^{1-k} z\delta(\rho z)) + O(\rho z \log z).$$

Now, we choose

$$(3.7) \qquad \rho = \rho(x) = \{\delta(x^{1/2k})\}^{1/k},$$

and write

$$(3.8) \qquad f(x) = \log^{3/5}(x^{1/2k})\{\log\log(x^{1/2k})\}^{-1/5}$$
$$= \left(\frac{1}{2k}\right)^{3/5} U^{3/5}(V - \log 2k)^{-1/5},$$

where $U = \log x$ and $V = \log\log x$.
(3.9) For $V \geqq 2\log 2k$, that is $U \geqq 4k^2$, $x \geqq \exp(4k^2)$, we have

$$V^{-1/5} \leqq (V - \log 2k)^{-1/5} \leqq \left(\frac{V}{2}\right)^{-1/5},$$

and therefore

$$(3.10) \qquad (1/2)k^{-3/5}U^{3/5}V^{-1/5} \leqq f(x) \leqq k^{-3/5}U^{3/5}V^{-1/5}.$$

(3.11) We assume without loss of generality that the constant A in (2.10) is less than unity.

By (3.7), (2.10) and (3.8), we have

$$(3.12) \qquad \rho = \exp\left\{-\frac{A}{k}f(x)\right\}.$$

By (3.9), we have $k^{-8/5}U^{3/5}V^{-1/5} \leqq (U/2k)$. Hence by (3.10), (3.11), (3.12) and the above

$$\rho \geqq \exp\{-Ak^{-8/5}U^{3/5}V^{-1/5}\} \geqq \exp\{-k^{-8/5}U^{3/5}V^{-1/5}\}$$

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$$\geqq \exp\left(-\frac{U}{2k}\right) = \exp\left(-\frac{\log x}{2k}\right),$$

so that $\rho \geqq x^{-1/2k}$.

Hence $\rho z \geqq x^{1/2k}$. Since $\delta(x)$ is monotonic decreasing, we have by (3.7), $\delta(\rho z) \leqq \delta(x^{1/2k}) = \rho^k$. Hence by (3.10) and (3.12), we have

$$\rho^{1-k}\delta(\rho z) \leqq \rho \leqq \exp\left\{-\frac{A}{2}k^{-8/5}U^{3/5}V^{-1/5}\right\}, \tag{3.13}$$

so that each of the first and second O-terms in (3.6) is

$$O\left(x^{1/k}\exp\left\{-\frac{A}{2}k^{-8/5}U^{3/5}V^{-1/5}\right\}\log x\right).$$

Hence, if $\Delta_k^*(x)$ denotes the error term in the asymptotic formula (3.6), then we have

$$\Delta_k^*(x) = O(x^{1/k}\exp\{-B\log^{3/5}x(\log\log x)^{-1/5}\}), \tag{3.14}$$

where $B(0 < B < (A/2)k^{-8/5})$ is a positive constant. Hence Theorem 3.1 follows.

COROLLARY 3.1.1 ($k = 2$). *If $Q^*(x)$ denotes the number of exponentially odd integers $\leqq x$, then for $x \geqq 3$,*

$$Q^*(x) = \alpha x + O(x^{1/2}\delta(x)), \tag{3.15}$$

where

$$\alpha = \zeta(2)\prod_p\left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) = \prod_p\left(1 - \frac{1}{p(p+1)}\right)$$

and $\delta(x)$ is given by (2.10).

REMARK. 3.1. *This is clearly an improvement in the order estimate of the error term for $Q^*(x)$ obtained by Cohen* (1960; *Theorem* 6.1).

THEOREM 3.2. *If the Riemann hypothesis is true, then for $x \geqq 3$,*

$$Q_k^*(x) = \alpha_k x + O(x^{2/2k+1}\omega(x)), \tag{3.16}$$

where α_k is given by (2.2) *and $\omega(x)$ is given by* (2.18).

PROOF. Following the same procedure adopted in proving theorem 3.1 and making use of Lemmas 2.16 and 2.19 instead of Lemmas 2.8 and 2.14, we get the following instead of (3.6):

$$Q_k^*(x) = \alpha_k x + O(\rho^{1/2-k}z^{1/2}\omega(\rho z)\log z) + O(\rho z\log z). \tag{3.17}$$

Now, choosing $\rho = z^{-1/2k+1}$, we see that $0 < \rho < 1$ and $\rho^{1/2-k}z^{1/2} = \rho z = x^{2/2k+1}$. Since $\omega(x)$ is monotonic increasing, we have $\omega(\rho z) \leqq \omega(z)$. Hence the first and second O-terms in (3.17) are equal to $O(x^{2/2k+1}\omega(x^{1/k})\log x) = O(x^{2/2k+1}\omega(x))$. Hence Theorem 3.2 follows.

COROLLARY 3.2.1 ($k = 2$). *If the Riemann hypothesis is true, then for* $x \geqq 3$,

$$Q^*(x) = \alpha x + O(x^{2/5}\omega(x)), \tag{3.18}$$

where α *is the constant given in Corollary* 3.1.1 *and* $\omega(x)$ *is given by* (2.18).

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