This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to I. G. Connell, Department of Mathematics, McGill University, Montreal, P. Q.

A THEOREM ON HARMONIC HOMOLOGIES

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Introduction. A collineation is a one-one mapping of a projective plane onto itself, taking points into points, lines into lines and preserving incidence. ([1], p. 247). A perspective collineation (sometimes called a central collineation) is a collineation which leaves invariant all points on a line h called the axis, and all lines through a point H called the centre. The perspective collineation is an elation if H is incident with h; otherwise it is a homology.

A harmonic homology is a homology of period two. ([1]), p. 248). This note finds necessary and sufficient conditions under which a product of harmonic homologies and certain sub-products are also harmonic homologies. It is a generalization of a well-known theorem involving the product of two harmonic homologies ([2], p. 64). Although the plane in question is assumed to be Desarguesian, W. Jonsson has pointed out that the theorem can be proved in more general planes as well.

THEOREM. Let (H_k, h_k) denote harmonic homologies with centres H_k and axes h_k for $k = 1, 2, \ldots, n$. Then the products

$$(H_1, h_1)(H_2, h_2) \ldots (H_j, h_j), \quad j = 2, \ldots, n$$

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https://doi.org/10.4153/CMB-1965-028-5 Published online by Cambridge University Press

are all harmonic homologies if and only if the following conditions hold:

(a) $n = 2$ (i) h_2 passes through H_1

(ii) H_2 lies on h_1

(b) $n = 3$ (i) the conditions of (a) hold

(ii) h_1, h_2, h_3 are concurrent

(iii) H_1, H_2, H_3 are collinear

(c) $n > 3$ (i) the conditions of (b) hold

(ii) h_k passes through $h_{k-1} \cdot (H_{k-1} H_{k-2})$

$k = 4, 5, \ldots, n$

(iii) H_k lies on $H_{k-1}(h_{k-1} \cdot h_{k-2})$

$k = 4, 5, \ldots, n$.

Proof. (a) For a proof see [2], pp. 64-65. Note in connection with this reference that in a group the product of two elements of period two is again of that type if and only if the elements commute. Note also that this proof is valid in any (Desarguesian) projective plane ([1], p. 230), not simply in the real projective plane.

(b) Suppose $(H_1, h_1)(H_2, h_2) = (J, j)$ and $(H_1, h_1)(H_2, h_2)(H_3, h_3) = (K, k)$ or $(J, j)(H_3, h_3) = (K, k)$. By (a), H_3 is on j and h_3 passes through J. But $j = H_1 H_2$ and $J = h_1 h_2$, also by (a).

Conversely $(J, j)(H_3, h_3)$ being a harmonic homology implies H_3 is on $j = H_1 H_2$ and h_3 passes through $J = h_1 \cdot h_2$.

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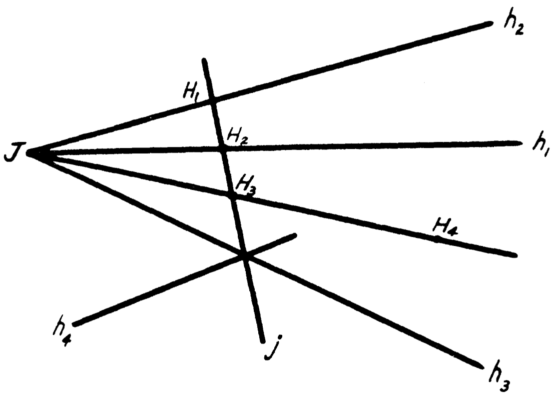
The case (c) proceeds in a way similar to (b) with repeated use of (a).



Note that for $n > 2$, the harmonic homologies appearing may not be permuted amongst themselves.

Finally, it is interesting to note the symmetry in the statement of the theorem. The theorem is, in fact, its own dual.

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https://doi.org/10.4153/CMB-1965-028-5 Published online by Cambridge University Press